THE GENERALIZED BI-PERIODIC FIBONACCI QUATERNIONS AND OCTONIONS
Murat Sahin\textsuperscript{1}, Elif Tan\textsuperscript{2,3} and Semih Yilmaz\textsuperscript{4}

Abstract. In this paper, we present a further generalization of the bi-periodic Fibonacci quaternions and octonions. We give the generating function, the Binet formula, and some basic properties of these quaternions and octonions. The results of this paper not only give a generalization of the bi-periodic Fibonacci quaternions and octonions, but also include new results such as the matrix representation and the norm value of the generalized bi-periodic Fibonacci sequence.

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1. Introduction

There has been a growing interest in quaternions that have been extensively studied in both applied and theoretical sciences. In particular, quaternions are very good at representing rotations in three-dimensional space. The octonions are invented as an analog to the quaternions, and related to the exceptional Lie algebra. Also they have applications in areas such as super string theory, projective geometry, topology, and Jordan algebras. For more details about quaternions and octonions we refer to \cite{1,2,24}.

The quaternion algebra

\begin{equation}
\mathbb{H} = \left\{ \sum_{l=0}^{3} a_l e_l : a_l \in \mathbb{R} \right\}
\end{equation}

is a four dimensional non-commutative vector space over $\mathbb{R}$ and the basis satisfies the following multiplication rules:

\begin{align}
e_l^2 &= -1, \ l \in \{1, 2, 3\}; \\
e_1e_2 &= -e_2e_1 = e_3, \ e_2e_3 = -e_3e_2 = e_1, \ e_3e_1 = -e_1e_3 = e_2.
\end{align}

($e_0$ can be identified with real number 1). Also, the quaternion algebra $\mathbb{H}$ is isomorphic to the Clifford algebra $C\ell_{0,2}$. There are several studies on different

\textsuperscript{1}Department of Mathematics, Faculty of Sciences, Ankara University, 06100 Tandogan, Ankara, TURKEY, e-mail: msahin@ankara.edu.tr
\textsuperscript{2}Department of Mathematics, Faculty of Sciences, Ankara University, 06100 Tandogan, Ankara, TURKEY, e-mail: etan@ankara.edu.tr
\textsuperscript{3}Corresponding author
\textsuperscript{4}Department of Actuarial Sciences, Kirikkale University, 71450 Kirikkale, TURKEY, e-mail: syilmaz@kku.edu.tr
types of sequences over quaternion algebra. For a survey on these researches we refer to [9, 11, 10, 12, 13, 7, 8, 6, 1, 20, 17].

Recently, Tan and et. al. [23, 22] introduced a new generalization of the Fibonacci and Lucas quaternions, named as the bi-periodic Fibonacci and Lucas quaternions. They are emerged as a generalization of the best known quaternions in the literature, such as classical Fibonacci and Lucas quaternions in [9], Pell and Pell-Lucas quaternions in [5], $k$–Fibonacci and $k$–Lucas quaternions in [17].

For $n \geq 0$, the bi-periodic Fibonacci and Lucas quaternions defined as

\[ Q_n = \sum_{l=0}^{3} q_{n+l}e_l \quad \text{and} \quad P_n = \sum_{l=0}^{3} p_{n+l}e_l, \]

respectively. Note that $q_n$ is the $n$th bi-periodic Fibonacci number and defined by

\[ q_n = \begin{cases} 
  aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even}, \\
  bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd}
\end{cases}, \quad n \geq 2
\]

with initial values $q_0 = 0$, $q_1 = 1$ and $a$, $b$ are nonzero real numbers and $p_n$ is the $n$th bi-periodic Lucas number and defined by

\[ p_n = \begin{cases} 
  bp_{n-1} + p_{n-2}, & \text{if } n \text{ is even}, \\
  ap_{n-1} + p_{n-2}, & \text{if } n \text{ is odd}
\end{cases}, \quad n \geq 2
\]

with the initial conditions $p_0 = 2$, $p_1 = a$.

