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NEW FORMS OF STRONG WEAKLY μ-COMPACT IN TERMS OF HEREDITARY CLASSES

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Abstract. The aim of this paper is to introduce and study new types of strong weakly μ-compact spaces in generalized topological spaces with a hereditary class, called weakly $\mathcal{S}\mu\mathcal{H}$-compact and weakly $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact spaces. Some fundamental properties of these spaces are given. Also, we investigate the invariants of weakly $\mathcal{S}\mu\mathcal{H}$-compact and weakly $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact spaces under functions.

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1. Introduction and Preliminaries

In 2007, Á. Császár [4] defined a class of subsets of a nonempty set called a hereditary class and studied a modification of the generalized topology with hereditary classes. In this paper, we introduce and study strong forms of weakly μ-compact spaces with respect to a hereditary class which was introduced by Qahis et al. in [8].

Let X be a nonempty set and $p(X)$ the power set of X. A subfamily μ of $p(X)$ is called a generalized topology [2] if $\phi \in \mu$ and the arbitrary union of members of μ is again in μ. The pair (X, μ) is called a generalized topological space (briefly GTS). The elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [2, 3]). A nonempty subcollection $\mathcal{H}$ of $p(X)$ is called a hereditary class (briefly HC) (see [4, 10, 5, 14]) if $A \subset B$, $B \in \mathcal{H}$ implies $A \in \mathcal{H}$. An HC $\mathcal{H}$ is called an ideal if $\mathcal{H}$ satisfies the additional condition: $A, B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$ [6]. Some useful hereditary classes in X are: $p(A)$, where $A \subseteq X$, $\mathcal{H}_f$, the HC of all finite subsets of X, and $\mathcal{H}_c$, the HC of all countable subsets of X. We introduced the notion of weakly $\mu\mathcal{H}$-compact spaces as follows: A subset A of

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X is said to be weakly $\mu\mathcal{H}$-compact [8] (resp. $\mu\mathcal{H}$-compact [1]) if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by μ-open sets, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{c_\mu(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$ (resp. $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$). If $A = X$, then (X, μ) is called a weakly $\mu\mathcal{H}$-compact (resp. $\mu\mathcal{H}$-compact) space. A subset A of a GTS (X, μ) is said to be weakly μ-compact [13] if any cover of A by μ-open sets of X has a finite subfamily, the union of the μ-closures of whose members covers A. If $A = X$, then (X, μ) is called a weakly μ-compact space. Given a generalized topological space (X, μ) with an HC $\mathcal{H}$, for a subset A of X, the generalized local function of A with respect to $\mathcal{H}$ and μ [4] is defined as follows: $A^*(\mathcal{H}, \mu) = \{x \in X : U \cap A \notin \mathcal{H} \quad \text{for all} \quad U \in \mu_x\}$, where $\mu_x = \{U : x \in U \quad \text{and} \quad U \in \mu\}$. Also, for a subset A of X, $c^*_\mu(A)$ is defined by $c^*_\mu(A) = A \cup A^*$. The family $\mu^* = \{A \subset X : c^*_\mu(X \setminus A) = X \setminus A\}$ is a GT on X which is finer than μ [4]. The elements of μ^* are called μ^*-open and the complement of a μ^*-open set is called a μ^*-closed set. It is clear that a subset A is μ^*-closed if and only if $A^* \subset A$. We call $(X, \mu, \mathcal{H})$ a hereditary generalized topological space and briefly we denote it by HGTS.

Theorem 1.1. *[4] Let (X, μ) be a GTS, $\mathcal{H}$ a hereditary class on X and A a subset of X. If A is μ^*-open, then for each $x \in A$ there exist $U \in \mu_x$ and $H \in \mathcal{H}$ such that $x \in U \setminus H \subset A$.*

Definition 1.2. [13] A GTS (X, μ) is said to be μ-regular if for each μ-open subset U of X and each $x \in U$, there exist a μ-open subset V of X and a μ-closed subset F of X such that $x \in V \subset F \subset U$.

Definition 1.3. [13] Let A be a subset of a GTS (X, μ). A point $x \in X$ is called a θ_μ-accumulation point of A if $c_\mu(V) \cap A \neq \emptyset$ for every μ-open subset V of X that contains x. The set of all θ_μ-accumulation points of A is called the θ_μ-closure of A and is denoted by $(c_\mu)_\theta(A)$. A is called μ_θ-closed if $(c_\mu)_\theta(A) = A$. The complement of a μ_θ-closed set is said to be μ_θ-open.

It is clear that A is μ_θ-open if and only if for each $x \in A$, there exists a μ-open set V such that $x \in V \subset c_\mu(V) \subset A$.

Definition 1.4. [13] Let A be a subset of a space (X, μ). Then A is said to be:

1. μ-regular closed if $A = c_\mu(i_\mu(A))$,

2. μ-regular open if $X \setminus A$ is μ-regular closed.

