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GENERALIZED PSEUDO-BOOLEAN FUNCTIONS ON FINITE SETS

In [2] *S.* Rudeanu gives some theorems about Boolean transformations $F:D\to B^m$, where $(B, U, \cdot, +, 1, 0)$ is a Boolean algebra, $D\subseteq B^n$, D is the set of solutions of a Boolean equation $d(X)=1$, $F=(f_1, f_2, \dots, f_m)$ where f_i are Boolean functions $f_i:B^n\to B$ $(i=1, 2, \dots m)$.

In this paper the idea of *S.* Rudeanu will be spread on generalized pseudo-Boolean transformations $F:D\to L^m$, $F=(f_1, \dots f_m)$ is a vector of generalized pseudo-Boolean functions

$$f_i:L^n\to L \ (i=1, 2, \dots, m).$$

$(P, +, \cdot)$ is a commutative ring with an identity element 1, L is a finite subset of P, D is the set of solutions of pseudo-Boolean equation $d(X)=1$, D is a finite subset of P.

We can de f ine a relation on the commutative ring $(P, +, \cdot)$ with an identity element 1

(1)
$$x_{ii}^{a}=\begin{cases}1, & x_i=a_i\\ 0, & x_i\neq a_i, \qquad x_i, \qquad a_i\in L\end{cases}$$

$X=(x_1, x_2, \dots x_n)$ and $A=(a_1, a_2, \dots, a_n)$ ar vectors with components from L.

Moreover,

(1')
$$(F^C X)=f_1^{c_1}(X)\dots f_m^{c_m}(X), \quad \text{for} \quad C=(c_1, \dots, c_m)$$

instead of $[f(x)]^C$ we write $f^C(x)$.

Directly from (1) it follows:

1 $\sum\limits_{A\in L_n} X^A=1$

II $X^A\cdot X^B=X^A$ if and only if $A=B$, A, $B\in L^n$

III $X^A\cdot X^B=0$ if and only if $A\neq$B, A, $B\in L^n$

IV $\sum\limits_A b_A X^A+\sum\limits_A c_A X^A=\sum\limits_A (b_A+c_A) X^A$, b_A, $c_A\in L$, $A\in L^n$

V $\sum_A (b_A X^A)^a = \sum_A b_A^a X^A,\ b_A,\ a \in L,\ A \in L^n$

VI $\sum_A b_A X^A \cdot \sum_A c_A X^A = \sum_A (b_A c_A)\, X^A,\ b_A,\ c_A \in L,\ A \in L^n.$

A pseudo-Boolean function $f: L^n \to P$ can be written

(2) $$f(X) = \sum_{A \in L^n} f(A)\, X^A.$$

The proof of (2) is given in [1].

Let

(3) $$d: L^n \to P$$

(4) $$f_i: L^n \to L\ (i = 1, 2, \ldots, m)$$

be generalized pseudo-Boolean functions.

In this case the generalized pseduo-Boolean transformation is:

(4′) $$F: D \to L^m,$$

D is the set of solutions of a pseduo-Boolean equation $d(X)=1$ and $F=(f_1, \ldots, f_m)$ is a vector of generalised pseudo-Boolean functions on the domain D.

From (1), (2) and (3) follows that for every $B \in L_p^m$

$$F^B(X) = \sum_{A \in L^s} f_1^{b_1}(A) \ldots f_n^{b_m}(A) X^A \tag{5}$$

i.e.

$$F^B(X) = \sum_{B \in L^n} F^B(A)\, X^A. \tag{6}$$

Theorem. *A generalished speudo-Boolean transformation $F: D \to L_m$ is injective if and only if*

$$F(D) = \{Y \mid \sum_{C \in L^m} (\sum_{A \in L^n} d(A)\, F^C(A))\, Y^C = 1\}.$$

Proof. If a generalized pseudo-Boolean transformation $F: D \to L^m$ is injective then for every $Y \in F(D) \subseteq L^m$ there exists A vector $X \in D$ such that $d(X)=1$ and $Y = F(X)$. (4); (4′); (2) implies that there exists a vector $X \in D$ such that

$$\sum_{A \in L^m} d(A) X^A = 1 \quad \text{and} \quad y_i = f_i(X), \qquad i = 1, 2, \ldots, m.$$

Moreover

$$f_i^{Y_i}(X) = 1, \qquad i = 1, \ldots, m, \qquad \text{i.e.} \qquad \prod_{i=1}^{m} f_i^{Y_i}(X) = 1 \text{ by (1).}$$

All together implies that there exists a vector $X \in D$ such that

$$\sum_{A \in L^n} (d(A) \prod_{i=1}^{m} f_i^{Y_i}(A) X^A = 1,$$

i.e.

$$\sum_{A\in L^n}(d(A)F^Y(A)=1. \tag{7}$$

For every $Y\in L^m$ (7) is satisfied therefore

$$F(D)=\{Y\mid \sum_{C\in L^m}(\sum_{A\in L^n} d(A)f^C(A))\, Y^C=1\}. \tag{8}$$

Let us suppose that the condition (8) is satisfied and the function $F:D\to L^m$ is not injective. Then there exist two vectors A, $B\in L^m$, $A\neq B$ such thad $d(A)=$ $=d(B)=1$, and $F(A)=c_1$ $F(B)=c_2$, $c_1=c_2$ i.e.

$$d(A)F^{c_1}(A)+d(A)F^{c_2}(B)\neq 1.$$

Relation (9) implies

$$\sum_{A\in L^n} d(A)F^{(A)}(A)\neq 1.$$

It is a contradiction with (7) and (8).

The proof is complete.

REFERENCES

[1] C. Ghilezan, *L'espace vectoriel pseudo-booleén généralisé*, Publ. Inst. Math. Beograd, Tome 19 (33), 1975.

[2] S. Rudeanu, *On the range of a Boolean transformation*, Publ. Inst. Math. Beograd, Tome 19 (33), 1977.

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GENERALISANE PSEUDO-BULOVE TRANSFORMACIJE NA KONAČNOM SKUPU

Rezime

Dat je potreban i dovoljan uslov (8) da generalisana pseudo-Bulova transformacija $F:D\to L^m$ bude injektivna, gde je domen D skup rešenja pseudo-Bulove jednačine $d(X)=1$. $D\subseteq L^n$.