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THE GENERALIZED BI-PERIODIC FIBONACCI QUATERNIONS AND OCTONIONS

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Abstract. In this paper, we present a further generalization of the bi-periodic Fibonacci quaternions and octonions. We give the generating function, the Binet formula, and some basic properties of these quaternions and octonions. The results of this paper not only give a generalization of the bi-periodic Fibonacci quaternions and octonions, but also include new results such as the matrix representation and the norm value of the generalized bi-periodic Fibonacci sequence.

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1. Introduction

There has been a growing interest in quaternions that have been extensively studied in both applied and theoretical sciences. In particular, quaternions are very good at representing rotations in three-dimensional space. The octonions are invented as an analog to the quaternions, and related to the exceptional Lie algebra. Also they have applications in areas such as super string theory, projective geometry, topology, and Jordan algebras. For more details about quaternions and octonions we refer to [1, 2, 24].

The quaternion algebra

$$\mathbf{H} = \{\sum_{l=0}^{3} a_l e_l : a_l \in \mathbb{R}\} \tag{1.1}$$

is a four dimensional non-commutative vector space over $\mathbb{R}$ and the basis satisfies the following multiplication rules:

$$\begin{aligned} e_l^2 &= -1,\ l \in \{1,2,3\}; \\ e_1e_2 &= -e_2e_1 = e_3,\ e_2e_3 = -e_3e_2 = e_1,\ e_3e_1 = -e_1e_3 = e_2. \end{aligned} \tag{1.2}$$

(e_0 can be identified with real number 1). Also, the quaternion algebra $\mathbf{H}$ is isomorphic to the Clifford algebra $C\ell_{0,2}$. There are several studies on different

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types of sequences over quaternion algebra. For a survey on these researches we refer to [9, 11, 10, 12, 13, 7, 8, 5, 4, 20, 17].

Recently, Tan and et. al. [23, 22] introduced a new generalization of the Fibonacci and Lucas quaternions, named as the bi-periodic Fibonacci and Lucas quaternions. They are emerged as a generalization of the best known quaternions in the literature, such as classical Fibonacci and Lucas quaternions in [9], Pell and Pell-Lucas quaternions in [5], $k-$Fibonacci and $k-$Lucas quaternions in [17].

For $n \geq 0$, the bi-periodic Fibonacci and Lucas quaternions defined as

$$Q_n = \sum_{l=0}^{3} q_{n+l} e_l \quad \text{and} \quad P_n = \sum_{l=0}^{3} p_{n+l} e_l, \tag{1.3}$$

respectively. Note that q_n is the nth bi-periodic Fibonacci number and defined by

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \; n \geq 2 \tag{1.4}$$

with initial values $q_0 = 0$, $q_1 = 1$ and a, b are nonzero real numbers and p_n is the nth bi-periodic Lucas number and defined by

$$p_n = \begin{cases} bp_{n-1} + p_{n-2}, & \text{if } n \text{ is even} \\ ap_{n-1} + p_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \; n \geq 2 \tag{1.5}$$

with the initial conditions $p_0 = 2$, $p_1 = a$.

The Binet formula for the bi-periodic Fibonacci quaternion is given by

$$Q_n = \begin{cases} \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \frac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta}, & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \frac{\alpha^{**} \alpha^n - \beta^{**} \beta^n}{\alpha - \beta}, & \text{if } n \text{ is odd} \end{cases} \tag{1.6}$$

and the Binet formula for the bi-periodic Lucas quaternion is

$$P_n = \begin{cases} \frac{1}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\alpha^{**} \alpha^n + \beta^{**} \beta^n \right), & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\alpha^* \alpha^n + \beta^* \beta^n \right), & \text{if } n \text{ is odd} \end{cases} \tag{1.7}$$

where

$$\begin{aligned} \alpha &: = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2}, \beta := \frac{ab - \sqrt{a^2b^2 + 4ab}}{2} \\ \alpha^* &: = \sum_{l=0}^{3} \frac{a^{\zeta(l+1)}}{(ab)^{\lfloor \frac{l}{2} \rfloor}} \alpha^l e_l, \; \beta^* := \sum_{l=0}^{3} \frac{a^{\zeta(l+1)}}{(ab)^{\lfloor \frac{l}{2} \rfloor}} \beta^l e_l \\ \alpha^{**} &: = \sum_{l=0}^{3} \frac{a^{\zeta(l)}}{(ab)^{\lfloor \frac{l+1}{2} \rfloor}} \alpha^l e_l, \; \beta^{**} := \sum_{l=0}^{3} \frac{a^{\zeta(l)}}{(ab)^{\lfloor \frac{l+1}{2} \rfloor}} \beta^l e_l. \end{aligned} \tag{1.8}$$

Here $\zeta(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function, i.e., $\zeta(n) = 0$ when n is even and $\zeta(n) = 1$ when n is odd. Assume that $a^2b^2 + 4ab \neq 0$. Also we have $\alpha + \beta = ab$, $\alpha - \beta = \sqrt{a^2b^2 + 4ab}$ and $\alpha\beta = -ab$. For the details of the bi-periodic Fibonacci and Lucas sequences see [25, 6, 3, 18, 16].