Definition 1.5. Let (X, μ) and (Y, ν) be two GTSs, then a function $f : (X, \mu) \to (Y, \nu)$ is said to be.
(1) (μ, ν)-continuous [2] if $U \in \nu$ implies $f^{-1}(U) \in \mu$.
(2) almost (μ, ν)-continuous [7] if for each $x \in X$ and each ν-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(U) \subseteq i_\nu(c_\nu(V))$.
(3) $\theta(\mu, \nu)$-continuous [2] if for every $x \in X$ and every ν-open subset V of Y containing $f(x)$, there exists a μ-open subset U in X containing x such that $f(c_\mu(U)) \subseteq c_\nu(V)$.

(4) (μ,ν)-open (or μ-open) [12] if $U \in \mu$ implies $f(U) \in \nu$.
(5) (μ,ν)-closed (or μ-closed) [11] if $f(F)$ is ν-closed in Y for each μ-closed set F of X.

Lemma 1.6. *[13] Let $f : (X,\mu) \to (Y,\nu)$ be a function. Then the following are equivalent:*

1. *f is (μ,ν)-continuous;*
2. *for every $x \in X$ and every ν-open set V containing $f(x)$, there exists a μ-open set U containing x such that $f(U) \subset V$;*
3. *$f(c_\mu(A)) \subset c_\nu(f(A))$ for every subset A of X;*
4. *$c_\mu(f^{-1}(B)) \subset f^{-1}(c_\nu(B))$ for every subset B of Y.*

Definition 1.7. A subset A of X is said to be $\mu\mathcal{H}$-compact [1] if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of A by μ-open sets, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. If $A = X$, then (X,μ) is called a $\mu\mathcal{H}$-compact space.

2. Weakly $\mathcal{S}\mu\mathcal{H}$-Compact and Weakly $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-Compact Spaces

In this section we define strong forms of weakly $\mu\mathcal{H}$-compact spaces, called weakly $\mathcal{S}\mu\mathcal{H}$-compact and weakly $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact spaces as follows:

Definition 2.1. Let (X,μ) be a GTS with HC. A subset A of an HGTS $(X,\mu,\mathcal{H})$ is said to be:

1. weakly $\mathcal{S}\mu\mathcal{H}$-compact if for every family $\{V_\alpha : \alpha \in \Delta\}$ of μ-open sets with $A \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $A \setminus \cup_{\alpha\in\Delta_0} c_\mu(V_\alpha) \in \mathcal{H}$. If $A = X$, then (X,μ) is called a weakly $\mathcal{S}\mu\mathcal{H}$-compact space;
2. weakly $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact if for every family $\{V_\alpha : \alpha \in \Delta\}$ of μ-open sets with $A \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $A \subseteq \cup_{\alpha\in\Delta_0} c_\mu(V_\alpha)$. If $A = X$, then (X,μ) is called a weakly $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact space.

Remark 2.2. (1) The following properties are equivalent by Definition 2.1:
(i) (X,μ) is weakly μ-compact;
(ii) $(X,\mu,\{\emptyset\})$ is weakly $\mathcal{S}\mu\{\emptyset\}$-compact;
(iii) $(X,\mu,\{\emptyset\})$ is weakly $\mathbf{S}-\mathcal{S}\mu\{\emptyset\}$-compact;
(iv) (X,μ) is weakly $\mu\{\emptyset\}$-compact.
(2) The following diagram holds:

$$\begin{array}{ccc} \textit{weakly } \mathbf{S}-\mathcal{S}\mu\mathcal{H}-\text{compact} & \Rightarrow & \textit{weakly } \mathcal{S}\mu\mathcal{H}-\text{compact} \\ \Downarrow & & \Downarrow \\ \textit{weakly } \mu-\text{compact} & \Rightarrow & \textit{weakly } \mu\mathcal{H}-\text{compact} \end{array}.$$

Example 2.3. Let μ be the Khalimsky topology, i.e., the topology on the set of integers $\mathbb{Z}$ generated by the set of all triplets of the form $\{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}$ as subbase and the hereditary class $\mathcal{H} = \{A : A \subseteq \mathbb{Z}\}$. Now it is clear that $(\mathbb{Z}, \mu)$ is not weakly μ-compact but it is evidently weakly $\mathcal{S}\mu\mathcal{H}$-compact.

A hereditary class $\mathcal{H}$ is said to be μ-condense [4] if $\mu \cap \mathcal{H} = \emptyset$.