The Binet formula for the bi-periodic Fibonacci quaternion is given by

\[ Q_n = \begin{cases} 
  \frac{1}{(ab)[\frac{n}{2}]} \alpha^{n} - \beta^{n}, & \text{if } n \text{ is even} \\
  \frac{1}{(ab)[\frac{n}{2}]} \alpha^{n} - \beta^{n}, & \text{if } n \text{ is odd}
\end{cases}
\]

and the Binet formula for the bi-periodic Lucas quaternion is

\[ P_n = \begin{cases} 
  \frac{1}{(ab)[\frac{n+1}{2}]} (\alpha^{n} + \beta^{n}), & \text{if } n \text{ is even} \\
  \frac{1}{(ab)[\frac{n+1}{2}]} (\alpha^{n} + \beta^{n}), & \text{if } n \text{ is odd}
\end{cases}
\]

where

\[ \alpha := \frac{ab + \sqrt{a^2b^2 + 4ab}}{2}, \quad \beta := \frac{ab - \sqrt{a^2b^2 + 4ab}}{2} \]

\[ \alpha^{*} := \sum_{l=0}^{3} \frac{a^{*}(l+1)}{(ab)[\frac{l}{2}]} \alpha^{l}e_l, \quad \beta^{*} := \sum_{l=0}^{3} \frac{a^{*}(l+1)}{(ab)[\frac{l}{2}]} \beta^{l}e_l \]

\[ \alpha^{**} := \sum_{l=0}^{3} \frac{a^{**}(l)}{(ab)[\frac{l+1}{2}]} \alpha^{l}e_l, \quad \beta^{**} := \sum_{l=0}^{3} \frac{a^{**}(l)}{(ab)[\frac{l+1}{2}]} \beta^{l}e_l. \]
Here $\zeta(n) = n - 2 \left\lfloor \frac{n}{2} \right\rfloor$ is the parity function, i.e., $\zeta(n) = 0$ when $n$ is even and $\zeta(n) = 1$ when $n$ is odd. Assume that $a^2b^2 + 4ab \neq 0$. Also we have $\alpha + \beta = ab$, $\alpha - \beta = \sqrt{a^2b^2 + 4ab}$ and $\alpha\beta = -ab$. For the details of the bi-periodic Fibonacci and Lucas sequences see \[25, 13, 3, 18, 16\].

The octonion algebra

\begin{equation}
O = \{ \sum_{l=0}^{7} a_le_l : a_l \in \mathbb{R} \}
\end{equation}

is an eight dimensional non-commutative and non-associative vector space over $\mathbb{R}$, and the multiplication rules can be derived from the following table:

|   | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
---|---|---|---|---|---|---|---|
1 | 1 | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ |
$e_1$ | $e_1$ | $-1$ | $e_3$ | $-e_2$ | $e_5$ | $-e_4$ | $-e_7$ |
$e_2$ | $e_2$ | $-e_3$ | $-1$ | $e_1$ | $e_6$ | $e_7$ | $-e_4$ |
$e_3$ | $e_3$ | $e_2$ | $-e_1$ | $-1$ | $e_7$ | $-e_6$ | $e_5$ |
$e_4$ | $e_4$ | $-e_5$ | $-e_6$ | $-e_7$ | $-1$ | $e_1$ | $e_2$ |
$e_5$ | $e_5$ | $e_4$ | $-e_7$ | $e_6$ | $-e_1$ | $-1$ | $-e_3$ |
$e_6$ | $e_6$ | $e_7$ | $e_4$ | $-e_5$ | $-e_2$ | $e_3$ | $-1$ |
$e_7$ | $e_7$ | $-e_6$ | $e_5$ | $e_4$ | $-e_3$ | $-e_2$ | $e_1$ |

\textbf{Table 1: Octonion Multiplication table}

Motivated by the results in \[23, 22\], Yilmaz and et. al. \[20, 27\] introduced the bi-periodic Fibonacci and Lucas octonions as

\begin{equation}
OQ_n = \sum_{l=0}^{7} q_{n+l}e_l \quad \text{and} \quad OP_n = \sum_{l=0}^{7} p_{n+l}e_l,
\end{equation}

respectively.