The octonion algebra

$$\mathbf{O} = \{\sum_{l=0}^{7} a_l e_l : a_l \in \mathbb{R}\} \tag{1.9}$$

is an eight dimensional non-commutative and non-associative vector space over $\mathbb{R}$, and the multiplication rules can be derived from the following table :

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

Table 1: Octonion Multiplication table

Motivated by the results in [23, 22], Yilmaz and et. al. [26, 27] introduced the bi-periodic Fibonacci and Lucas octonions as

$$OQ_n = \sum_{l=0}^{7} q_{n+l} e_l \quad \text{and} \quad OP_n = \sum_{l=0}^{7} p_{n+l} e_l, \tag{1.10}$$

respectively.

The Binet formula for the bi-periodic Fibonacci octonion is

$$OQ_n = \begin{cases} \frac{1}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \frac{\gamma^* \alpha^n - \delta^* \beta^n}{\alpha - \beta}, & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \frac{\gamma^{**} \alpha^n - \delta^{**} \beta^n}{\alpha - \beta}, & \text{if } n \text{ is odd} \end{cases} \tag{1.11}$$

and the Binet formula for the bi-periodic Lucas octonion is

$$OP_n = \begin{cases} \frac{1}{(ab)^{\left\lfloor \frac{n+1}{2} \right\rfloor}} \left(\gamma^{**} \alpha^n + \delta^{**} \beta^n\right), & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\left\lfloor \frac{n+1}{2} \right\rfloor}} \left(\gamma^* \alpha^n + \delta^* \beta^n\right), & \text{if } n \text{ is odd} \end{cases} \tag{1.12}$$

where

$$\begin{aligned} \gamma^* &:= \sum_{l=0}^{7} \frac{a^{\zeta(l+1)}}{(ab)^{\left\lfloor \frac{l}{2} \right\rfloor}} \alpha^l e_l, \ \delta^* := \sum_{l=0}^{7} \frac{a^{\zeta(l+1)}}{(ab)^{\left\lfloor \frac{l}{2} \right\rfloor}} \beta^l e_l \\ \gamma^{**} &:= \sum_{l=0}^{7} \frac{a^{\zeta(l)}}{(ab)^{\left\lfloor \frac{l+1}{2} \right\rfloor}} \alpha^l e_l, \ \delta^{**} := \sum_{l=0}^{7} \frac{a^{\zeta(l)}}{(ab)^{\left\lfloor \frac{l+1}{2} \right\rfloor}} \beta^l e_l. \end{aligned} \tag{1.13}$$

For related studies on different types of sequences over octonion algebra, we refer to [15, 19, 20, 4, 14].

In this paper, we present a further generalization of the bi-periodic Fibonacci quaternions and octonions. We give the generating function, the Binet formula, and some basic properties of these quaternions and octonions. This new generalization can be seen as a generalization of the notions given in [23, 22, 26, 27, 4]. The results of this paper not only give a generalization of the bi-periodic Fibonacci quaternions and octonions, but also include new results such as the matrix representation and the norm value of the generalized bi-periodic Fibonacci sequence. The main contribution of this study is that one can get a great number of distinct quaternion and octonion sequences by providing the initial values in the generalized bi-periodic Fibonacci sequence. To this end, first consider the generalized bi-periodic Fibonacci sequence, $\{w_n\}$, which is defined in [6] as:

$$w_n = \begin{cases} aw_{n-1} + w_{n-2}, & \text{if } n \text{ is even} \\ bw_{n-1} + w_{n-2}, & \text{if } n \text{ is odd} \end{cases}, n \geq 2 \tag{1.14}$$

with arbitrary initial conditions w_0, w_1 where w_0, w_1, a, b are nonzero real numbers. Note that, if we take $w_0 = 0, w_1 = 1$ in $\{w_n\}$, we get the bi-periodic Fibonacci sequence $\{q_n\}$ in (1.4). If we take $w_0 = 2, w_1 = b$, and switch a and b in $\{w_n\}$, we get the bi-periodic Lucas sequence $\{p_n\}$ in (1.5).

In [21], the Binet formula of the sequence $\{w_n\}$ is given by

$$w_n = \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(A\alpha^{n-1} - B\beta^{n-1}\right) \tag{1.15}$$

where

$$A := \frac{\alpha w_1 + b w_0}{\alpha - \beta} \text{ and } B := \frac{\beta w_1 + b w_0}{\alpha - \beta}. \tag{1.16}$$

For more results related to the sequence $\{w_n\}$, we refer to [21].

2. The generalized bi-periodic Fibonacci quaternions

In this section, we introduce the generalized bi-periodic Fibonacci quaternions and give some basic properties of them. These results can be seen as a generalization of the results in [23, 22, 4].