Theorem 2.4. *Let $(X, \mu, \mathcal{H})$ be an HGTS. Then the following properties hold.*

1. *If $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-compact and $\mathcal{H}$ is μ-codense, then (X, μ) is weakly μ-compact.*

2. *If $(X, \mu, \mathcal{H})$ is weakly $\mathcal{S}\mu\mathcal{H}$-compact and $\mathcal{H}$ is μ-codense, then $(X, \mu, \mathcal{H})$ is weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact.*

Proof. (1) Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of μ-open subsets of X. Then there exists a finite subset Δ_0 of Δ such that $X \setminus \cup_{\alpha\in\Delta_0} c_\mu(V_\alpha) \in \mathcal{H}$. Since $\mathcal{H}$ is μ-codense, then $i_\mu(X \setminus \cup_{\alpha\in\Delta_0} c_\mu(V_\alpha)) = X \setminus \cup_{\alpha\in\Delta_0} c_\mu(V_\alpha) = \emptyset$ which implies $X \subseteq \cup_{\alpha\in\Delta_0} c_\mu(V_\alpha)$. Hence (X, μ) is weakly μ-compact.
(2) Let $\{V_\alpha : \alpha \in \Delta\}$ be a family of μ-open subsets of X such that $X \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$. There exists a finite subset Δ_0 of Δ such that $X \setminus \cup_{\alpha\in\Delta_0} c_\alpha(V_\alpha) \in \mathcal{H}$. Since, $\mathcal{H}$ is μ-codense, then $i_\mu(X \setminus \cup_{\alpha\in\Delta_0} c_\alpha(V_\alpha)) = X \setminus \cup_{\alpha\in\Delta_0} c_\alpha(V_\alpha) = \emptyset$. It follows that $X \subseteq \cup_{\alpha\in\Delta_0} c_\alpha(V_\alpha)$ and hence $(X, \mu, \mathcal{H})$ is weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact. □

Proposition 2.5. *For an HGTS $(X, \mu, \mathcal{H})$, the following properties hold.*

1. *$(X, \mu, \mathcal{H})$ is weakly $\mathcal{S}\mu\mathcal{H}$-compact if and only if for any family $\{V_\alpha : \alpha \in \Delta\}$ of μ-regular open subsets of X such that $X \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $X \setminus \cup_{\alpha\in\Delta_0} c_\mu(V_\alpha) \in \mathcal{H}$.*

2. *$(X, \mu, \mathcal{H})$ is weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact if and only if for any family of $\{V_\alpha : \alpha \in \Delta\}$ of μ-regular open subsets of X such that $X \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $X \subseteq \cup_{\alpha\in\Delta_0} c_\mu(V_\alpha)$.*

Proof. (1) Necessity is obvious from the definition. To show sufficiency, assume $\{V_\alpha : \alpha \in \Delta\}$ is a family of μ-open subsets of X such that $X \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$. Then $\{i_\mu(c_\mu(V_\alpha)) : \alpha \in \Delta\}$ is a family of μ-regular open sets. Since $V_\alpha \subseteq i_\mu(c_\mu(V_\alpha))$, then $X \setminus \cup_{\alpha\in\Delta} i_\mu(c_\mu(V_\alpha)) \in \mathcal{H}$. Thus there exists a finite subset Δ_0 of Δ such that $X \setminus \cup_{\alpha\in\Delta_0} c_\mu(i_\mu(c_\mu(V_\alpha))) \in \mathcal{H}$. Since $X \setminus \cup_{\alpha\in\Delta_0} c_\mu(V_\alpha) \subseteq X \setminus \cup_{\alpha\in\Delta_0} c_\mu(i_\mu(c_\mu(V_\alpha)))$, then $X \setminus \cup_{\alpha\in\Delta_0} c_\mu(V_\alpha) \in \mathcal{H}$. This implies that $(X, \mu, \mathcal{H})$ is weakly $\mathcal{S}\mu\mathcal{H}$-compact.
(2) The proof is similar to (1) □

Proposition 2.6. *For an HGTS $(X, \mu, \mathcal{H})$, the following properties are equivalent:*

1. *$(X, \mu, \mathcal{H})$ is weakly $\mathcal{S}\mu\mathcal{H}$-compact;*

2. *For any family $\{F_\alpha : \alpha \in \Delta\}$ of μ-closed subsets of X such that $\cap_{\alpha\in\Delta} F_\alpha \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $\cap_{\alpha\in\Delta_0} i_\mu(F_\alpha) \in \mathcal{H}$;*

3. *For any family $\{F_\alpha : \alpha \in \Delta\}$ of μ-regular closed subsets of X such that $\cap_{\alpha\in\Delta}F_\alpha \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $\cap_{\alpha\in\Delta_0} i_\mu (F_\alpha) \in \mathcal{H}$.*

Proof. (1) $\Rightarrow$ (2): Let $\{F_\alpha : \alpha \in \Delta\}$ be a family of μ-closed subsets of X such that $\cap_{\alpha\in\Delta}F_\alpha \in \mathcal{H}$. Then $\{X \setminus F_\alpha : \alpha \in \Delta\}$ is a family of μ-open subsets of X. Since

$$\cap_{\alpha\in\Delta}F_\alpha = X \setminus \cup_{\alpha\in\Delta}(X \setminus F_\alpha) \in \mathcal{H},$$

there exists a finite subset Δ_0 of Δ such that $X \setminus \cup_{\alpha\in\Delta_0} c_\mu (X \setminus F_\alpha) \in \mathcal{H}$. Now we have

$$\begin{aligned} X \setminus \cup_{\alpha\in\Delta_0} c_\mu (X \setminus F_\alpha) &= \cap_{\alpha\in\Delta_0}(X \setminus c_\mu (X \setminus F_\alpha)) \\ &= \cap_{\alpha\in\Delta_0} i_\mu (X \setminus (X \setminus F_\alpha)) = \cap_{\alpha\in\Delta_0} i_\mu (F_\alpha) \in \mathcal{H}. \end{aligned}$$

