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SUFFICIENT CONDITIONS FOR PERIODICITY OF MEROMORPHIC FUNCTION AND ITS SHIFT OPERATOR SHARING ONE OR MORE SETS WITH FINITE WEIGHT

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Abstract. In this paper, we investigate the uniqueness property of meromorphic functions together with its shift counterpart sharing one or two sets. With the help of the range set introduced in [2], we have improved the result of Bhusnurmath-Kabbur [3] and obtain the unique range set corresponding to shift operators. Our paper also improves the result of Frank-Reinder's [5] in some sense.

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1. Introduction Definitions and Results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities).

We have used the standard notations from Nevalinna's theory of value distribution of meromorphic functions as in [6]. We recall that $T(r, f)$ denotes the Nevanlinna characteristic function of the non-constant meromorphic function and $N(r, a; f)$ ($\overline{N}(r, a; f)$) denotes the counting function (reduced counting function) of a-points of meromorphic function f. A meromorphic function a is said to be a small function of f provided that $T(r, a) = S(r, f)$, that is $T(r, a) = o(T(r, f))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure. Let $S(f)$ be the set of all small functions of $f(z)$. For a set $S \subset S(f)$, we define the following:

$$E_f(S) = \bigcup_{a \in S} \{z | f(z) - a(z) = 0, counting\, multiplicities\},$$

$$\overline{E}_f(S) = \bigcup_{a \in S} \{z | f(z) - a(z) = 0, ignoring\, multiplicities\}.$$

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In 2001, an idea of gradation of sharing known as weighted sharing has been introduced in [8], [9] which measure how close a shared value is to being shared CM or to being shared IM. In the following definition we explain the notion.

Definition 1.1. [8] Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f, where an a point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Definition 1.2. [8] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a non-negative integer or ∞. We denote by $E_f(S, k)$ the set $\bigcup_{a \in S} E_k(a, f)$. If $E_f(S, k) = E_g(S, k)$, then we say f, g, share the set S with weight k.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

The unicity of meromorphic functions sharing sets is an important topic of the uniqueness theory. First of all, we state the following result of by Li-Yang[10].

Theorem A. *[10] Let $m \geq 2$ and $n > 2m + 6$ with n and $n - m$ having no common factors. Let a and b be two non-zero constants such that the equation $\omega^n + a\omega^{n-m} + b = 0$ has no multiple roots. Let $S = \{\omega \mid \omega^n + a\omega^{n-m} + b = 0\}$. Then for any two non constant meromorphic functions f and g, the conditions $E_f(S, \infty) = E_g(S, \infty)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$ imply $f \equiv g$.*

Yi-Lin [12] considered the case $m = 1$ with the condition that two meromorphic functions share three sets and got the result as follows.

Theorem B. *[12] Let $S_1 = \{\omega \mid \omega^n + a\omega^{n-1} + b = 0\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$, where a and b are non-zero constants such that $\omega^n + a\omega^{n-1} + b = 0$ has no repeated root and $n(\geq 4)$ an integer. If for two non-constant meromorphic functions f and g, $E_f(S_j, \infty) = E_g(S_j, \infty)$ for $j = 1, 2, 3$ and $\Theta(\infty; f) > 0$, then $f \equiv g$.*

Though the standard definitions and notations of the value distribution theory are available in [6], we explain some definitions and notations which are used in the paper.

Definition 1.3. [7] For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f. For a positive integer m, we denote by $N(r, a; f \mid\leq m)$ $(N(r, a; f \mid\geq m))$ the counting function of those a-point of f whose multiplicities are not greater (less) than m, where each a-point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid\leq m)$ $(\overline{N}(r, a; f \mid\geq m))$ are defined similarly except that in counting the a-points of f we ignore the multiplicity. Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ and $\overline{N}(r, a; f \mid> m)$ are defined similarly.

Definition 1.4. [9] For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2)$.

Definition 1.5. [9] Let f and g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

For, a non-zero complex constant c we define the shift of $f(z)$ by $f(z+c)$.

Recently, a number of papers have focused on shift analogues of the Nevalinna theory. In particular, there has been an increasing interest in studying the uniqueness problems related to meromorphic functions and their shift operators.

In 2010, Zhang [14] considered a meromorphic function $f(z)$ sharing sets with its shift $f(z+c)$ and proved the following result.

Theorem C. *[14] Let $m \geq 2$ and $n \geq 2m+4$ with n and $n-m$ having no common factors. Let a and b be two non-zero complex constants such that the equation $\omega^n + a\omega^{n-m} + b = 0$ has no multiple roots. Let $S = \{\omega \mid \omega^n + a\omega^{n-m} + b = 0\}$. Suppose that $f(z)$ is a non-constant meromorphic function of finite order. Then $E_{f(z)}(S, \infty) = E_{f(z+c)}(S, \infty)$ and $E_{f(z)}(\{\infty\}, \infty) = E_{f(z+c)}(\{\infty\}, \infty)$ imply $f(z) \equiv f(z+c)$.*

Earlier in 1998, Frank-Reinders [5] obtained a result. To demonstrate their result, we first require the following.

Let the polynomial P_* be defined as

$$P_*(\omega) = \frac{(n-1)(n-2)}{2}\omega^n - n(n-2)\omega^{n-1} + \frac{n(n-1)}{2}\omega^{n-2} - c,$$

where $n(\geq 3)$ is an integer and $c(\neq 0, 1)$ is a constant.

Theorem D. *[5] Let $S = \{\omega \mid P_*(\omega) = 0\}$, where $P_*(\omega)$ is as defined above and $n(\geq 11)$ be an integer. Then for any two non-constant meromorphic functions f and g the condition $E_{f(z)}(S, \infty) = E_{g(z)}(S, \infty)$ implies $f \equiv g$.*

In 2013, Bhusnurmath-Kabbur [3] considered the shift analogue of the above result with some additional supposition and obtained the following result.

Theorem E. *[3] Let $n \geq 8$ be an integer and $S = \{\omega \mid P_*(\omega) = 0\}$. Suppose that f is a non-constant meromorphic function of finite order. Then $E_{f(z)}(S, \infty) = E_{f(z+c)}(S, \infty)$ and $E_{f(z)}(\{\infty\}, \infty) = E_{f(z+c)}(\{\infty\}, \infty)$ implies $f(z) \equiv f(z+c)$.*

Though our main intention is to improve the results of Bhusnurmath-Kabbur [3] and Frank-Reinders [5] in some sense, we have also explored rigorously corresponding three set sharing problems and presented some relevant issues. In this respect, we have also presented some examples in the last section.

Regarding Theorem E, the following question is inevitable.

Question 1.6. In Theorem E, whether the sharing of the range sets can further be relaxed?