Definition 2.1. The generalized bi-periodic Fibonacci quaternions $\{W_n\}$ are defined by

$$W_n = \sum_{l=0}^{3} w_{n+l} e_l, \tag{2.1}$$

where w_n is defined in (1.14).

In the following, we give several different sequences which are special cases of $\{W_n\}$:

1. If we take the initial conditions $w_0 = 0$ and $w_1 = 1$, we get the bi-periodic Fibonacci quaternions in [23].
2. If we take the initial conditions $w_0 = 2$ and $w_1 = b$, we get the bi-periodic Lucas quaternions in [22]. (Note that we switch a and b).
3. If we take the initial conditions $w_0 = w_1 = 1$ and $a = b = 2$ in $\{w_n\}$, we get the modified Pell quaternion numbers in [4].
4. If we take $a = b = 1$ in $\{w_n\}$, we get the Horadam quaternion numbers in [8] with the case of $q = 1$.

Theorem 2.2. *The generating function for the generalized bi-periodic Fibonacci quaternions W_n is*

$$G(t) = \frac{W_0 + (W_1 - bW_0)\,t + (a-b)\sum_{s=0}^{3} R(t,s)\,e_s}{1 - bt - t^2} \tag{2.2}$$

where

$$R(t,s) := \left(f(t) - \sum_{k=1}^{\left\lfloor \frac{s+1}{2} \right\rfloor} w_{2k-1} t^{2k-1} \right) t^{1-s}, \tag{2.3}$$

$$f(t) := \sum_{n=1}^{\infty} w_{2n-1} t^{2n-1} = \frac{w_1 t + (bw_0 - w_1)\,t^3}{1 - (ab+2)\,t^2 + t^4}. \tag{2.4}$$

Proof. By using a similar method as in [23, Theorem 1] and considering the relation

$$w_{2n-1} = (ab+2)\,w_{2n-3} - w_{2n-5},$$

we get the desired result. $\square$

In the following theorem, we state the Binet formula for the generalized bi-periodic Fibonacci quaternions and so derive some well-known mathematical properties such as Catalan-like identity and Cassini-like identity.

Theorem 2.3. *The Binet formula for the generalized bi-periodic Fibonacci quaternion is*

$$W_n = \begin{cases} \frac{1}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(A\alpha^* \alpha^{n-1} - B\beta^* \beta^{n-1} \right), & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(A\alpha^{**} \alpha^{n-1} - B\beta^{**} \beta^{n-1} \right), & \text{if } n \text{ is odd} \end{cases} \tag{2.5}$$

where $A, B, \alpha^*, \beta^*, \alpha^{**}$, and β^{**} are defined in (1.8) and (1.16).

Proof. By using the definition of the sequence $\{w_n\}$ and the Binet formula in (1.15), we can easily obtain the desired result. □

By using the Binet formula for the generalized bi-periodic Fibonacci quaternion sequences, we obtain the following identity.

Theorem 2.4. *(Catalan-like identity) For nonnegative integer number n and even integer r, such that $r \leq n$, we have*

$$W_{n-r}W_{n+r} - W_n^2 = \begin{cases} \frac{AB(\alpha^r-\beta^r)}{(\alpha\beta)^{r+1}} \left[\alpha^*\beta^*\beta^r - \beta^*\alpha^*\alpha^r\right], & \text{if } n \text{ is even} \\ \frac{AB(\alpha^r-\beta^r)}{(\alpha\beta)^{r}} \left[\alpha^{**}\beta^{**}\beta^r - \beta^{**}\alpha^{**}\alpha^r\right], & \text{if } n \text{ is odd} \end{cases}. \tag{2.6}$$

Proof. For even n, we have

$$\begin{aligned}
&W_{n-r}W_{n+r} - W_n^2 \\
&= \frac{1}{(ab)^n}\left(A\alpha^*\alpha^{n-r-1} - B\beta^*\beta^{n-r-1}\right)\left(A\alpha^*\alpha^{n+r-1} - B\beta^*\beta^{n+r-1}\right) \\
&\quad - \frac{1}{(ab)^n}\left(A\alpha^*\alpha^{n-1} - B\beta^*\beta^{n-1}\right)\left(A\alpha^*\alpha^{n-1} - B\beta^*\beta^{n-1}\right) \\
&= \frac{1}{(ab)^n}\left[AB\alpha^*\beta^*\left(\alpha\beta\right)^{n-1}\left(1-\frac{\beta^r}{\alpha^r}\right) + BA\beta^*\alpha^*\left(\alpha\beta\right)^{n-1}\left(1-\frac{\alpha^r}{\beta^r}\right)\right] \\
&= \frac{\left(\alpha\beta\right)^{n-1}AB}{(ab)^n}\left[\alpha^*\beta^*\left(1-\frac{\beta^r}{\alpha^r}\right) + \beta^*\alpha^*\left(1-\frac{\alpha^r}{\beta^r}\right)\right] \\
&= \frac{AB}{\left(-1\right)^n\alpha\beta}\left[\alpha^*\beta^*\left(\frac{\alpha^r-\beta^r}{\alpha^r}\right) + \beta^*\alpha^*\left(\frac{\beta^r-\alpha^r}{\beta^r}\right)\right] \\
&= \frac{AB}{\left(-1\right)^n\left(\alpha\beta\right)^{r+1}}\left[\alpha^*\beta^*\beta^r\left(\alpha^r-\beta^r\right) + \beta^*\alpha^*\alpha^r\left(\beta^r-\alpha^r\right)\right] \\
&= \frac{AB\left(\alpha^r-\beta^r\right)}{\left(\alpha\beta\right)^{r+1}}\left[\alpha^*\beta^*\beta^r - \beta^*\alpha^*\alpha^r\right].
\end{aligned}$$