(2) $\Rightarrow$ (3): It is obvious
(3) $\Rightarrow$ (1): Let $\{V_\alpha : \alpha \in \Delta\}$ be any family of μ-open subsets of X such that $X \setminus \cup_{\alpha\in\Delta}V_\alpha \in \mathcal{H}$. Now $\{X \setminus i_\mu (c_\mu(V_\alpha)) : \alpha \in \Delta\}$ is a family of μ-regular closed sets and

$$\cap_{\alpha\in\Delta}(X \setminus i_\mu (c_\mu (V_\alpha))) = \cap_{\alpha\in\Delta} c_\mu (i_\mu (X \setminus V_\alpha)) \in \mathcal{H}.$$

By assumption there exists a finite subset Δ_0 of Δ such that

$$\cap_{\alpha\in\Delta_0} i_\mu(c_\mu(i_\mu(X \setminus V_\alpha))) \in \mathcal{H}.$$

Now

$$\begin{aligned} \cap_{\alpha\in\Delta_0} i_\mu(c_\mu(i_\mu(X \setminus V_\alpha))) &\supset \cap_{\alpha\in\Delta_0} i_\mu(X \setminus V_\alpha) \\ &= \cap_{\alpha\in\Delta_0}(X \setminus c_\mu(V_\alpha)) = X \setminus \cup_{\alpha\in\Delta_0} c_\mu(V_\alpha). \end{aligned}$$

Therefore, $X \setminus \cup_{\alpha\in\Delta_0} c_\mu(V_\alpha) \in \mathcal{H}$. Hence, $(X, \mu, \mathcal{H})$ is weakly $\mathcal{S}\mu\mathcal{H}$-compact. □

Proposition 2.7. *For an HGTS $(X, \mu, \mathcal{H})$, the following properties are equivalent:*

1. *$(X, \mu, \mathcal{H})$ is weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact;*
2. *For any family $\{F_\alpha : \alpha \in \Delta\}$ of μ-closed subsets of X such that $\cap_{\alpha\in\Delta}F_\alpha \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $\cap_{\alpha\in\Delta_0} i_\mu (F_\alpha) = \emptyset$;*
3. *For any family $\{F_\alpha : \alpha \in \Delta\}$ of μ-regular closed subsets of X such that $\cap_{\alpha\in\Delta}F_\alpha \in \mathcal{H}$, there exists a finite subset Δ_0 of Δ such that $\cap_{\alpha\in\Delta_0} i_\mu (F_\alpha) = \emptyset$.*

Proof. The proof is similar to Proposition 2.6. □

Theorem 2.8. *Let (X, μ) be a μ-regular GTS. If $(X, \mu, \mathcal{H})$ is weakly $\mathcal{S}\mu\mathcal{H}$-compact (resp. weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact), then $(X, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-compact.*

Proof. We prove for weakly $\mathcal{S}\mu\mathcal{H}$-compact only and the proof for the other one is similar. Suppose X is μ-regular, weakly $\mathcal{S}\mu\mathcal{H}$-compact and $\{V_\alpha : \alpha \in \Delta\}$ is a cover of μ-open subsets of X. Then for each $x \in X$, there exists $\alpha_x \in \Delta$ such that $x \in V_{\alpha_x}$. Since X is μ-regular, there exists a μ-open set U_x containing x such that $U_x \subset c_\mu(U_x) \subset V_{\alpha_x}$. Then $\{U_x : x \in X\}$ is a cover of μ-open subsets of X and $X \setminus \cup_{x\in X} U_x = \emptyset \in \mathcal{H}$. By hypothesis, there exists a finite subset X_0 of X such that $X \setminus \cup_{x\in X_0} c_\mu (U_x) \in \mathcal{H}$. Since $X \setminus \cup_{x\in X_0} V_{\alpha_x} \subset X \setminus \cup_{x\in X_0} c_\mu (U_x)$, then $X \setminus \cup_{x\in X_0} V_{\alpha_x} \in \mathcal{H}$. Hence, $(X, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-compact. □

Theorem 2.9. *If a HGTS $(X, \mu, \mathcal{H})$ is weakly $\mathcal{S}\mu\mathcal{H}$-compact (resp. weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact), then for every cover $\{V_\alpha : \alpha \in \Delta\}$ of X by μ_θ-open sets, there exists a finite subset Δ_0 of Δ such that $X \setminus \cup_{\alpha\in\Delta_0} V_\alpha \in \mathcal{H}$ (resp. $X \subseteq \cup_{\alpha\in\Delta_0} V_\alpha$).*