To seek the possible answer of the above question is the motivation of the paper. To this end, the following polynomial introduced in [2] renders an useful resource. Let for $d \in \mathbb{C}$,

$$P(z) = z^n - \frac{2n}{n-m} z^{n-m} + \frac{n}{n-2m} z^{n-2m} - d. \tag{1.1}$$

Then

$$P^{'}(z) = nz^{n-2m-1}(z^m - 1)^2 = nz^{n-2m-1} \prod_{j=0}^{m-1} (z - \omega_j)^2,$$

where $\omega_j = \cos \frac{2j\pi}{m} + i \sin \frac{2j\pi}{m}$, $j = 0, 1, \ldots, m-1$.

Therefore,

$$P(0) = -d$$

and

$$\begin{aligned} P(\omega_j) \qquad &= \omega_j^n - \frac{2n}{n-m}\omega_j^{n-m} + \frac{n}{n-2m}\omega_j^{n-2m} - d \\ &= \omega_j^n \Big(1 - \frac{2n}{n-m} + \frac{n}{n-2m}\Big) - d \\ &= \frac{2m^2\omega_j^n}{(n-m)(n-2m)} - d \\ &= \gamma_j - d, \end{aligned}$$

where $\gamma_j = \frac{2m^2\omega_j^n}{(n-m)(n-2m)}$, $j = 0, 1, 2, \ldots, m-1$. Therefore, if $d \neq 0, \gamma_j$, $j = 0, 1, \ldots, m-1$ all the zeros of the polynomial $P(z)$ given by (1.1) are simple.

Now it is clear that $P(z) - P(\omega_j) = (z - \omega_j)^3 Q_{n-3}(z)$, where $Q_{n-3}(z)$ is a polynomial of degree $n-3$, $j = 0, 1, \ldots, m-1$. Hence,

$$P(f) - P(\omega_j) = (f - \omega_j)^3 Q_{n-3}(f).$$

i.e.,

$$dF - d - (\gamma_j - d) = (f - \omega_j)^3 Q_{n-3}(f),$$

where

$$F = \frac{f(z)^{n-2m}(f(z)^{2m} - \frac{2n}{n-m} f(z)^m + \frac{n}{n-2m})}{d}.$$

i.e.,

$$F - \frac{\gamma_j}{d} = \frac{1}{d}(f - \omega_j)^3 Q_{n-3}(f). \tag{1.2}$$

i.e.,

$$F - \beta_j = \frac{1}{d}(f - \omega_j)^3 Q_{n-3}(f),$$

where

$$\beta_j = \frac{\gamma_j}{d},\ j = 0, 1, \ldots, m-1. \tag{1.3}$$

Throughout the paper we shall denote by $a = 3m + 2$, $b = 4 + 2m + \frac{(4m+2)(7n-3)}{(n-1)(3n-1)}$, $q = 2m + \frac{8m+2}{n-1} + \frac{(n-2m+2)(4m+1)}{(n-2m-1)(nk+n-1)}$, $r = 2m + \frac{(n-2m+2)(4m+1)}{(n-2m-1)(nk+n-1)} + \frac{4m+1}{n-1}$.

Let us define χ_n as follows:

$$\chi_n = \begin{cases} 1, & if\ n \geq 11 \\ 0, & otherwise. \end{cases}$$

The following four theorems are the main results of the paper.

Theorem 1.7. *Let $S = \{z \mid P(z) = 0\}$, where $P(z)$ is a polynomial given by (1.1), $n(\geq 1)$, $m(\geq 1)$, with $gcd(n, m) = 1$ be two positive integers and $d \in \mathbb{C} \setminus \{0, \gamma_0, \gamma_1, \ldots, \gamma_{m-1}\}$. Let $f(z)$ be a transcendental meromorphic function of finite order and c be a non-zero complex constant. If $E_{f(z)}(S, 2) = E_{f(z+c)}(S, 2)$ and $n > max\{3m + 2, 2m + 8\}$, then*

$$f(z) \equiv f(z + c).$$

Putting $m = 1$ in the above theorem we can easily obtain the following corollary.

Corollary 1.8. *Let $n(\geq 1)$ be a positive integer and $d_1 = \frac{(n-1)(n-2)}{2}d$, where d is a non-zero complex constant such that $d_1 \neq 0, 1, \frac{1}{2}$. Let*

$$S = \{z : \frac{(n-1)(n-2)}{2}z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2} - d_1 = 0\}.$$

Let $f(z)$ be a transcendental meromorphic function of finite order and c be a non-zero complex constant. Suppose $E_{f(z)}(S, 2) = E_{f(z+c)}(S, 2)$ and $n \geq 11$. Then

$$f(z) \equiv f(z + c).$$

Theorem 1.9. *Let $S = \{z \mid P(z) = 0\}$, where $P(z)$ is a polynomial given by (1.1) and $n(\geq 8)$, $m(\geq 1)$ with $gcd(n, m) = 1$, be two positive integers and $d(\neq 0, \gamma_j,\ j = 0, 1, \ldots, m-1)$ be a complex number. Let $f(z)$ be a transcendental meromorphic function of finite order and c be a non-zero complex constant. If $E_{f(z)}(\{\infty\}, 2) = E_{f(z+c)}(\{\infty\}, 2)$ and $E_{f(z)}(S, 2) = E_{f(z+c)}(S, 2)$ and $n > max\{\chi_n a, b\}$, then*

$$f(z) \equiv f(z + c).$$

Putting $m = 1$ in the above theorem, we can easily deduce the following corollary.

Corollary 1.10. *Let $n(\geq 8)$ be a positive integer and $d_1 = \frac{(n-1)(n-2)}{2}d$, where d is a non-zero complex constant such that $d_1 \neq 0, 1, \frac{1}{2}$. Let S be defined as in Corollary 1.8. Let $f(z)$ be a transcendental meromorphic function of finite order and c be a non-zero complex constant. If $E_{f(z)}(\{\infty\}, 2) = E_{f(z+c)}(\{\infty\}, 2)$ and $E_{f(z)}(S, 2) = E_{f(z+c)}(S, 2)$, then*

$$f(z) \equiv f(z+c).$$

Theorem 1.11. *Let $S = \{z \mid P(z) = 0\}$, where $P(z)$ is a polynomial given by (1.1) and $n(\geq 1)$, $m(\geq 1)$, k, t with $gcd(n, m) = 1$ be four positive integers. Let $f(z)$ be a transcendental meromorphic function of finite order and c be a non-zero complex constant. Suppose $f(z)$, $f(z+c)$ share $(0, 0)$, (∞, k), $E_{f(z)}(S, t) = E_{f(z+c)}(S, t)$, where $1 \leq k < \infty$, $t > \frac{3}{2} - \frac{3}{n-2m-1} - \frac{2}{n-1} - \frac{n-2m+2}{(n-2m-1)(nk+n-1)}$. If one of the following conditions hold:*

(i) $m = 1$, $n \geq 5$ and $d \neq 0, \dfrac{2}{(n-1)(n-2)}, \dfrac{1}{(n-1)(n-2)}$ or

(ii) $m \geq 2$, $n > max\{3m, q\}$ and $d \in \mathbb{C} \setminus \{0, \gamma_0, \gamma_1, \ldots, \gamma_{m-1}\}$, then

$$f(z) \equiv f(z+c).$$

Putting $m = 1$, $t = 4$ and $k = 5$ in the above theorem we obtain the following corollary.