The Binet formula for the bi-periodic Fibonacci octonion is

\begin{equation}
OQ_n = \begin{cases} 
\frac{1}{(ab)\left\lfloor \frac{n}{2} \right\rfloor} \gamma^* \alpha^n - \delta^* \beta^n, & \text{if } n \text{ is even} \\
\frac{1}{(ab)\left\lfloor \frac{n}{2} \right\rfloor} \gamma^{**} \alpha^n - \delta^{**} \beta^n, & \text{if } n \text{ is odd}
\end{cases}
\end{equation}

and the Binet formula for the bi-periodic Lucas octonion is

\begin{equation}
OP_n = \begin{cases} 
\frac{1}{(ab)\left\lfloor \frac{n+1}{2} \right\rfloor} (\gamma^* \alpha^n + \delta^* \beta^n), & \text{if } n \text{ is even} \\
\frac{1}{(ab)\left\lfloor \frac{n+1}{2} \right\rfloor} (\gamma^{**} \alpha^n + \delta^{**} \beta^n), & \text{if } n \text{ is odd}
\end{cases}
\end{equation}

where

\begin{align}
\gamma^* & : = \sum_{l=0}^{7} \frac{a^{\zeta(l+1)}}{(ab)^{\left\lfloor \frac{l}{2} \right\rfloor}} \alpha^l e_l, \quad \delta^* := \sum_{l=0}^{7} \frac{a^{\zeta(l)+1}}{(ab)^{\left\lfloor \frac{l}{2} \right\rfloor}} \beta^l e_l \\
\gamma^{**} & : = \sum_{l=0}^{7} \frac{a^{\zeta(l+1)}}{(ab)^{\left\lfloor \frac{l+1}{2} \right\rfloor}} \alpha^l e_l, \quad \delta^{**} := \sum_{l=0}^{7} \frac{a^{\zeta(l)}}{(ab)^{\left\lfloor \frac{l+1}{2} \right\rfloor}} \beta^l e_l.
\end{align}
For related studies on different types of sequences over octonion algebra, we refer to [13, 16, 20, 4, 14].

In this paper, we present a further generalization of the bi-periodic Fibonacci quaternions and octonions. We give the generating function, the Binet formula, and some basic properties of these quaternions and octonions. This new generalization can be seen as a generalization of the notions given in [23, 22, 26, 27, 4]. The results of this paper not only give a generalization of the bi-periodic Fibonacci quaternions and octonions, but also include new results such as the matrix representation and the norm value of the generalized bi-periodic Fibonacci sequence. The main contribution of this study is that one can get a great number of distinct quaternion and octonion sequences by providing the initial values in the generalized bi-periodic Fibonacci sequence. To this end, first consider the generalized bi-periodic Fibonacci sequence, \( \{w_n\} \), which is defined in (6) as:

\[
    w_n = \begin{cases} 
        aw_{n-1} + w_{n-2}, & \text{if } n \text{ is even} \\
        bw_{n-1} + w_{n-2}, & \text{if } n \text{ is odd}
    \end{cases}, \quad n \geq 2
\]

with arbitrary initial conditions \( w_0, w_1 \) where \( w_0, w_1, a, b \) are nonzero real numbers. Note that, if we take \( w_0 = 0, w_1 = 1 \) in \( \{w_n\} \), we get the bi-periodic Fibonacci sequence \( \{q_n\} \) in (1.3). If we take \( w_0 = 2, w_1 = b \), and switch \( a \) and \( b \) in \( \{w_n\} \), we get the bi-periodic Lucas sequence \( \{p_n\} \) in (1.5).

In [21], the Binet formula of the sequence \( \{w_n\} \) is given by

\[
    w_n = \frac{a^\left\lfloor \frac{n}{2} \right\rfloor}{(ab)^\left\lfloor \frac{n}{2} \right\rfloor} \left( A\alpha^{n-1} - B\beta^{n-1} \right)
\]

where

\[
    A := \frac{\alpha w_1 + bw_0}{\alpha - \beta} \quad \text{and} \quad B := \frac{\beta w_1 + bw_0}{\alpha - \beta}.
\]

For more results related to the sequence \( \{w_n\} \), we refer to [31].

2. The generalized bi-periodic Fibonacci quaternions

In this section, we introduce the generalized bi-periodic Fibonacci quaternions and give some basic properties of them. These results can be seen as a generalization of the results in [23, 22, 3].