Proof. We prove for weakly $\mathcal{S}\mu\mathcal{H}$-compact only and the proof for the other one is similar. Let $\{V_\alpha : \alpha \in \Delta\}$ be a cover of X by μ_θ-open sets. For each $x \in X$, there exists $\alpha_x \in \Delta$ such that $x \in V_{\alpha_x}$. Since V_{α_x} is μ_θ-open, there exists a μ-open set U_{α_x} such that $x \in U_{\alpha_x} \subset c_\mu(U_{\alpha_x}) \subset V_{\alpha_x}$. Then $\{U_{\alpha_x} : \alpha_x \in \Delta\}$ is a cover of X by μ-open subsets and so $X \setminus \cup_{\alpha_x\in\Delta} U_{\alpha_x} = \emptyset \in \mathcal{H}$. By hypothesis, there exists a finite subset Δ_0 of Δ such that $X \setminus \cup_{\alpha_x\in\Delta_0} c_\mu(U_{\alpha_x}) \in \mathcal{H}$. Since $X \setminus \cup_{\alpha_x\in\Delta_0} V_{\alpha_x} \subset X \setminus \cup_{\alpha_x\in\Delta_0} c_\mu(U_{\alpha_x})$, then $X \setminus \cup_{\alpha_x\in\Delta_0} V_{\alpha_x} \in \mathcal{H}$. □

Theorem 2.10. *Every μ_θ-closed subset of a weakly $\mathcal{S}\mu\mathcal{H}$-compact (resp. weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact) space $(X, \mu, \mathcal{H})$ is weakly $\mathcal{S}\mu\mathcal{H}$-compact (resp. weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact).*

Proof. We prove for weakly $\mathcal{S}\mu\mathcal{H}$-compact only and the proof for the other one is similar. Let F be a μ_θ-closed subset of X, $\{V_\alpha : \alpha \in \Delta\}$ be a family of μ-open subsets of X such that $F \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$. Since $X \setminus F$ is μ_θ-open, for each $x \in X \setminus F$, there exists a μ-open set U_x such that $x \in U_x \subset c_\mu(U_x) \subset X \setminus F$. Then $\{V_\alpha : \alpha \in \Delta\} \cup \{U_x : x \in X \setminus F\}$ is a collection of μ-open subsets of X and

$$\begin{aligned} X\setminus[(\cup_{\alpha\in\Delta} V_\alpha) \cup (\cup_{x\in X\setminus F} U_x)] &= X \setminus [(\cup_{\alpha\in\Delta} V_\alpha) \cup (X \setminus F)] \\ &= (X \setminus (\cup_{\alpha\in\Delta} V_\alpha)) \cap F = F \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}. \end{aligned}$$

By hypothesis, there exists a finite subset Δ_0 of Δ and finite points, say $x_1, x_2, ..., x_n \in X \setminus F$, such that $X \setminus [(\cup_{\alpha\in\Delta_0} c_\mu (V_\alpha)) \cup (\cup_{i=1}^n c_\mu (U_{x_i}))] \in \mathcal{H}$. Then

$$\begin{aligned} & X \setminus [(\cup_{\alpha\in\Delta_0} c_\mu (V_\alpha)) \cup (\cup_{i=1}^n c_\mu (U_{x_i}))] \\ = \quad & (X \setminus \cup_{\alpha\in\Delta_0} c_\mu (V_\alpha)) \cap (X \setminus \cup_{i=1}^n c_\mu (U_{x_i})) \\ \supset \quad & (X \setminus \cup_{\alpha\in\Delta_0} c_\mu (V_\alpha)) \cap X \setminus (X \setminus F) \\ = \quad & (X \setminus \cup_{\alpha\in\Delta_0} c_\mu (V_\alpha)) \cap F \\ = \quad & F \setminus \cup_{\alpha\in\Delta_0} c_\mu (V_\alpha), \end{aligned}$$

which implies $F\setminus\cup_{\alpha\in\Delta_0} c_\mu (V_\alpha) \in \mathcal{H}$. Therefore, F is weakly $\mathcal{S}\mu\mathcal{H}$-compact. □

Theorem 2.11. *For an HGTS $(X, \mu, \mathcal{H})$, the following properties hold.*

1. *If A_1 and A_2 are weakly $\mathcal{S}\mu\mathcal{H}$-compact subsets of $(X, \mu, \mathcal{H})$ and $\mathcal{H}$ is an ideal, then $A_1 \cup A_2$ is weakly $\mathcal{S}\mu\mathcal{H}$-compact.*