Corollary 1.12. *Let $n(\geq 5)$ be a positive integer and $d_1 = \frac{(n-1)(n-2)}{2}d$, where d is a non-zero complex constant such that $d_1 \neq 0, 1, \frac{1}{2}$. Let S be defined as in Corollary 1.8. Let $f(z)$ be a transcendental meromorphic function of finite order and c be a non-zero complex constant. Suppose $f(z)$, $f(z+c)$ share $(0, 0)$, $(\infty, 5)$ and $E_{f(z)}(S, 4) = E_{f(z+c)}(S, 4)$. Then*

$$f(z) \equiv f(z+c).$$

Theorem 1.13. *Let $S = \{z \mid P(z) = 0\}$, where $P(z)$ is a polynomial given by (1.1) and $m(\geq 1)$, k, t are positive integers such that $gcd(n, m) = 1$. Let $f(z)$ be a transcendental meromorphic function of finite order and c be a non-zero complex constant. Suppose $f(z)$, $f(z+c)$ share $(0, \infty)$, (∞, k), $E_{f(z)}(S, t) = E_{f(z+c)}(S, t)$, where $1 \leq k < \infty$, $t > \frac{3}{2} - \frac{2}{n-2m-1} - \frac{2}{n-1} - \frac{n-2m+1}{(n-2m-1)(nk+n-1)}$. If one of the following conditions hold:*

(i) $m = 1$, $n \geq 5$ and $d \neq 0, \dfrac{2}{(n-1)(n-2)}, \dfrac{1}{(n-1)(n-2)}$ or

(ii) $m \geq 2$, $n > max\{3m, r\}$, $d(\neq 0, \gamma_j, j = 0, 1, \ldots, m-1)$ be a complex number, then

$$f(z) \equiv f(z+c).$$

Putting $m = 1$, $t = 4$ and $k = 1$ in the above theorem, we get the following corollary.

Corollary 1.14. *Let $n(\geq 5)$ be a positive integer and $d_1 = \frac{(n-1)(n-2)}{2}d$, where d is a non-zero complex constant such that $d_1 \neq 0, 1, \frac{1}{2}$. Let S be defined as* *in Corollary 1.8. Let $f(z)$ be a transcendental meromorphic function of finite order and c be a non-zero complex constant. Suppose $f(z)$, $f(z+c)$ share $(0,\infty)$, $(\infty,1)$ and $E_{f(z)}(S,4)=E_{f(z+c)}(S,4)$. Then*

$$f(z)\equiv f(z+c).$$

2. Lemmas

In this section, we present some lemmas which will be needed in the sequel. Let f and g be two non-constant meromorphic functions defined in $\mathbb{C}$. Let us also define two functions, F and G, in $\mathbb{C}$ by

$$F=\frac{f^{n-2m}(f^{2m}-\frac{2n}{n-m}f^m+\frac{n}{n-2m})}{d}, \tag{2.1}$$

$$G=\frac{f(z+c)^{n-2m}(f(z+c)^{2m}-\frac{2n}{n-m}f(z+c)^m+\frac{n}{n-2m})}{d}. \tag{2.2}$$

We also denote by H, V, H_1, V_1 and Φ, the following functions

$$H=\Big(\frac{f''}{f'}-\frac{2f'}{f-1}\Big)-\Big(\frac{g''}{g'}-\frac{2g'}{g-1}\Big),$$

$$V=\frac{f'}{f(f-1)}-\frac{g'}{g(g-1)},$$

$$H_1=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right),$$

$$V_1=\frac{F'}{F(F-1)}-\frac{G'}{G(G-1)}$$

and

$$\Phi=\frac{F'}{F-1}-\frac{G'}{G-1}.$$

Lemma 2.1. *[9] Let f, g be two non-constant meromorphic functions such that they share $(1,1)$ and $H\not\equiv 0$. Then*

$$N(r,1;f\mid=1)=N(r,1;g\mid=1)\le N(r,H)+S(r,f)+S(r,g).$$

Lemma 2.2. *[1] Let f, g be two non-constant meromorphic functions sharing $(1,t)$, where $1\le t<\infty$. Then*

$$\overline{N}(r,1;f)+\overline{N}(r,1;g)-\overline{N}(r,1;f\mid=1)+\left(t-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)$$
$$\le\ \frac{1}{2}[N(r,1;f)+N(r,1;g)].$$

Lemma 2.3. *Suppose f, g share $(1,0)$, $(\infty,0)$, $(0,0)$ and β_j, defined as in* (1.3)*, are non-zero complex numbers. If $H \not\equiv 0$, then*

$$\begin{aligned} N(r,H) \quad & \leq \overline{N}_*(r,0;f,g) + \sum_{j=0}^{m-1} \overline{N}(r,\beta_j;f \mid\geq 2) + \sum_{j=0}^{m-1} \overline{N}(r,\beta_j;g \mid\geq 2) \\ & + \overline{N}_*(r,1;f,g) + \overline{N}_*(r,\infty;f,g) + \overline{N}_0(r,0;f^{'}) + \overline{N}_0(r,0;g^{'}) \\ & + S(r,f) + S(r,g), \end{aligned}$$

where $\overline{N}_0(r,0;f^{'})$ is the reduced counting function of those zeros of $f^{'}$ which are not the zeros of $f(f-1)\prod_{j=0}^{m-1}(f-\beta_j)$ and $\overline{N}_0(r,0;g^{'})$ is similarly defined.

Proof. By the definition of H we verify that the possible poles of H occur from the following six cases: (i) The common zeros of f and g of different multiplicities. (ii) The multiple β_j- points of f and g for each $j = 0, 1, 2, \ldots, m-1$. (iii) Those common poles of f and g, where each such pole of f and g has different multiplicities related to f and g. (iv) Those common 1-points of f and g, where each such point has different multiplicities related to f and g. (v) The zeros of $f^{'}$ which are not zeros of $f(f-1)\prod_{j=0}^{m-1}(f-\beta_j)$. (vi) The zeros of $g^{'}$ which are not zeros of $g(g-1)\prod_{j=0}^{m-1}(g-\beta_j)$. Since all poles of H are simple, the lemma follows. □

The next two lemmas are very much similar to the Lemma 2.3. So we only write the statement of the lemmas in the following.