**Definition 2.1.** The generalized bi-periodic Fibonacci quaternions \( \{W_n\} \) are defined by

\[
    W_n = \sum_{l=0}^{3} w_{n+l} e_l,
\]

where \( w_n \) is defined in (1.4).
In the following, we give several different sequences which are special cases of \( \{W_n\} : \\

1. If we take the initial conditions \( w_0 = 0 \) and \( w_1 = 1 \), we get the bi-periodic Fibonacci quaternions in [23]. \\
2. If we take the initial conditions \( w_0 = 2 \) and \( w_1 = b \), we get the bi-periodic Lucas quaternions in [22]. (Note that we switch \( a \) and \( b \)). \\
3. If we take the initial conditions \( w_0 = w_1 = 1 \) and \( a = b = 2 \) in \( \{w_n\} \), we get the modified Pell quaternion numbers in [4]. \\
4. If we take \( a = b = 1 \) in \( \{w_n\} \), we get the Horadam quaternion numbers in [8] with the case of \( q = 1 \).

**Theorem 2.2.** The generating function for the generalized bi-periodic Fibonacci quaternions \( W_n \) is

\[
G(t) = \frac{W_0 + (W_1 - bW_0) t + (a - b) \sum_{s=0}^{3} R(t,s) e_s}{1 - bt - t^2}
\]

where

\[
R(t,s) := \left( f(t) - \sum_{k=1}^{\lfloor s+1/2 \rfloor} w_{2k-1} t^{2k-1} \right) t^{1-s},
\]

\[
f(t) := \sum_{n=1}^{\infty} w_{2n-1} t^{2n-1} = \frac{w_1 t + (bw_0 - w_1) t^3}{1 - (ab + 2) t^2 + t^4}.
\]

**Proof.** By using a similar method as in [23, Theorem 1] and considering the relation

\[ w_{2n-1} = (ab + 2)w_{2n-3} - w_{2n-5}, \]

we get the desired result.

In the following theorem, we state the Binet formula for the generalized bi-periodic Fibonacci quaternions and so derive some well-known mathematical properties such as Catalan-like identity and Cassini-like identity.

**Theorem 2.3.** The Binet formula for the generalized bi-periodic Fibonacci quaternion is

\[
W_n = \begin{cases} 
\frac{1}{(ab)\left(\frac{n}{2}\right)} \left( \alpha^* \alpha^{n-1} - B \beta^* \beta^{n-1} \right), & \text{if } n \text{ is even} \\
\frac{1}{(ab)\left(\frac{n}{2}\right)} \left( \alpha^{\#} \alpha^{n-1} - B \beta^{\#} \beta^{n-1} \right), & \text{if } n \text{ is odd}
\end{cases}
\]

where \( A, B, \alpha^*, \beta^*, \alpha^{\#}, \text{ and } \beta^{\#} \) are defined in [18] and [17].
Proof. By using the definition of the sequence \(\{w_n\}\) and the Binet formula in (1.15), we can easily obtain the desired result.

By using the Binet formula for the generalized bi-periodic Fibonacci quaternion sequences, we obtain the following identity.

**Theorem 2.4.** (Catalan-like identity) For nonnegative integer number \(n\) and even integer \(r\), such that \(r \leq n\), we have

\[
W_{n-r}W_{n+r} - W_n^2 = \begin{cases} 
\frac{AB(\alpha^r - \beta^r)}{(\alpha \beta)^{r+1}} [\alpha^* \beta^* \beta^r - \beta^* \alpha^* \alpha^r], & \text{if } n \text{ is even} \\
\frac{AB(\alpha^r - \beta^r)}{(\alpha \beta)^r} [\alpha^* \beta^* \beta^r - \beta^* \alpha^* \alpha^r], & \text{if } n \text{ is odd} 
\end{cases}.
\]

Proof. For even \(n\), we have

\[
W_{n-r}W_{n+r} - W_n^2 = \frac{1}{(ab)^n} (A\alpha^* \alpha^{n-r-1} - B\beta^* \beta^{n-r-1}) (A\alpha^* \alpha^{n+r-1} - B\beta^* \beta^{n+r-1})
\]

\[
- \frac{1}{(ab)^n} (A\alpha^* \alpha^{n-1} - B\beta^* \beta^{n-1}) (A\alpha^* \alpha^{n-1} - B\beta^* \beta^{n-1})
\]

\[
= \frac{1}{(ab)^n} [AB\alpha^* \beta^* (\alpha \beta)^{n-1} \left(1 - \frac{\beta^r}{\alpha^r}\right) + BA\beta^* \alpha^* (\alpha \beta)^{n-1} \left(1 - \frac{\alpha^r}{\beta^r}\right)]
\]