2. *If A_1 and A_2 are weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact subsets of $(X, \mu, \mathcal{H})$, then $A_1 \cup A_2$ is weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be a family of μ-open subsets of X such that $(A_1 \cup A_2) \backslash \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$. Since $A_1 \backslash \cup_{\alpha\in\Delta} V_\alpha \subseteq (A_1 \cup A_2) \backslash \cup_{\alpha\in\Delta} V_\alpha$ and $A_2 \backslash \cup_{\alpha\in\Delta} V_\alpha \subseteq (A_1 \cup A_2) \setminus \cup_{\alpha\in\Delta} V_\alpha$, then $A_1 \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$ and $A_2 \setminus \cup_{\alpha\in\Delta} V_\alpha \in \mathcal{H}$.
(1) Since A_1 and A_2 are weakly $\mathcal{S}\mu\mathcal{H}$-compact, then there exist finite subsets Δ_0 and Δ_1 of Δ with $A_1 \setminus \cup_{\alpha\in\Delta_0} c_\mu(V_\alpha) \in \mathcal{H}$ and $A_2 \setminus \cup_{\alpha\in\Delta_1} c_\mu(V_\alpha) \in \mathcal{H}$. This implies that $A_1 \setminus \cup_{\alpha\in\Delta_0\cup\Delta_1} c_\mu(V_\alpha) \in \mathcal{H}$ and $A_2 \setminus \cup_{\alpha\in\Delta_0\cup\Delta_1} c_\mu(V_\alpha) \in \mathcal{H}$ and since $\mathcal{H}$ is an ideal we have that $(A_1 \cup A_2) \setminus \cup_{\alpha\in\Delta_0\cup\Delta_1} c_\mu(V_\alpha) \in \mathcal{H}$. Hence $A_1 \cup A_2$ is weakly $\mathcal{S}\mu\mathcal{H}$-compact.
(2) Since A_1 and A_2 are weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact, there exist finite subsets Δ_0 and Δ_1 of Δ such that $A_1 \subseteq \cup_{\alpha\in\Delta_0} c_\mu(V_\alpha)$ and $A_2 \subseteq \cup_{\alpha\in\Delta_1} c_\mu(V_\alpha)$. This implies that $A_1 \subseteq \cup_{\alpha\in\Delta_0\cup\Delta_1} c_\mu(V_\alpha)$ and $A_2 \subseteq \cup_{\alpha\in\Delta_0\cup\Delta_1} c_\mu(V_\alpha)$ and hence $A_1 \cup A_2 \subseteq \cup_{\alpha\in\Delta_0\cup\Delta_1} c_\mu(V_\alpha)$. Thus $A_1 \cup A_2$ is weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact. □

The following example shows that the first part of the previous theorem does not hold when $\mathcal{H}$ is just a hereditary class, not an ideal.

Example 2.12. Let $\mathbb{R}$ be the set of real numbers, μ the standard topology and the hereditary class $\mathcal{H} = \{H \subset \mathbb{R} : H \subset (0, 1) \quad or \quad H \subset (1, 2)\}$. Observe that $H_1 = (0, 1)$ and $H_2 = (1, 2)$ are weakly $\mathcal{S}\mu\mathcal{H}$-compact sets. But $H_1 \cup H_2$ is not weakly $\mathcal{S}\mu\mathcal{H}$-compact. Note that $\{(\frac{1}{n}, 2 - \frac{1}{n}) : n \in \mathbb{Z}^+\}$ is a family of μ-open subsets of X and $(H_1 \cup H_2) \setminus \cup_{n>1}(\frac{1}{n}, 2 - \frac{1}{n}) = \emptyset \in \mathcal{H}$. Let $\{n_1, n_2, ..., n_k\}$ be any finite subset of the positive integer $\mathbb{Z}^+$ and let $N = max\{n_1, n_2, ..., n_k\}$. Then $(H_1 \cup H_2) \setminus \cup_{i=1}^k c_\mu(\frac{1}{n_i},\ 2 - \frac{1}{n_i}) = (H_1 \cup H_2) \setminus \cup_{i=1}^k [\frac{1}{n_i},\ 2 - \frac{1}{n_i}] = (H_1 \cup H_2) \setminus \left[\frac{1}{N},\ 2 - \frac{1}{N}\right] = \left(0, \frac{1}{N}\right) \cup \left(2 - \frac{1}{N}, 2\right) \notin \mathcal{H}$.

3. Invariants Under Functions

In this section we investigate the invariants of weakly $\mu\mathcal{H}$-compact (resp. weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact) spaces by functions. Note that if $\mathcal{H}$ is a hereditary class on a set X and $f : X \to Y$ is a function, then $f(\mathcal{H}) = \{f(H) : H \in \mathcal{H}\}$ is a hereditary class on Y [1].