Lemma 2.4. *Suppose f, g share $(1,0)$, $(\infty,0)$ and β_j, defined as in* (1.3)*, are non-zero complex numbers. If $H \not\equiv 0$, then,*

$$\begin{aligned} N(r,H) \quad & \leq N(r,0;f \mid\geq 2) + N(r,0;g \mid\geq 2) + \sum_{j=0}^{m-1} \overline{N}(r,\beta_j;f \mid\geq 2) \\ & + \sum_{j=0}^{m-1} \overline{N}(r,\beta_j;g \mid\geq 2) + \overline{N}_*(r,\infty;f,g) + \overline{N}_0(r,0;f^{'}) + \overline{N}_0(r,0;g^{'}) \\ & + \overline{N}_*(r,1;f,g) + S(r,f) + S(r,g), \end{aligned}$$

where $\overline{N}_0(r,0;f^{'})$ is the reduced counting function of those zeros of $f^{'}$ which are not the zeros of $f(f-1)\prod_{j=0}^{m-1}(f-\beta_j)$ and $\overline{N}_0(r,0;g^{'})$ is similarly defined.

Lemma 2.5. *Suppose f, g be two non-constant meromorphic functions sharing $(1,0)$ and β_j, defined as in* (1.3)*, are non-zero complex numbers. If $H \not\equiv 0$,* *then*

$$\begin{aligned} & N(r,H) \\ \leq\ & N(r,0;f\mid\geq 2)+N(r,0;g\mid\geq 2)+\sum_{j=0}^{m-1}\overline{N}(r,\beta_j;f\mid\geq 2)+\overline{N}_*(r,1;f,g) \\ & +\sum_{j=0}^{m-1}\overline{N}(r,\beta_j;g\mid\geq 2)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_0(r,0;f^{'}) \\ & +\overline{N}_0(r,0;g^{'})+S(r,f)+S(r,g), \end{aligned}$$

where $\overline{N}_0(r,0;f^{'})$ *is the reduced counting function of those zeros of* $f^{'}$ *which are not the zeros of* $f(f-1)\prod_{j=0}^{m-1}(f-\beta_j)$ *and* $\overline{N}_0(r,0;g^{'})$ *is similarly defined.*

Lemma 2.6. *[11] Let* f *be a non-constant meromorphic function and* $P(f)=a_0+a_1f+a_2f^2+\ldots+a_nf^n$*, where* $a_0,a_1,a_2,\ldots,a_n$ *are constants and* $a_n\neq 0$*. Then*

$$T(r,P(f))=nT(r,f)+O(1).$$

Lemma 2.7. *[4] Let* f *be a transcendental meromorphic function of finite order and* $c\in\mathbb{C}-\{0\}$ *be fixed. Then*

$$T(r,f(z+c))=T(r,f(z))+S(r,f(z)).$$

Lemma 2.8. *Let* f *be a transcendental meromorphic function of finite order and* $c\in\mathbb{C}-\{0\}$ *be fixed. Then*

$$S(r,f(z+c))=S(r,f(z)).$$

Proof. Using Lemma 2.6, it can be easily seen that

$$S(r,f(z+c))=o(T(r,f(z+c)))=o(T(r,f(z)))=S(r,f(z)).$$

□

Lemma 2.9. *Let* F *and* G *be given by* (2.1) *and* (2.2)*,* $n(\geq 1)$ *an integer and* $\Phi\not\equiv 0$*. If* F *and* G *share* $(1,m)$*,* $f(z)$ *and* $f(z+c)$ *share* $(0,p)$*,* (∞,k)*, where* $0\leq k<\infty$*, then*

$$\begin{aligned} & \{(n-2m)(p+1)-1\}\overline{N}(r,0;f\mid\geq p+1) \\ =\ & \{(n-2m)(p+1)-1\}\overline{N}(r,0;f(z+c)\mid\geq p+1) \\ \leq\ & \overline{N}_*(r,1;F,G)+\overline{N}_*(r,\infty;F,G)+S(r,f)+S(r,f(z+c)). \end{aligned}$$

Proof. Suppose 0 is an e.v.p. of $f(z)$ and $f(z+c)$. Then the lemma follows immediately. Next suppose 0 is not an e.v.p. of $f(z)$ and $f(z+c)$. Let z_0 be a zero of f with multiplicity q and a zero of $f(z+c)$ with multiplicity r. Then from (2.1) and (2.2), we know that z_0 is a zero of F with multiplicity $(n-2m)q$ and a zero of G with multiplicity $(n-2m)r$. We note that F and G have no zero of multiplicity t, where $(n-2m)p<t<(n-2m)(p+1)$.

So, from definition of Φ, it is clear that z_0 is a zero of Φ with multiplicity at least $(n-2m)(p+1)-1$.

So, we have,

$$\begin{aligned} & \{(n-2m)(p+1)-1\}\overline{N}(r,0;f(z)\mid\geq p+1) \\ = & \{(n-2m)(p+1)-1\}\overline{N}(r,0;f(z+c)\mid\geq p+1) \\ \leq & \overline{N}_*(r,\infty;f(z),f(z+c))+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,f(z+c)). \end{aligned}$$

□

Lemma 2.10. *Let* F, G *be given by* (2.1) *and* (2.2), *where* $n(\geq 8)$ *is an integer and* $H_1 \not\equiv 0$. *Suppose* $\alpha_1, \alpha_2, \ldots, \alpha_{2m}$ *are the roots of the equation* $z^{2m} - \frac{2n}{n-m}z^m + \frac{n}{n-2m} = 0$. *Suppose also that* F, G *share* $(1,t)$ *and* $f(z)$, $f(z+c)$ *share* (∞,k), $(0,0)$, *where* $2 \leq t < \infty$. *Then, for the complex numbers* β_j *given by (1.3), we have*

$$\begin{aligned} & n\left(m+\frac{1}{2}\right)\{T(r,f(z))+T(r,f(z+c))\} \\ \leq & \overline{N}(r,0;f(z))+\overline{N}(r,0;f(z+c))+\overline{N}_*(r,0;f(z),f(z+c))+\overline{N}(r,\infty;f) \\ & +\sum_{j=1}^{2m}N_2(r,\alpha_j;f(z))+\sum_{j=1}^{2m}N_2(r,\alpha_j;f(z+c))+\overline{N}(r,\infty;f(z+c)) \\ & +\sum_{j=0}^{m-1}N_2(r,\beta_j;F)+\sum_{j=0}^{m-1}N_2(r,\beta_j;G)+\overline{N}_*(r,\infty;f(z),f(z+c)) \\ & -\left(t-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f(z))+S(r,f(z+c)). \end{aligned}$$