\[
= \frac{(\alpha \beta)^{n-1} AB}{(ab)^n} \left[\alpha^* \beta^* \left(1 - \frac{\beta^r}{\alpha^r}\right) + \beta^* \alpha^* \left(1 - \frac{\alpha^r}{\beta^r}\right)\right]
\]

\[
= \frac{AB}{(-1)^n (\alpha \beta)^{r+1}} \left[\alpha^* \beta^* \beta^r (\alpha^r - \beta^r) + \beta^* \alpha^* \alpha^r (\beta^r - \alpha^r)\right]
\]

\[
= \frac{AB (\alpha^r - \beta^r)}{(\alpha \beta)^{r+1}} [\alpha^* \beta^* \beta^r - \beta^* \alpha^* \alpha^r].
\]

Similarly, it can be proven for odd \(n\). \hfill \Box

If we take the initial conditions \(w_0 = 0\) and \(w_1 = 1\) in (2.6), we get the result in [23, Theorem 5], and if we take the initial conditions \(w_0 = 2\) and \(w_1 = b\) in (2.6), we get the result in [22, Theorem 5]. Also, it is clear that if we take \(r = 2\) in the above theorem we obtain the following result.

**Corollary 2.5.** (Cassini-like identity) For nonnegative even integer number \(n\), we have

\[
W_{n-2}W_{n+2} - W_n^2 = \frac{AB (\alpha^2 - \beta^2)}{(\alpha \beta)^3} [\alpha^* \beta^* \beta^2 - \beta^* \alpha^* \alpha^2].
\]
If we take the initial conditions \( w_0 = 0 \) and \( w_1 = 1 \) in (2.7), we get the result in [23, Theorem 3], and if we take the initial conditions \( w_0 = 2 \) and \( w_1 = b \) in (2.7), we get the result in [22, Theorem 3].

To present the Cassini-like identity in a different manner, now we give a matrix representation for the even indices terms of the generalized bi-periodic Fibonacci quaternions.

**Theorem 2.6.** For \( n \geq 1 \), we have

\[
(2.8) \quad \begin{bmatrix} W_{2n} & W_{2(n-1)} \\ W_{2(n+1)} & W_{2n} \end{bmatrix} = \begin{bmatrix} W_2 & W_0 \\ W_4 & W_2 \end{bmatrix} \begin{bmatrix} ab + 2 & 1 \\ -1 & 0 \end{bmatrix}^{n-1}.
\]

**Proof.** We prove it by using induction on \( n \). It is clear that the result is true when \( n = 1 \). Assume that it is true for any integer \( m \) such that \( 1 \leq m \leq n \). Then by using inductive assumption, we have

\[
\begin{bmatrix} W_2 & W_0 \\ W_4 & W_2 \end{bmatrix} \begin{bmatrix} ab + 2 & 1 \\ -1 & 0 \end{bmatrix}^n = \begin{bmatrix} W_2 & W_0 \\ W_4 & W_2 \end{bmatrix} \begin{bmatrix} ab + 2 & 1 \\ -1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} ab + 2 & 1 \\ -1 & 0 \end{bmatrix}^{n-1} = \begin{bmatrix} (ab + 2) W_{2n} - W_{2(n-1)} & W_{2n} \\ (ab + 2) W_{2(n+1)} - W_{2n} & W_{2(n+1)} \end{bmatrix} = \begin{bmatrix} W_{2(n+1)} & W_{2n} \\ W_{2(n+2)} & W_{2(n+1)} \end{bmatrix}
\]

which completes the proof. \( \square \)

**Corollary 2.7.** For \( n \geq 1 \), we have

\[
(2.9) \quad W_{2(n-1)} W_{2(n+1)} - W_{2n}^2 = W_0 W_4 - W_2^2.
\]

By means of this formula we can state the Cassini-like identity for the bi-periodic Fibonacci quaternions as:

\[
Q_{2(n-1)} Q_{2(n+1)} - Q_{2n}^2 = Q_0 Q_4 - Q_2^2,
\]

which was not known before.

Following result gives the relation between the generalized bi-periodic Fibonacci quaternions \( \{W_n\} \) and the bi-periodic Fibonacci quaternions \( \{Q_n\} \).