Similarly, it can be proven for odd n. □

If we take the initial conditions $w_0 = 0$ and $w_1 = 1$ in (2.6), we get the result in [23, Theorem 5], and if we take the initial conditions $w_0 = 2$ and $w_1 = b$ in (2.6), we get the result in [22, Theorem 5]. Also, it is clear that if we take $r = 2$ in the above theorem we obtain the following result.

Corollary 2.5. *(Cassini-like identity) For nonnegative even integer number n, we have*

$$W_{n-2}W_{n+2} - W_n^2 = \frac{AB\left(\alpha^2-\beta^2\right)}{\left(\alpha\beta\right)^3}\left[\alpha^*\beta^*\beta^2 - \beta^*\alpha^*\alpha^2\right]. \tag{2.7}$$

If we take the initial conditions $w_0 = 0$ and $w_1 = 1$ in (2.7), we get the result in [23, Theorem 3], and if we take the initial conditions $w_0 = 2$ and $w_1 = b$ in (2.7), we get the result in [22, Theorem 3].

To present the Cassini-like identity in a different manner, now we give a matrix representation for the even indices terms of the generalized bi-periodic Fibonacci quaternions.

Theorem 2.6. *For $n \geq 1$, we have*

$$\left[\begin{array}{cc} W_{2n} & W_{2(n-1)} \\ W_{2(n+1)} & W_{2n} \end{array}\right] = \left[\begin{array}{cc} W_2 & W_0 \\ W_4 & W_2 \end{array}\right] \left[\begin{array}{cc} ab+2 & 1 \\ -1 & 0 \end{array}\right]^{n-1}. \tag{2.8}$$

Proof. We prove it by using induction on n. It is clear that the result is true when $n = 1$. Assume that it is true for any integer m such that $1 \leq m \leq n$. Then by using inductive assumption, we have

$$\begin{aligned} & \left[\begin{array}{cc} W_2 & W_0 \\ W_4 & W_2 \end{array}\right] \left[\begin{array}{cc} ab+2 & 1 \\ -1 & 0 \end{array}\right]^{n} \\ &= \left[\begin{array}{cc} W_2 & W_0 \\ W_4 & W_2 \end{array}\right] \left[\begin{array}{cc} ab+2 & 1 \\ -1 & 0 \end{array}\right]^{n-1} \left[\begin{array}{cc} ab+2 & 1 \\ -1 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc} W_{2n} & W_{2(n-1)} \\ W_{2(n+1)} & W_{2n} \end{array}\right] \left[\begin{array}{cc} ab+2 & 1 \\ -1 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc} (ab+2)\,W_{2n} - W_{2(n-1)} & W_{2n} \\ (ab+2)\,W_{2(n+1)} - W_{2n} & W_{2(n+1)} \end{array}\right] = \left[\begin{array}{cc} W_{2(n+1)} & W_{2n} \\ W_{2(n+2)} & W_{2(n+1)} \end{array}\right] \end{aligned}$$

which completes the proof. □

Corollary 2.7. *For $n \geq 1$, we have*

$$W_{2(n-1)}W_{2(n+1)} - W_{2n}^2 = W_0W_4 - W_2^2. \tag{2.9}$$

By means of this formula we can state the Cassini-like identity for the bi-periodic Fibonacci quaternions as:

$$Q_{2(n-1)}Q_{2(n+1)} - Q_{2n}^2 = Q_0Q_4 - Q_2^2,$$

which was not known before.

Following result gives the relation between the generalized bi-periodic Fibonacci quaternions $\{W_n\}$ and the bi-periodic Fibonacci quaternions $\{Q_n\}$.