Proof. By the Second Fundamental Theorem of Nevalinna, we have

$$\begin{aligned} & (m+1)\{T(r,F)+T(r,G)\} \\ \leq & \overline{N}(r,0;F)+\overline{N}(r,1;F)+\overline{N}(r,\infty;F)+\sum_{j=0}^{m-1}\overline{N}(r,\beta_j;F)+\overline{N}(r,0;G) \\ & +\overline{N}(r,1;G)+\overline{N}(r,\infty;G)+\sum_{j=0}^{m-1}\overline{N}(r,\beta_j;G)-\overline{N}_0(r,0;F^{'})-\overline{N}_0(r,0;G^{'}) \\ & +S(r,F)+S(r,G). \end{aligned}$$

Now using Lemma 2.1, Lemma 2.2 and Lemma 2.3 and Lemma 2.6, we have

$$
\begin{aligned}
& n\left(m+\frac{1}{2}\right)\{T(r,f(z))+T(r,f(z+c))\} \\
\leq\ & \overline{N}(r,0;f(z))+\overline{N}(r,0;f(z+c))+\overline{N}_*(r,0;f(z),f(z+c))+\overline{N}(r,\infty;f) \\
& +\sum_{j=1}^{2m}N_2(r,\alpha_j;f(z))+\sum_{j=1}^{2m}N_2(r,\alpha_j;f(z+c))+\overline{N}(r,\infty;f(z+c)) \\
& +\sum_{j=0}^{m-1}N_2(r,\beta_j;F)+\sum_{j=0}^{m-1}N_2(r,\beta_j;G)+\overline{N}_*(r,\infty;f(z),f(z+c)) \\
& -\left(t-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f(z))+S(r,f(z+c)).
\end{aligned}
$$

□

The next two lemmas are very much similar to the Lemma 2.10. So we only write the statement of the lemmas in the following.

Lemma 2.11. *Let F, G be given by* (2.1) *and* (2.2), *where* $n(\geq 8)$ *is an integer and* $H_1 \not\equiv 0$. *Suppose* $\alpha_1,\alpha_2,\dots,\alpha_{2m}$ *are the same as defined in Lemma 2.10. Suppose also that F, G share* $(1,t)$ *and* $f(z)$, $f(z+c)$ *share* (∞,k), *where* $2\leq t<\infty$. *Then, for the complex numbers* β_j *given by (1.3), we have*

$$
\begin{aligned}
& n\left(m+\frac{1}{2}\right)\{T(r,f(z))+T(r,f(z+c))\} \\
\leq\ & 2\{\overline{N}(r,0;f(z))+\overline{N}(r,0;f(z+c))\}+\sum_{j=1}^{2m}N_2(r,\alpha_j;f(z))+\overline{N}(r,\infty;f) \\
& +\sum_{j=1}^{2m}N_2(r,\alpha_j;f(z+c))+\overline{N}(r,\infty;f(z+c))+\sum_{j=0}^{m-1}N_2(r,\beta_j;F) \\
& +\sum_{j=0}^{m-1}N_2(r,\beta_j;G)+\overline{N}_*(r,\infty;f(z),f(z+c))-\left(t-\frac{3}{2}\right)\overline{N}_*(r,1;F,G) \\
& +S(r,f(z))+S(r,f(z+c)).
\end{aligned}
$$

Lemma 2.12. *Let F, G be given by (2.1) and (2.2), where* $n\geq 8$ *is an integer and* $H_1\not\equiv 0$ *and let* $\alpha_1,\alpha_2,\dots,\alpha_{2m}$ *be the same as defined in Lemma 2.10. Suppose F, G share* $(1,t)$. *Then, for the complex numbers* β_j *as given by* (1.3), *we have*

$$\begin{aligned} & n\left(m+\frac{1}{2}\right)\{T(r,f(z))+T(r,f(z+c))\} \\ \leq \quad & 2\{\overline{N}(r,0;f(z))+\overline{N}(r,0;f(z+c))\} \\ & +\sum_{j=1}^{2m} N_2(r,\alpha_j;f(z))+\sum_{j=0}^{m-1} N_2(r,\beta_j;F) \\ & +\sum_{j=1}^{2m} N_2(r,\alpha_j;f(z+c))+2\{\overline{N}(r,\infty;f(z))+\overline{N}(r,\infty;f(z+c))\} \\ & +\sum_{j=0}^{m-1} N_2(r,\beta_j;G)-\left(t-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f(z))+S(r,f(z+c)). \end{aligned}$$

Lemma 2.13. *Let F, G be given by (2.1) and (2.2), $n \geq 8$ is an integer and $V_1 \not\equiv 0$. Suppose also F, G share $(1,t)$, and $f(z)$, $f(z+c)$ share (∞,k), $(0,0)$, where t, k and p are non-negative integers. Then the poles of F and G are zeros of V_1 and*

$$\begin{aligned} & (nk+n-1)\overline{N}(r,\infty;f(z)\mid\geq k+1) \\ = \quad & (nk+n-1)\overline{N}(r,\infty;f(z+c))\mid\geq k+1) \\ \leq \quad & \overline{N}_*(r,0;f(z),f(z+c))+\sum_{j=1}^{2m}\overline{N}(r,\alpha_j;f(z))+\sum_{j=1}^{2m}\overline{N}(r,\alpha_j;f(z+c)) \\ & +\overline{N}_*(r,1;F,G)+S(r,f(z))+S(r,f(z+c)), \end{aligned}$$

where α_i, $i=1,2,\ldots,2m$ has the same meaning as in Lemma 2.11.

Proof. Since $f(z)$, $f(z+c)$ share $(\infty;k)$, it follows that F, G share $(\infty;nk)$ and so a pole of F with multiplicity $p(\geq nk+1)$ is a pole of G with multiplicity $r(\geq nk+1)$ and vice versa. We note that F and G have no pole of multiplicity q where $nk<q<nk+n$. Now using the Milloux Theorem [[6], p. 55], we get from the definition of V_1,

$$m(r,V_1)=S(r,f(z))+S(r,f(z+c)).$$

Hence

$$\begin{aligned}
& (nk+n-1)\overline{N}(r,\infty;f\mid\geq k+1)\\
= & (nk+n-1)\overline{N}(r,\infty;f(z+c)\mid\geq k+1)\\
& \leq N(r,0;V_1)\\
& \leq T(r,V_1)+O(1)\\
& \leq N(r,\infty;V_1)+m(r,V_1)+O(1)\\
& \leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)+S(r,f(z))+S(r,f(z+c))\\
& \leq \overline{N}_*(r,0;f(z),f(z+c))+\sum_{j=1}^{2m}\overline{N}(r,\alpha_j;f(z))+\sum_{j=1}^{2m}\overline{N}(r,\alpha_j;f(z+c))\\
& \quad +\overline{N}_*(r,1;F,G)+S(r,f(z))+S(r,f(z+c)),
\end{aligned}$$

where α_i, $i=1,2,\ldots,2m$ has the same meaning as in Lemma 2.11. □

The proof of the following lemma is similar to that of Lemma 2.13. So we omit the details.