**Theorem 2.8.** For any natural number \( n \), we have

\[
(2.10) \quad W_{2(n+1)} Q_{2n} - W_{2n} Q_{2(n+1)} = \frac{1}{ab} [A\alpha^* \beta^* \beta - B\beta^* \alpha^* \alpha].
\]

**Proof.** By using the Binet formula for the generalized bi-periodic Fibonacci quaternions and the bi-periodic Fibonacci quaternions, we can easily obtain the desired result. \( \square \)
The norm value of the generalized bi-periodic Fibonacci quaternions is

\[ Nr(W_n) := W_n \overline{W}_n, \]

where \( W_n := w_n e_0 - w_{n+1} e_1 - w_{n+2} e_2 - w_{n+3} e_3 \) is the conjugate of the generalized bi-periodic Fibonacci quaternion. Thus we have

\[ Nr(W_n) = w_n^2 + w_{n+1}^2 + w_{n+2}^2 + w_{n+3}^2. \]

By using the definition of the norm value and the Binet formula of the sequence \( \{w_n\} \), then by making some necessary calculations, we obtain the following result.

**Theorem 2.9.** The norm value of the generalized bi-periodic Fibonacci quaternions can be stated as

\[ Nr(W_n) = T(n) + T(n + 1), \]

where

\[ T(n) := \frac{a^{2\zeta(n+1)}}{(ab)^{n-\zeta(n)}(\alpha\beta)^2(\alpha - \beta)^2} \left[ w_1^2 X + 2 w_0 w_1 b Y + w_0^2 b^2 Z \right], \]

and

\[ X := \alpha^{2n} \left( \alpha^4 + (\alpha\beta)^2 \right) + \beta^{2n} \left( \beta^4 + (\alpha\beta)^2 \right) - 4 (\alpha\beta)^{n+2}, \]

\[ Y := \alpha^{2n-1} \left( \alpha^4 + (\alpha\beta)^2 \right) + \beta^{2n-1} \left( \beta^4 + (\alpha\beta)^2 \right) + 2 (\alpha\beta)^{n+2}, \]

\[ Z := \alpha^{2n-2} \left( \alpha^4 + (\alpha\beta)^2 \right) + \beta^{2n-2} \left( \beta^4 + (\alpha\beta)^2 \right) - 4 (\alpha\beta)^{n+1}. \]

Note that, if we take the initial conditions \( w_0 = 0, w_1 = 1 \) and \( a = b = 2 \), we get the norm value of the Pell quaternions in [20, Theorem 3.1], and if we take \( a = b \) in \( \{w_n\} \), we get the norm value of the Horadam quaternion numbers in [8] with the case of \( q = 1 \).

Finally, we give some summation formulas for the generalized bi-periodic Fibonacci quaternions.

**Theorem 2.10.** For \( n \geq 1 \), we have

\[ (i) \sum_{r=0}^{n-1} W_r = \frac{W_n - W_{n-2} + W_{n+1} - W_{n-1}}{ab} - \frac{A\alpha^* \beta^2 - B\beta^* \alpha^2 - ab (A\alpha** \beta - B\beta** \alpha)}{(ab)^2}, \]

\[ (ii) \sum_{r=0}^{n-1} W_{2r} = \frac{W_{2n} - W_{2n-2}}{ab} - \frac{A\alpha^* \beta^2 - B\beta^* \alpha^2}{(ab)^2}, \]

\[ (iii) \sum_{r=0}^{n-1} W_{2r+1} = \frac{W_{2n+1} - W_{2n-1}}{ab} + \frac{A\alpha** \beta - B\beta** \alpha}{ab}. \]
Proof. (i) If $n$ is odd,

$$
\sum_{r=0}^{n-1} W_r = \sum_{r=0}^{n-1} W_{2r} + \sum_{r=0}^{n-3} W_{2r+1}
$$

$$
= \sum_{r=0}^{n-1} \left( \frac{1}{ab} \right)^r (A\alpha^r - B\beta^r) + \sum_{r=0}^{n-3} \left( \frac{1}{ab} \right)^r (A\alpha^2 - B\beta^2)
$$

$$
= A\alpha^r \sum_{r=0}^{n-1} \left( \frac{\alpha^2}{ab} \right)^r - B\beta^r \sum_{r=0}^{n-1} \left( \frac{\beta^2}{ab} \right)^r
$$