Theorem 3.1. *Let $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ be a (μ, ν)-continuous surjection. Then the following properties hold.*

1. *If $(X, \mu, \mathcal{H})$ is weakly $\mathcal{S}\mu\mathcal{H}$-compact, then $(Y, \nu, f(\mathcal{H}))$ is weakly $\mathcal{S}\nu f(\mathcal{H})$-compact.*

2. *If $(X, \mu, \mathcal{H})$ is weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact, then $(Y, \nu, f(\mathcal{H}))$ is weakly $\mathbf{S} - \mathcal{S}\nu f(\mathcal{H})$-compact.*

Proof. (1) Let $\{V_\alpha : \alpha \in \Delta\}$ be a family of ν-open subsets of Y such that $Y \setminus \cup_{\alpha\in\Delta} V_\alpha \in f(\mathcal{H})$. Since f is (μ,ν)-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a family of μ-open subsets of X and $(X,\mu,\mathcal{H})$ is weakly $\mathcal{S}\mu\mathcal{H}$-compact. Then there exists a finite subset Δ_0 of Λ such that $X \setminus \cup_{\alpha\in\Delta_0} c_\mu\left(f^{-1}(V_\alpha)\right) \in \mathcal{H}$. Since f is (μ,ν)-continuous, $c_\mu(f^{-1}(V_\alpha)) \subset f^{-1}(c_\nu(V_\alpha))$. This implies,

$$X \setminus \cup_{\alpha\in\Delta_0} f^{-1}(c_\nu(V_\alpha)) \subset X \setminus \cup_{\alpha\in\Delta_0} c_\mu(f^{-1}(V_\alpha)) \in \mathcal{H}.$$

Hence

$$\begin{aligned} X \setminus \cup_{\alpha\in\Delta_0} f^{-1}(c_\nu(V_\alpha)) &= X \setminus f^{-1}(\cup_{\alpha\in\Delta_0} c_\nu(V_\alpha)) \\ &= f^{-1}(Y \setminus \cup_{\alpha\in\Delta_0} c_\nu(V_\alpha)) \in \mathcal{H}, \end{aligned}$$

and hence

$$f(f^{-1}(Y \setminus \cup_{\alpha\in\Delta_0} c_\nu(V_\alpha))) = Y \setminus \cup_{\alpha\in\Delta_0} c_\nu(V_\alpha) \in f(\mathcal{H}).$$

Hence $(Y,\nu,f(\mathcal{H}))$ is weakly $\mathcal{S}\nu f(\mathcal{H})$-compact.
(2) Let $\{V_\alpha : \alpha \in \Delta\}$ be a family of ν-open subsets of Y such that $Y \setminus \cup_{\alpha\in\Delta} V_\alpha \in f(\mathcal{H})$. Since f is (μ,ν)-continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is a family of μ-open subsets of X and $(X,\mu,\mathcal{H})$ is weakly $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact. Then there exists a finite subset Δ_0 of Δ such that $X = \cup_{\alpha\in\Delta_0} c_\mu\left(f^{-1}(V_\alpha)\right)$. Since f is (μ,ν)-continuous, it follows from Lemma 1.7 (4) that $c_\mu(f^{-1}(V_\alpha)) \subset f^{-1}(c_\nu(V_\alpha))$. Therefore,

$$\begin{aligned} Y = f(X) = f(\cup_{\alpha\in\Delta_0} c_\mu\left(f^{-1}(V_\alpha)\right)) &\subseteq f(\cup_{\alpha\in\Delta_0} f^{-1}\left(c_\nu\left(V_\alpha\right)\right)) \\ &= \cup_{\alpha\in\Delta_0} f(f^{-1}\left(c_\nu\left(V_\alpha\right)\right)) \subseteq \cup_{\alpha\in\Delta_0} c_\nu\left(V_\alpha\right). \end{aligned}$$

This implies that $(Y,\nu,f(\mathcal{H}))$ is weakly $\mathbf{S}-\mathcal{S}\nu f(\mathcal{H})$-compact. □

Corollary 3.2. *The following properties hold.*

1. *The (μ,ν)-continuous image of a weakly $\mathcal{S}\mu\mathcal{H}$-compact space is weakly $\mathcal{S}\nu f(\mathcal{H})$-compact.*

2. *The (μ,ν)-continuous image of a weakly $\mathbf{S}-\mathcal{S}\mu\mathcal{H}$-compact space is weakly $\mathbf{S}-\mathcal{S}\nu f(\mathcal{H})$-compact.*

Corollary 3.3. *Let $f : (X,\mu) \to (Y,\nu,\mathcal{G})$ be a (μ,ν)-open bijective function. Then*

1. *If $(Y,\nu,\mathcal{G})$ is weakly $\mathcal{S}\nu\mathcal{G}$-compact, then (X,μ) is weakly $\mathcal{S}\mu f^{-1}(\mathcal{G})$-compact.*

2. *If $(Y,\nu,\mathcal{G})$ is weakly $\mathbf{S}-\mathcal{S}\nu\mathcal{G}$-compact, then (X,μ) is weakly $\mathbf{S}-\mathcal{S}\mu f^{-1}(\mathcal{G})$-compact.*

Proof. The proof is clear from Theorem 3.1. □

Theorem 3.4. *Let $f : (X,\mu,\mathcal{H}) \to (Y,\nu)$ be a $\theta(\mu,\nu)$-continuous surjection. Then, following properties hold.*