Lemma 2.14. *Let F, G be given by (2.1) and (2.2), $n\geq 8$ is an integer and $V_1\not\equiv 0$. If F, G share $(1,t)$, and $f(z)$, $f(z+c)$ share (∞,k), where $0\leq k<\infty$, then the poles of F and G are zeros of V_1 and*

$$\begin{aligned}
& (nk+n-1)\overline{N}(r,\infty;f(z)\mid\geq k+1)\\
= & (nk+n-1)\overline{N}(r,\infty;f(z+c))\mid\geq k+1)\\
\leq & \overline{N}(r,0;f(z))+\overline{N}(r,0;f(z+c))+\sum_{j=1}^{2m}\overline{N}(r,\alpha_j;f(z))+\overline{N}_*(r,1;F,G)\\
& +\sum_{j=1}^{2m}\overline{N}(r,\alpha_j;f(z+c))+S(r,f(z))+S(r,f(z+c)),
\end{aligned}$$

where α_i, $i=1,2,\ldots,2m$ has the same meaning as in Lemma 2.11.

Lemma 2.15. *Let F and G be defined as in* (2.1) *and* (2.2). *Then $FG\not\equiv 1$ for $n\geq 5$.*

Proof. Suppose on the contrary $FG\equiv 1$. Then by Mokhon'ko's Lemma

$$T(r,f(z))=T(r,f(z+c))+O(1).$$

Also

$$(f(z))^{n-2m}\prod_{j=1}^{2m}(f(z)-\alpha_j)(f(z+c))^{n-2m}\prod_{j=1}^{2m}(f(z+c)-\alpha_j)\equiv d^2,$$

where $\alpha_1,\alpha_2,\ldots,\alpha_{2m}$ have the same meaning as in Lemma 2.10.

Let z_0 be a α_j-point of $f(z)$ of order p. Then z_0 is a pole of $f(z+c)$ of order q such that $p = nq \geq n$. Therefore,

$$\overline{N}(r, \alpha_j; f(z)) \leq \frac{1}{n} N(r, \alpha_j; f(z)).$$

Again let z_0 be a zero of $f(z)$ of order t. Then z_0 is a pole of $f(z+c)$ of order s such that

$$(n-2m)t = ns.$$

This implies $t > s$ and $2ms = (n-2m)(t-s) \geq n-2m$. Therefore, $(n-2m)t = ns$ gives $t \geq \frac{n}{2m}$. So

$$\overline{N}(r, 0; f(z)) \leq \frac{2m}{n} N(r, 0; f(z)).$$

Again

$$\begin{aligned}
\overline{N}(r, \infty; f(z)) &\leq \overline{N}(r, 0; f(z+c)) + \sum_{j=1}^{2m} \overline{N}(r, \alpha_j; f(z+c)) \\
&\leq \frac{2m}{n} N(r, 0; f(z+c)) + \frac{1}{n} \sum_{j=1}^{2m} N(r, \alpha_j; f(z+c)) \\
&\leq \frac{4m}{n} T(r, f(z+c)).
\end{aligned}$$

Therefore, by the Second Fundamental Theorem of Nevanlinna, we get

$$\begin{aligned}
& 2mT(r, f(z)) \\
\leq\ & \overline{N}(r, \infty; f(z)) + \overline{N}(r, 0; f(z)) + \sum_{j=1}^{2m} \overline{N}(r, \alpha_j; f(z)) + S(r, f(z)) \\
\leq\ & \frac{8m}{n} T(r, f) + S(r, f),
\end{aligned}$$

which is a contradiction for $n \geq 5$. □

Lemma 2.16. *Let $m(\geq 1)$ and $n(> 2m)$ be two positive integers. Then the polynomial*

$$\phi(h) = (n-m)^2(h^n - 1)(h^{n-2m} - 1) - n(n-2m)(h^{n-m} - 1)^2$$

of degree $2n - 2m$ has m roots of multiplicity 4 and all other zeros are simple.

Proof. Let $F(t) = \frac{1}{2}\phi(e^t)e^{-(n-m)t}$ for $t \in \mathbb{C}$.

An elementary calculation gives

$$F(t) = m^2 \cosh(n-m)t - (n-m)^2 \cosh mt + n(n-2m).$$

Assume that $\phi(\omega) = \phi^{'}(\omega) = 0$ for some $\omega \in \mathbb{C}$.

Then $F(t) = F^{'}(t) = 0$ for every $t \in \mathbb{C}$ satisfying $e^t = \omega$. From $F(t) = 0$, we get

$$m^2 \cosh(n-m)t = (n-m)^2 \cosh mt - n(n-2m) \tag{2.3}$$

From $F^{'}(t) = 0$, we get

$$m^2 \sinh(n-m)t = m(n-m) \sinh mt \tag{2.4}$$

Therefore, from (2.3) and (2.4) we have

$$\begin{aligned} m^4 &= \{(n-m)^2 \cosh mt - n(n-2m)\}^2 - \{m(n-m)\sinh mt\}^2 \\ &= (n-m)^4 \cosh^2 mt + \{n(n-2m)\}^2 - 2n(n-2m)(n-m)^2 \cosh mt \\ &\quad -\{m(n-m)\}^2(\cosh^2 mt - 1) \\ &= \{(n-m)^4 - m^2(n-m)^2\} \cosh^2 mt + \{n(n-2m)\}^2 + m^2(n-m)^2 \\ &\quad -2n(n-2m)(n-m)^2 \cosh mt \end{aligned}$$

or,

$$\begin{aligned} &(n-m)^2 n(n-2m) \cosh^2 mt + \{n(n-2m)\}^2 + m^2(n-m)^2 \\ &-2n(n-2m)(n-m)^2 \cosh mt - m^4 + \{n(n-2m)\}^2 = 0 \end{aligned}$$

or,

$$\begin{aligned} &(n-m)^2 n(n-2m)(\cosh mt - 1)^2 - (n-m)^2 n(n-2m) + m^2(n-m)^2 \\ &-m^4 + \{n(n-2m)\}^2 = 0 \end{aligned}$$

or,

$$\begin{aligned} &(n-m)^2 n(n-2m)(\cosh mt - 1)^2 - n(n-2m)\{(n-m)^2 - n(n-2m)\} \\ &+m^2 n(n-2m) = 0 \end{aligned}$$

or,

$$(n-m)^2 n(n-2m)(\cosh mt - 1)^2 = 0$$

or,

$$(\cosh mt - 1)^2 = 0$$

or,

$$\left(\frac{e^{mt} + e^{-mt}}{2} - 1\right)^2 = 0$$

or,

$$(\omega^m - 1)^4 = 0,$$

which shows that the roots of the equation $\omega^m = 1$ are of multiplicity 4.

Therefore, $\phi(h)$ has m zeros of multiplicity 4 and all other zeros are simple.