$$
+ A\alpha^2 \sum_{r=0}^{n-3} \left( \frac{\alpha^2}{ab} \right)^r - B\beta^2 \sum_{r=0}^{n-3} \left( \frac{\beta^2}{ab} \right)^r
$$

$$
= A\alpha^r \left( \frac{\alpha^{n-1}}{\alpha^2 - ab} \right) - B\beta^r \left( \frac{\beta^{n-1}}{\beta^2 - ab} \right)
$$

$$
+ A\alpha^2 \left( \frac{\alpha^{n-3}}{\alpha^2 - ab} \right) - B\beta^2 \left( \frac{\beta^{n-3}}{\beta^2 - ab} \right)
$$

$$
= A\alpha^r \left( \frac{\alpha^{n+1} - (ab)\frac{n+1}{2}}{\alpha^2 - ab} \right) - B\beta^r \left( \frac{\beta^{n+1} - (ab)\frac{n+1}{2}}{\beta^2 - ab} \right)
$$

$$
+ A\alpha^2 \left( \frac{\alpha^{n-1} - (ab)\frac{n-1}{2}}{\alpha^2 - ab} \right) - B\beta^2 \left( \frac{\beta^{n-1} - (ab)\frac{n-1}{2}}{\beta^2 - ab} \right)
$$

$$
= \frac{1}{ab} \left( A\alpha^r \left( \frac{\alpha^{n+1} - (ab)\frac{n+1}{2}}{\alpha^2} \right) - B\beta^r \left( \frac{\beta^{n+1} - (ab)\frac{n+1}{2}}{\beta^2} \right) \right)
$$

$$
+ \frac{1}{ab} \left( A\alpha^2 \left( \frac{\alpha^{n-1} - (ab)\frac{n-1}{2}}{\alpha} \right) - B\beta^2 \left( \frac{\beta^{n-1} - (ab)\frac{n-1}{2}}{\beta} \right) \right)
$$

$$
= \frac{1}{ab} \left( A\alpha^r \left( \frac{\alpha^{n+1} - B\beta^r \beta^{n-1} - A\alpha^2 \left( \frac{\alpha^{n+1}}{\alpha^2} \right) - \frac{B\beta^r \beta^{n-1}}{\beta^2} \right) \right)
$$
Similarly, it can be proven for even \( n \). Also, the same procedure can be applied for \((ii)\) and \((iii)\).

Note that, if we take the initial conditions \( w_0 = 0 \) and \( w_1 = 1 \) in the above theorem, we get the summation formulas for the bi-periodic Fibonacci quaternions, and by taking the initial conditions \( w_0 = 2 \) and \( w_1 = b \), we get the summation formulas for the bi-periodic Lucas quaternions which were not known before.

3. The generalized bi-periodic Fibonacci octonions

In this section, we introduce the generalized bi-periodic Fibonacci octonions and give some basic properties of them. These results can be seen as a generalization of the papers in [26] and [27]. Most of the results can be obtained analogously to the results for the generalized bi-periodic Fibonacci quaternions, so we omit some proofs.

**Definition 3.1.** The generalized bi-periodic Fibonacci octonions \( \{OW_n\} \) are defined by

\[
OW_n = \sum_{l=0}^{7} w_{n+l} c_l,
\]

where \( w_n \) is defined in (1.15).

Note that, if we take the initial conditions \( w_0 = 0 \) and \( w_1 = 1 \), we get the bi-periodic Fibonacci octonions in [26]. If we take the initial conditions \( w_0 = 2 \) and \( w_1 = b \), we get the bi-periodic Lucas octonions in [27]. Also, if we take \( a = b = 1 \) in \( \{w_n\} \), we get the Horadam octonion numbers in [14] with the case of \( q = 1 \).

**Theorem 3.2.** The generating function for the generalized bi-periodic Fibonacci octonion \( OW_n \) is

\[
G'(t) = \frac{OW_0 + (OW_1 - bOW_0) t + (a - b) \sum_{s=0}^{7} R'(t, s) e_s}{1 - bt - t^2}
\]
where

\[
R'(t, s) := \left( f(t) - \sum_{k=1}^{\left\lfloor \frac{s+1}{2} \right\rfloor} w_{2k-1} t^{2k-1} \right) t^{1-s}
\]

and \( f(t) \) is defined in (2.4).