Theorem 2.8. *For any natural number n, we have*

$$W_{2(n+1)}Q_{2n} - W_{2n}Q_{2(n+1)} = \frac{1}{ab}\left[A\alpha^*\beta^*\beta - B\beta^*\alpha^*\alpha\right]. \tag{2.10}$$

Proof. By using the Binet formula for the generalized bi-periodic Fibonacci quaternions and the bi-periodic Fibonacci quaternions, we can easily obtain the desired result. □

The norm value of the generalized bi-periodic Fibonacci quaternions is

$$Nr\left(W_n\right) := W_n\overline{W_n},$$

where $\overline{W_n} := w_n e_0 - w_{n+1}e_1 - w_{n+2}e_2 - w_{n+3}e_3$ is the conjugate of the generalized bi-periodic Fibonacci quaternion. Thus we have

$$Nr\left(W_n\right) = w_n^2 + w_{n+1}^2 + w_{n+2}^2 + w_{n+3}^2.$$

By using the definition of the norm value and the Binet formula of the sequence $\{w_n\}$, then by making some necessary calculations, we obtain the following result.

Theorem 2.9. *The norm value of the generalized bi-periodic Fibonacci quaternions can be stated as*

$$Nr\left(W_n\right) = T\left(n\right) + T\left(n+1\right), \tag{2.11}$$

where

$$T\left(n\right) := \frac{a^{2\zeta(n+1)}}{\left(ab\right)^{n-\zeta(n)}\left(\alpha\beta\right)^2\left(\alpha-\beta\right)^2}\left[w_1^2 X + 2w_0 w_1 bY + w_0^2 b^2 Z\right], \tag{2.12}$$

and

$$\begin{aligned} X &:= \alpha^{2n}\left(\alpha^4 + \left(\alpha\beta\right)^2\right) + \beta^{2n}\left(\beta^4 + \left(\alpha\beta\right)^2\right) - 4\left(\alpha\beta\right)^{n+2}, \\ Y &:= \alpha^{2n-1}\left(\alpha^4 + \left(\alpha\beta\right)^2\right) + \beta^{2n-1}\left(\beta^4 + \left(\alpha\beta\right)^2\right) + 2\left(\alpha\beta\right)^{n+2}, \\ Z &:= \alpha^{2n-2}\left(\alpha^4 + \left(\alpha\beta\right)^2\right) + \beta^{2n-2}\left(\beta^4 + \left(\alpha\beta\right)^2\right) - 4\left(\alpha\beta\right)^{n+1}. \end{aligned} \tag{2.13}$$

Note that, if we take the initial conditions $w_0 = 0, w_1 = 1$ and $a = b = 2$, we get the norm value of the Pell quaternions in [20, Theorem 3.1], and if we take $a = b$ in $\{w_n\}$, we get the norm value of the Horadam quaternion numbers in [8] with the case of $q = 1$.

Finally, we give some summation formulas for the generalized bi-periodic Fibonacci quaternions.

Theorem 2.10. *For $n \geq 1$, we have*

$$\begin{aligned} (i)\sum_{r=0}^{n-1} W_r &= \frac{W_n - W_{n-2} + W_{n+1} - W_{n-1}}{ab} \\ &\quad - \frac{A\alpha^*\beta^2 - B\beta^*\alpha^2 - ab\left(A\alpha^{**}\beta - B\beta^{**}\alpha\right)}{\left(ab\right)^2}, \end{aligned} \tag{2.14}$$

$$(ii)\sum_{r=0}^{n-1} W_{2r} = \frac{W_{2n} - W_{2n-2}}{ab} - \frac{A\alpha^*\beta^2 - B\beta^*\alpha^2}{\left(ab\right)^2}, \tag{2.15}$$

$$(iii)\sum_{r=0}^{n-1} W_{2r+1} = \frac{W_{2n+1} - W_{2n-1}}{ab} + \frac{A\alpha^{**}\beta - B\beta^{**}\alpha}{ab}. \tag{2.16}$$

Proof. (i)If n is odd,

$$\sum_{r=0}^{n-1} W_r = \sum_{r=0}^{\frac{n-1}{2}} W_{2r} + \sum_{r=0}^{\frac{n-3}{2}} W_{2r+1}$$

$$= \sum_{r=0}^{\frac{n-1}{2}} \frac{1}{(ab)^r} \left(A\alpha^* \alpha^{2r-1} - B\beta^* \beta^{2r-1}\right) + \sum_{r=0}^{\frac{n-3}{2}} \frac{1}{(ab)^r} \left(A\alpha^{**} \alpha^{2r} - B\beta^{**} \beta^{2r}\right)$$

$$= A\alpha^* \alpha^{-1} \sum_{r=0}^{\frac{n-1}{2}} \left(\frac{\alpha^2}{ab}\right)^r - B\beta^* \beta^{-1} \sum_{r=0}^{\frac{n-1}{2}} \left(\frac{\beta^2}{ab}\right)^r$$

$$+ A\alpha^{**} \sum_{r=0}^{\frac{n-3}{2}} \left(\frac{\alpha^2}{ab}\right)^r - B\beta^{**} \sum_{r=0}^{\frac{n-3}{2}} \left(\frac{\beta^2}{ab}\right)^r$$

$$= A\alpha^* \alpha^{-1} \frac{\left(\frac{\alpha^2}{ab}\right)^{\frac{n-1}{2}+1} - 1}{\frac{\alpha^2}{ab} - 1} - B\beta^* \beta^{-1} \frac{\left(\frac{\beta^2}{ab}\right)^{\frac{n-1}{2}+1} - 1}{\frac{\beta^2}{ab} - 1}$$