□

Lemma 2.17. *[13] Let f, g share $(\infty,0)$ and $V\equiv 0$. Then $f\equiv g$.*

Lemma 2.18. *Let $m(\geq 1)$ and $(2\leq n\leq 10)$ are integers. Then*

$$3m+2<4+2m+\frac{(4m+2)(7n-3)}{(n-1)(3n-1)}.$$

Proof. Let $P(m)$ be the statement

$$P(m): 3m+2<4+2m+\frac{(4m+2)(7n-3)}{(n-1)(3n-1)}.$$

Then, clearly $P(1)$ is true. Suppose that $P(k)$ is true for $k\geq 1$.
i.e.,

$$3k+2<4+2k+\frac{(4k+2)(7n-3)}{(n-1)(3n-1)}.$$

Now

$$\begin{aligned} P(k+1) &= 4+2(k+1)+\frac{(4(k+1)+2)(7n-3)}{(n-1)(3n-1)} \\ &= 4+2k+\frac{(4k+2)(7n-3)}{(n-1)(3n-1)}+2+\frac{4(7n-3)}{(n-1)(3n-1)} \\ &> 3k+2+(2+\frac{4(7n-3)}{(n-1)(3n-1)}) \\ &> 3k+2+3 \\ &= 3(k+1)+2. \end{aligned}$$

Hence $P(m)$ is true for all $m\in\mathbb{N}$. Therefore, the lemma follows. □

3. Proof of the theorems

Proof of Theorem 1.11. Let F and G be two functions defined in (2.1) and (2.2).

Since $E_{f(z)}(S,t)=E_{f(z+c)}(S,t)$ and $E_{f(z)}(\{\infty\},k)=E_{f(z+c)}(\{\infty\},k)$, it follows that F, G share $(1,t)$ and (∞,nk).

Since

$$F-\beta_j=\frac{1}{d}(f-\omega_j)^3Q_{n-3}(f),$$

where $Q_{n-3}(f)$ is a polynomial in f of degree $n-3$, for $j=0,1,2,\ldots,m-1$, we have

$$\begin{aligned} (3.1)\qquad N_2\Big(r,\beta_j;F\Big) &\leq 2\overline{N}(r,\omega_j;f)+N(r,0;Q_{n-3}(f)) \\ &\leq 2\overline{N}(r,\omega_j;f)+(n-3)T(r,f)+S(r,f). \end{aligned}$$

Similarly,

$$N_2\Big(r,\beta_j;G\Big) \leq 2\overline{N}(r,\omega_j;f(z+c))+(n-3)T(r,f(z+c)) +S(r,f(z+c)), \tag{3.2}$$

for each $j=0,1,2,\ldots,m-1$.

Case 1: Suppose $H_1\not\equiv 0$. Then $F\not\equiv G$. So, it follows from Lemma 2.17 that $V_1\not\equiv 0$.

Hence using (3.1), (3.2) and Lemma 2.1, Lemma 2.2, Lemma 2.3, Lemma 2.9 and Lemma 2.10 and Lemma 2.13, we have

$$\begin{aligned} & n\left(m+\frac{1}{2}\right)\{T(r,f(z))+f(z+c)\} \\ \leq\ & 3\overline{N}(r,0;f)+2\overline{N}(r,\infty;f)+\{2m+m(n-1)\}\{T(r,f)+T(r,f(z+c))\} \\ & +\overline{N}_*(r,\infty;f(z),f(z+c))-\left(t-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f) \\ & +S(r,f(z+c)) \end{aligned}$$

i.e.,

$$
\begin{aligned}
&\left(\frac{n}{2}-m\right)\{T(r,f(z))+T(r,f(z+c))\}\\
\leq\ & \frac{3}{n-2m-1}\{\overline{N}_*(r,\infty;f(z),f(z+c))+\overline{N}_*(r,1,F,G)\}\\
&+\frac{2}{n-1}\{\overline{N}_*(r,0;f(z),f(z+c))+\overline{N}_*(r,1;F,G)+2mT(r,f(z))\\
&+2mT(r,f(z+c))\}+\overline{N}_*(r,\infty;f(z),f(z+c))-\left(t-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)\\
&+S(r,f)+S(r,f(z+c))\\
\leq\ & \left(1+\frac{3}{n-2m-1}\right)\overline{N}_*(r,\infty;f(z),f(z+c))+\frac{2}{n-1}\left(2m+\frac{1}{2}\right)\{T(r,f)\\
&+T(r,f(z+c))\}-\left(t-\frac{3}{2}-\frac{3}{n-2m-1}-\frac{2}{n-1}\right)\overline{N}_*(r,1;F,G)\\
&+S(r,f)+S(r,f(z+c))\\
\leq\ & \frac{n-2m+2}{(n-2m-1)(nk+n-1)}\{\overline{N}_*(r,0;f(z),f(z+c)+2m(T(r,f)\\
&+T(r,f(z+c)))+\overline{N}_*(r,1;F,G)\}+\frac{4m+1}{n-1}\{T(r,f)+T(r,f(z+c))\}\\
&-\left(t-\frac{3}{2}-\frac{3}{n-2m-1}-\frac{2}{n-1}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,f(z+c))\\
\leq\ & \frac{n-2m+2}{(n-2m-1)(nk+n-1)}\{\overline{N}_*(r,0;f(z),f(z+c)+2m(T(r,f(z+c))\\
&+T(r,f(z)))\}+\frac{4m+1}{n-1}\{T(r,f)+T(r,f(z+c))\}-\left(t-\frac{3}{2}-\frac{2}{n-1}\right.\\
&\left.-\frac{3}{n-2m-1}-\frac{n-2m+2}{(n-2m-1)(nk+n-1)}\right)\overline{N}_*(r,1;F,G)+S(r,f)\\
&+S(r,f(z+c)).
\end{aligned}
$$

Therefore, from the condition over t and k in the theorem, we get from above

$$
\begin{aligned}
&\left\{\frac{n}{2}-m-\frac{4m+1}{n-1}-\frac{(n-2m+2)(4m+1)}{2(n-2m-1)(nk+n-1)}\right\}\{T(r,f)\\
&+T(r,f(z+c))\}\leq S(r,f)+S(r,f(z+c)),
\end{aligned}
$$

which is a contradiction.

Case 2: Suppose $H_1\equiv 0$. Then by integration we have

$$F\equiv\frac{AG+B}{CG+D}, \tag{3.3}$$

where A,B,C,D are complex constants satisfying $AD-BC\neq 0$.

Therefore, from (3.3), F, G share $(1,\infty)$. Since F, G share (∞,nk), it follows that F, G share (∞,∞). Also from Lemma 2.9, we obtain $\overline{N}(r,0;f(z))=N(r,0;f(z+c))=S(r,f)+S(r,f(z+c))$.