Note that, if we take the initial conditions \( w_0 = 0 \) and \( w_1 = 1 \), we get the generating function of the bi-periodic Fibonacci octonions in [26, Theorem 2.4].

**Theorem 3.3.** The Binet formula for the generalized bi-periodic Fibonacci octonion is

\[
OW_n = \begin{cases} \frac{1}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left( A\gamma^*\alpha^{n-1} - B\delta^*\beta^{n-1} \right), & \text{if } n \text{ is even} \\ \frac{1}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left( A\gamma^{**}\alpha^{n-1} - B\delta^{**}\beta^{n-1} \right), & \text{if } n \text{ is odd} \end{cases}
\]

where \( A, B, \gamma^*, \delta^*, \gamma^{**}, \text{ and } \delta^{**} \) defined in (1.13) and (1.16).

**Theorem 3.4.** (Catalan-like identity) For nonnegative integer number \( n \) and odd integer \( r \) such that \( r \leq n \), we have

\[
OW_{n-r}OW_{n+r} - OW_n^2 = \begin{cases} \frac{AB(\alpha^{-r} - \beta^{-r})}{(ab)^{r+1}} \left[ \gamma^*\delta^*\beta^r - \delta^*\gamma^*\alpha^r \right], & \text{if } n \text{ is even} \\ \frac{AB(\alpha^{-r} - \beta^{-r})}{(ab)^{r}} \left[ \gamma^{**}\delta^{**}\beta^r - \delta^{**}\gamma^{**}\alpha^r \right], & \text{if } n \text{ is odd} \end{cases}
\]

It is clear that, if we take \( r = 2 \) in the above theorem we obtain the Cassini-like identity.

**Theorem 3.5.** For any natural number \( n \), we have

\[
OW_{2(n+1)}OQ_{2n} - OW_{2n}OQ_{2(n+1)} = \frac{1}{ab} \left[ A\gamma^*\delta^*\beta - B\delta^{**}\gamma^*\alpha \right].
\]

**Theorem 3.6.** For the generalized bi-periodic Fibonacci octonions, we have

\[
(i) \sum_{r=0}^{n-1} OW_r = \frac{OW_n - OW_{n-2} + OW_{n+1} - OW_{n-1}}{ab} - \frac{A\gamma^*\beta^2 - B\delta^*\alpha^2 - ab(A\gamma^{**}\beta - B\delta^{**}\alpha)}{(ab)^2}
\]

\[
(ii) \sum_{r=0}^{n-1} OW_{2r} = \frac{OW_{2n} - OW_{2n-2}}{ab} - \frac{A\gamma^*\beta^2 - B\delta^*\alpha^2}{(ab)^2}
\]

\[
(iii) \sum_{r=0}^{n-1} OW_{2r+1} = \frac{OW_{2n+1} - OW_{2n-1}}{ab} + \frac{A\gamma^{**}\beta - B\delta^{**}\alpha}{ab}.
\]

If we take the initial conditions \( w_0 = 0 \) and \( w_1 = 1 \) in the above theorem, we obtain the results in [26, Theorem 2.5]. If we take the initial conditions \( w_0 = 2 \) and \( w_1 = b \) in the above theorem, we obtain the results in [27, Theorem 2.6].
4. Conclusion

In this paper, we presented the generalized bi-periodic Fibonacci quaternions \( \{W_n\} \), which is defined by \( W_n = w_n e_0 + w_1 e_1 + w_2 e_2 + w_3 e_3 \), where \( w_n = aw_{n-1} + w_{n-2} \), if \( n \) is even, \( w_n = bw_{n-1} + w_{n-2} \), if \( n \) is odd with arbitrary initial conditions \( w_0, w_1 \) and nonzero numbers \( a, b \). Analogously, we defined the generalized bi-periodic Fibonacci octonions and gave some basic properties of them. Our results not only gave a generalization of the papers in \([23, 22, 20, 27, 4]\), but also included new results. The main contribution of this research is one can get a great number of distinct quaternion and octonion sequences by providing the initial values in the generalized bi-periodic Fibonacci sequence \( \{w_n\} \).

References


The generalized bi-periodic Fibonacci quaternions and octonions


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