$$+ A\alpha^{**} \frac{\left(\frac{\alpha^2}{ab}\right)^{\frac{n-3}{2}+1} - 1}{\frac{\alpha^2}{ab} - 1} - B\beta^{**} \frac{\left(\frac{\beta^2}{ab}\right)^{\frac{n-3}{2}+1} - 1}{\frac{\beta^2}{ab} - 1}$$

$$= A\alpha^* \alpha^{-1} \frac{\alpha^{n+1} - (ab)^{\frac{n+1}{2}}}{(\alpha^2 - ab)(ab)^{\frac{n-1}{2}}} - B\beta^* \beta^{-1} \frac{\beta^{n+1} - (ab)^{\frac{n+1}{2}}}{(\beta^2 - ab)(ab)^{\frac{n-1}{2}}}$$

$$+ A\alpha^{**} \frac{\alpha^{n-1} - (ab)^{\frac{n-1}{2}}}{(\alpha^2 - ab)(ab)^{\frac{n-3}{2}}} - B\beta^{**} \frac{\beta^{n-1} - (ab)^{\frac{n-1}{2}}}{(\beta^2 - ab)(ab)^{\frac{n-3}{2}}}$$

$$= A\alpha^* \frac{\alpha^{n+1} - (ab)^{\frac{n+1}{2}}}{\alpha^2 (ab)^{\frac{n+1}{2}}} - B\beta^* \frac{\beta^{n+1} - (ab)^{\frac{n+1}{2}}}{\beta^2 (ab)^{\frac{n+1}{2}}}$$

$$+ A\alpha^{**} \frac{\alpha^{n-1} - (ab)^{\frac{n-1}{2}}}{\alpha (ab)^{\frac{n-1}{2}}} - B\beta^{**} \frac{\beta^{n-1} - (ab)^{\frac{n-1}{2}}}{\beta (ab)^{\frac{n-1}{2}}}$$

$$= \frac{1}{(ab)^{\frac{n+1}{2}}} \left(A\alpha^* \frac{\alpha^{n+1} - (ab)^{\frac{n+1}{2}}}{\alpha^2} - B\beta^* \frac{\beta^{n+1} - (ab)^{\frac{n+1}{2}}}{\beta^2} \right)$$

$$+ \frac{1}{(ab)^{\frac{n-1}{2}}} \left(A\alpha^{**} \frac{\alpha^{n-1} - (ab)^{\frac{n-1}{2}}}{\alpha} - B\beta^{**} \frac{\beta^{n-1} - (ab)^{\frac{n-1}{2}}}{\beta} \right)$$

$$= \frac{1}{(ab)^{\frac{n+1}{2}}} \left(A\alpha^* \alpha^{n-1} - B\beta^* \beta^{n-1} - \frac{A\alpha^* (ab)^{\frac{n+1}{2}}}{\alpha^2} - \frac{B\beta^* (ab)^{\frac{n+1}{2}}}{\beta^2} \right)$$

$$+\frac{1}{(ab)^{\frac{n-1}{2}}}\left(A\alpha^{**}\alpha^{n-2}-B\beta^{**}\beta^{n-2}-\frac{A\alpha^{**}(ab)^{\frac{n-1}{2}}}{\alpha}+\frac{B\beta^{**}(ab)^{\frac{n-1}{2}}}{\beta}\right)$$

$$=\frac{1}{(ab)^{\frac{n+1}{2}}}\left(A\alpha^{*}\alpha^{n-1}-B\beta^{*}\beta^{n-1}\right)-\left(\frac{A\alpha^{*}}{\alpha^{2}}-\frac{B\beta^{*}}{\beta^{2}}\right)$$

$$+\frac{1}{(ab)^{\frac{n-1}{2}}}\left(A\alpha^{**}\alpha^{n-2}-B\beta^{**}\beta^{n-2}\right)-\left(\frac{A\alpha^{**}}{\alpha}-\frac{B\beta^{**}}{\beta}\right)$$

$$=\frac{W_{n+1}-W_{n-2}+W_{n}-W_{n-1}}{ab}-\frac{A\alpha^{*}\beta^{2}-B\beta^{*}\alpha^{2}-ab\left(A\alpha^{**}\beta-B\beta^{**}\alpha\right)}{(ab)^{2}}.$$

Similarly, it can be proven for even n. Also, the same procedure can be applied for (ii) and (iii). □

Note that, if we take the initial conditions $w_0 = 0$ and $w_1 = 1$ in the above theorem, we get the summation formulas for the bi-periodic Fibonacci quaternions, and by taking the initial conditions $w_0 = 2$ and $w_1 = b$, we get the summation formulas for the bi-periodic Lucas quaternions which were not known before.