1. *If $(X, \mu, \mathcal{H})$ is weakly $\mathcal{S}\mu\mathcal{H}$-compact, then $(Y, \nu, f(\mathcal{H}))$ is weakly $\mathcal{S}\nu f(\mathcal{H})$-compact.*

2. *If $(X, \mu, \mathcal{H})$ is weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact, then $(Y, \nu, f(\mathcal{H}))$ is weakly $\mathbf{S} - \mathcal{S}\nu f(\mathcal{H})$-compact.*

Proof. Let $\mathcal{V} = \{V_\alpha : \alpha \in \Delta\}$ be a family of ν-open subsets of Y such that $Y \setminus \cup_{\alpha\in\Delta} V_\alpha \in f(\mathcal{H})$. Let $x \in X$ and V_{α_x} be a ν-open set in Y such that $f(x) \in V_{\alpha_x}$. Since f is $\theta(\mu, \nu)$-continuous, there exists a μ-open set U_{α_x} of X containing x such that $f(c_\mu(U_{\alpha_x})) \subseteq c_\nu(V_{\alpha_x})$. Now $\{U_{\alpha_x} : x \in X\}$ is a cover of μ-open subsets of X.
(1) By hypothesis, there exists a finite subset X_0 of X such that $X \setminus \cup_{x\in X_0} c_\mu(U_{\alpha_x}) \in \mathcal{H}$. Now $f(X \setminus \cup_{x\in X_0} c_\mu(U_{\alpha_x})) \in f(\mathcal{H})$. We know $f(X) \setminus f(\cup_{x\in X_0} c_\mu(U_{\alpha_x})) \subseteq f(X \setminus \cup_{x\in X_0} c_\mu(U_{\alpha_x}))$. This implies $Y \setminus \cup_{x\in X_0} f(c_\mu(U_{\alpha_x})) \in f(\mathcal{H})$. Since $f(c_\mu(U_{\alpha_x})) \subseteq c_\mu(V_{\alpha_x})$ for each α_x, $Y \setminus \cup_{x\in X_0} c_\nu(V_{\alpha_x}) \subseteq Y \setminus \cup_{x\in X_0} f(c_\mu(U_{\alpha_x}))$. Thus $Y \setminus \cup_{x\in X_0} c_\nu(V_{\alpha_x}) \in f(\mathcal{H})$. This implies that $(Y, \nu, f(\mathcal{H}))$ is weakly $\mathcal{S}\nu f(\mathcal{H})$-compact.
(2) By hypothesis, there exists a finite subset X_0 of X such that $X = \cup_{x\in X_0} c_\mu(U_{\alpha_x})$. Therefore,

$$Y = f(X) = f(\cup_{x\in X_0} c_\mu(U_{\alpha_x})) = \cup_{x\in X_0} f(c_\mu(U_{\alpha_x})) \subseteq \cup_{x\in X_0} c_\nu(V_{\alpha_x}).$$

This implies that $(Y, \nu, f(\mathcal{H}))$ is weakly $\mathbf{S} - \mathcal{S}\nu f(\mathcal{H})$-compact. □

Corollary 3.5. *The following properties hold.*

1. *The $\theta(\mu, \nu)$-continuous image of a weakly $\mathcal{S}\mu\mathcal{H}$-compact space is weakly $\mathcal{S}\nu f(\mathcal{H})$-compact.*

2. *The $\theta(\mu, \nu)$-continuous image of a weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact space is weakly $\mathbf{S} - \mathcal{S}\nu f(\mathcal{H})$-compact.*

The following lemma is used in the proofs of corollaries stated below.

Lemma 3.6. *[9] If $f : (X, \mu) \to (Y, \nu)$ is almost (μ, ν)-continuous, then f is $\theta(\mu, \nu)$-continuous.*

Corollary 3.7. *Let $f : (X, \mu) \to (X, \nu)$ be an almost (μ, ν)-continuous surjection. Then, the following properties hold.*

1. *If $(X, \mu, \mathcal{H})$ is weakly $\mathcal{S}\mu\mathcal{H}$-compact, then $(Y, \nu, f(\mathcal{H}))$ is weakly $\mathcal{S}\nu f(\mathcal{H})$-compact.*

2. *If $(X, \mu, \mathcal{H})$ is weakly $\mathbf{S} - \mathcal{S}\mu\mathcal{H}$-compact, then $(Y, \nu, f(\mathcal{H}))$ is weakly $\mathbf{S} - \mathcal{S}\nu f(\mathcal{H})$-compact.*

Proof. The proof follows immediately from Lemma 3.6 and Corollary 3.5. □

Since every (μ, ν)-continuous function is almost (μ, ν)-continuous, we conclude the following corollary.

Corollary 3.8. *The following properties hold.*

1. *weakly $\mathcal{S}\mu\mathcal{H}$-compact property is a GT property.*

2. *weakly $\mathbf{S} - \mathcal{S}\nu f(\mathcal{H})$-compact property is a GT property.*

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