3. The generalized bi-periodic Fibonacci octonions

In this section, we introduce the generalized bi-periodic Fibonacci octonions and give some basic properties of them. These results can be seen as a generalization of the papers in [26] and [27]. Most of the results can be obtained analogously to the results for the generalized bi-periodic Fibonacci quaternions, so we omit some proofs.

Definition 3.1. The generalized bi-periodic Fibonacci octonions $\{OW_n\}$ are defined by

$$OW_n = \sum_{l=0}^{7} w_{n+l} e_l, \tag{3.1}$$

where w_n is defined in (1.15).

Note that, if we take the initial conditions $w_0 = 0$ and $w_1 = 1$, we get the bi-periodic Fibonacci octonions in [26]. If we take the initial conditions $w_0 = 2$ and $w_1 = b$, we get the bi-periodic Lucas octonions in [27]. Also, if we take $a = b = 1$ in $\{w_n\}$, we get the Horadam octonion numbers in [14] with the case of $q = 1$.

Theorem 3.2. *The generating function for the generalized bi-periodic Fibonacci octonion OW_n is*

$$G'(t) = \frac{OW_0 + (OW_1 - bOW_0)\,t + (a-b)\sum_{s=0}^{7} R'(t,s)\, e_s}{1 - bt - t^2} \tag{3.2}$$

where

$$R^{'}(t,s) := \left(f(t) - \sum_{k=1}^{\lfloor \frac{s+1}{2} \rfloor} w_{2k-1} t^{2k-1} \right) t^{1-s} \tag{3.3}$$

and $f(t)$ is defined in (2.4).

Note that, if we take the initial conditions $w_0 = 0$ and $w_1 = 1$, we get the generating function of the bi-periodic Fibonacci octonions in [26, Theorem 2.4].

Theorem 3.3. *The Binet formula for the generalized bi-periodic Fibonacci octonion is*

$$OW_n = \begin{cases} \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(A\gamma^* \alpha^{n-1} - B\delta^* \beta^{n-1} \right), & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(A\gamma^{**} \alpha^{n-1} - B\delta^{**} \beta^{n-1} \right), & \text{if } n \text{ is odd} \end{cases} \tag{3.4}$$

where $A, B, \gamma^*, \delta^*, \gamma^{**}$, and δ^{**} defined in (1.13)and (1.16).

Theorem 3.4. *(Catalan-like identity) For nonnegative integer number n and odd integer r, such that $r \leq n$, we have*

$$\begin{aligned} &OW_{n-r} OW_{n+r} - OW_n^2 \\ &= \begin{cases} \frac{AB(\alpha^r - \beta^r)}{(\alpha\beta)^{r+1}} \left[\gamma^* \delta^* \beta^r - \delta^* \gamma^* \alpha^r \right], & \text{if } n \text{ is even} \\ \frac{AB(\alpha^r - \beta^r)}{(\alpha\beta)^{r}} \left[\gamma^{**} \delta^{**} \beta^r - \delta^{**} \gamma^{**} \alpha^r \right], & \text{if } n \text{ is odd} \end{cases}. \end{aligned} \tag{3.5}$$

It is clear that, if we take $r = 2$ in the above theorem we obtain the Cassini-like identity.

Theorem 3.5. *For any natural number n, we have*

$$OW_{2(n+1)} OQ_{2n} - OW_{2n} OQ_{2(n+1)} = \frac{1}{ab} \left[A\gamma^* \delta^* \beta - B\delta^* \gamma^* \alpha \right]. \tag{3.6}$$

Theorem 3.6. *For the generalized bi-periodic Fibonacci octonions, we have*

$$\begin{aligned} (i) \sum_{r=0}^{n-1} OW_r &= \frac{OW_n - OW_{n-2} + OW_{n+1} - OW_{n-1}}{ab} \\ &\quad - \frac{A\gamma^* \beta^2 - B\delta^* \alpha^2 - ab\left(A\gamma^{**} \beta - B\delta^{**} \alpha \right)}{(ab)^2} \end{aligned} \tag{3.7}$$

$$(ii) \sum_{r=0}^{n-1} OW_{2r} = \frac{OW_{2n} - OW_{2n-2}}{ab} - \frac{A\gamma^* \beta^2 - B\delta^* \alpha^2}{(ab)^2} \tag{3.8}$$

$$(iii) \sum_{r=0}^{n-1} OW_{2r+1} = \frac{OW_{2n+1} - OW_{2n-1}}{ab} + \frac{A\gamma^{**} \beta - B\delta^{**} \alpha}{ab}. \tag{3.9}$$

If we take the initial conditions $w_0 = 0$ and $w_1 = 1$ in the above theorem, we obtain the results in [26, Theorem 2.5]. If we take the initial conditions $w_0 = 2$ and $w_1 = b$ in the above theorem, we obtain the results in [27, Theorem 2.6].

4. Conclusion

In this paper, we presented the generalized bi-periodic Fibonacci quaternions $\{W_n\}$, which is defined by $W_n = w_n e_0 + w_1 e_1 + w_2 e_2 + w_3 e_3$, where $w_n = a w_{n-1} + w_{n-2}$, if n is even, $w_n = b w_{n-1} + w_{n-2}$, if n is odd with arbitrary initial conditions w_0, w_1 and nonzero numbers a, b. Analogously, we defined the generalized bi-periodic Fibonacci octonions and gave some basic properties of them. Our results not only gave a generalization of the papers in [23, 22, 26, 27, 4], but also included new results. The main contribution of this research is one can get a great number of distinct quaternion and octonion sequences by providing the initial values in the generalized bi-periodic Fibonacci sequence $\{w_n\}$.

References

[1] Adler, S. L. *Quaternionic quantum mechanics and quantum fields*, vol. 88 of *International Series of Monographs on Physics.* The Clarendon Press, Oxford University Press, New York, 1995.

[2] Baez, J. C. The octonions. *Bull. Amer. Math. Soc. (N.S.) 39*, 2 (2002), 145–205.

[3] Bilgici, G. Two generalizations of Lucas sequence. *Appl. Math. Comput. 245* (2014), 526–538.

[4] Catarino, P. The modified Pell and the modified k-Pell quaternions and octonions. *Adv. Appl. Clifford Algebr. 26*, 2 (2016), 577–590.

[5] Çimen, C. B., and İpek, A. On Pell quaternions and Pell-Lucas quaternions. *Adv. Appl. Clifford Algebr. 26*, 1 (2016), 39–51.

[6] Edson, M., and Yayenie, O. A new generalization of Fibonacci sequence and extended Binet's formula. *Integers 9* (2009), A48, 639–654.

[7] Halici, S. On Fibonacci quaternions. *Adv. Appl. Clifford Algebr. 22*, 2 (2012), 321–327.

[8] Halici, S., and Karata¸s, A. On a generalization for Fibonacci quaternions. *Chaos Solitons Fractals 98* (2017), 178–182.

[9] Horadam, A. F. Complex Fibonacci numbers and Fibonacci quaternions. *Amer. Math. Monthly 70* (1963), 289–291.

[10] Iakin, A. L. Extended Binet forms for generalized quaternions of higher order. *Fibonacci Quart. 19*, 5 (1981), 410–413.

[11] Iakin, I. L. Generalized quaternions of higher order. *Fibonacci Quart. 15*, 4 (1977), 343–346.

[12] Iyer, M. R. A note on Fibonacci quaternions. *Fibonacci Quart. 7*, 3 (1969), 225–229.

[13] Iyer, M. R. Some results on Fibonacci quaternions. *Fibonacci Quart. 7* (1969), 201–210, 224.

[14] Karata¸s, A., and Halici, S. Horadam octonions. *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 25*, 3 (2017), 97–106.

[15] Keçilio˘ glu, O., and Akkus, I. The Fibonacci octonions. *Adv. Appl. Clifford Algebr. 25*, 1 (2015), 151–158.

[16] Panario, D., Sahin, M., and Wang, Q. A family of Fibonacci-like conditional sequences. *Integers 13* (2013), Paper No. A78, 14.

[17] Ramí rez, J. L. Some combinatorial properties of the k-Fibonacci and the k-Lucas quaternions. *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 23*, 2 (2015), 201–212.

[18] Sahin, M. The Gelin-Cesàro identity in some conditional sequences. *Hacet. J. Math. Stat. 40*, 6 (2011), 855–861.

[19] Savin, D. Some properties of Fibonacci numbers, Fibonacci octonions, and generalized Fibonacci-Lucas octonions. *Adv. Difference Equ.* (2015), 2015:298, 10.

[20] Szynal-Liana, A., and Wł och, I. The Pell quaternions and the Pell octonions. *Adv. Appl. Clifford Algebr. 26*, 1 (2016), 435–440.

[21] Tan, E. Some properties of the bi-periodic horadam sequences. *Notes Number Theory Discrete Math. 23*, 4 (2017), 56–65.

[22] Tan, E., Yilmaz, S., and Sahin, M. A note on bi-periodic Fibonacci and Lucas quaternions. *Chaos Solitons Fractals 85* (2016), 138–142.

[23] Tan, E., Yilmaz, S., and Sahin, M. On a new generalization of Fibonacci quaternions. *Chaos Solitons Fractals 82* (2016), 1–4.

[24] Ward, J. P. *Quaternions and Cayley numbers*, vol. 403 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1997. Algebra and applications.

[25] Yayenie, O. A note on generalized Fibonacci sequences. *Appl. Math. Comput. 217*, 12 (2011), 5603–5611.

[26] Yilmaz, N., Yazlik, Y., and Taskara, N. On the bi-periodic Fibonacci octonions. *ArXiv e-prints*, arXiv:1603.00681v2.

[27] Yilmaz, N., Yazlik, Y., and Taskara, N. On the bi-periodic Lucas octonions. *Adv. Appl. Clifford Algebr. 27*, 2 (2017), 1927–1937.

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