Subcase 2.1: Suppose $AC \neq 0$. Then $F - \frac{A}{C} = \frac{-(AD-BC)}{C(CG+D)} \neq 0$. So F omits the value $\frac{A}{C}$.

Therefore, by the Second Fundamental Theorem, we have

$$\begin{aligned} nT(r,f) &\leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,\frac{A}{C};F) + S(r,F) \\ &\leq (2m+1)T(r,f) + S(r,f). \end{aligned}$$

i.e.,

$$(n-2m-1)T(r,f) \leq S(r,f),$$

which is a contradiction.

Subcase 2.2: Suppose $AC = 0$. Since $AD - BC \neq 0$, both A and C can not be simultaneously zero.

Subcase 2.2.1: Suppose $A \neq 0$ and $C = 0$. Then (3.3) becomes

$$F \equiv \alpha G + \beta, \tag{3.4}$$

where $\alpha = \frac{A}{D}$ and $\beta = \frac{B}{D}$.

If F has no 1-point, then by the Second Fundamental Theorem of Nevalinna, we have

$$T(r,F) \leq \overline{N}(r,0;F) + \overline{N}(r,1;F) + \overline{N}(r,\infty;F) + S(r,F)$$

or,

$$(n-2m-1)T(r,f) \leq S(r,f),$$

which is not possible.

Let F has some 1-points. Then $\alpha + \beta = 1$. Therefore from (3.4), we have $F = \alpha G + 1 - \alpha$.

Subcase 2.2.1.1: Suppose $\alpha \neq 1$. We consider the following subcases.

Subcase 2.2.1.1.1: Suppose $m = 1$. So $\omega_0 = 1$. Noting that $n \geq 5$, from (1.3), we have $\beta_0 = \dfrac{\gamma_0}{d} = \dfrac{2\omega_0^n}{(n-1)(n-2)d} = \dfrac{2}{(n-1)(n-2)d}$ and therefore, in view of (1.2) we must have

$$F - \beta_0 = \frac{1}{d}(f(z)-1)^3 Q_{n-3}(f(z)),$$

where $Q_{n-3}(f(z))$ is a polynomial in $f(z)$ of degree $n-3$. Therefore, we have

$$\overline{N}(r,\beta_0;F) \leq \overline{N}(r,1;f(z)) + (n-3)T(r,f(z)) + S(r,f(z)).$$

In a similar manner, we write $G - \beta_0 = \dfrac{1}{d}(f(z+c)-1)^3 Q^*_{n-3}(f(z+c))$, $Q^*_{n-3}(f(z+c))$ is a polynomial in $f(z+c)$ of degree $n-3$ and

$$\overline{N}(r,\beta_0;G) \leq \overline{N}(r,1;f(z+c)) + (n-3)T(r,f(z+c)) + S(r,f(z+c)).$$

If $1-\alpha\neq\beta_0$, then by the Second Fundamental Theorem, Lemma 2.7 and Lemma 2.8, we have

$$\begin{aligned}
& 2T(r,F)\\
\leq\ & \overline{N}(r,0;F)+\overline{N}(r,1-\alpha;F)+\overline{N}(r,\beta_0;F)+\overline{N}(r,\infty;F)+S(r,F)\\
\leq\ & \overline{N}(r,0;f(z))+2T(r,f(z))+\overline{N}(r,0;f(z+c))+2T(r,f(z+c))\\
& +\overline{N}(r,1;f(z))+(n-3)T(r,f(z))+\overline{N}(r,\infty;f(z))+S(r,f)\\
\leq\ & (n+3)T(r,f)+S(r,f).
\end{aligned}$$

i.e.,

$$(n-3)T(r,f)\leq S(r,f),$$

which is not possible.

If $1-\alpha=\beta_0$, then we have from (3.4) that $F=(1-\beta_0)G+\beta_0$. Since $d\neq\frac{1}{(n-1)(n-2)}$, by the Second Fundamental Theorem, Lemma 2.7 and Lemma 2.8 we have,

$$\begin{aligned}
2T(r,G)\ \leq\ & \overline{N}(r,0;G)+\overline{N}(r,\frac{\beta_0}{\beta_0-1};G)+\overline{N}(r,\beta_0;G)+\overline{N}(r,\infty;G)\\
& +S(r,G)\\
\leq\ & \overline{N}(r,0;f(z+c))+2T(r,f(z+c))+\overline{N}(r,0;f)+2T(r,f)\\
& +\overline{N}(r,1;f(z+c))+(n-3)T(r,f(z+c))+\overline{N}(r,\infty;f(z+c))\\
& +S(r,f)+S(r,f(z+c)).
\end{aligned}$$

i.e.,

$$(n-3)T(r,f)\leq S(r,f),$$

which is not possible.

Subcase 2.2.1.1.2: Next suppose $m\geq 2$. Then by the Second Fundamental Theorem of Nevanlinna, Lemma 2.7 and Lemma 2.8, we have,

$$\begin{aligned}
& (m+1)T(r,F)\\
\leq\ & \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1-\alpha;F)+\sum_{j=0}^{m-1}\overline{N}(r,\beta_j;F)+S(r,F)\\
\leq\ & \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\sum_{j=0}^{m-1}\overline{N}(r,\beta_j;F)+S(r,F)\\
\leq\ & \overline{N}(r,0;f)+2mT(r,f)+T(r,f)+\overline{N}(r,0;f(z+c))+2mT(r,f(z+c))\\
& +m(n-2)T(r,f)+S(r,f).
\end{aligned}$$

i.e.,

$$(n-2m-1)T(r,f)\leq S(r,f),$$

which is not possible.

Subcase 2.2.1.2: Suppose $\alpha = 1$. Then $F \equiv G$.

i.e.,

$$f(z)^{n-2m}(f(z)^{2m} - \frac{2n}{n-m}f(z)^m + \frac{n}{n-2m}) \equiv f(z+c)^{n-2m}(f(z+c)^{2m} - \frac{2n}{n-m}f(z+c)^m + \frac{n}{n-2m}).$$

i.e.,

$$f(z+c)^n - \frac{2n}{n-m}f(z+c)^{n-m} + \frac{n}{n-2m}f(z+c)^{n-2m} \equiv f(z)^n - \frac{2n}{n-m}f(z)^{n-m} + \frac{n}{n-2m}f(z)^{n-2m}.$$

Suppose that $h(z) = \frac{f(z+c)}{f(z)}$. Then we have from above,

$$(h^n - 1)f^{2m} - \frac{2n}{n-m}(h^{n-m} - 1)f^m + \frac{n}{n-2m}(h^{n-2m} - 1) = 0.$$

i.e.,

(3.5)

$$\frac{(n-m)(n-2m)}{2}(h^n-1)g_1^2 - n(n-2m)(h^{n-m}-1)g_1 + \frac{n(n-m)}{2}(h^{n-2m}-1) = 0,$$

where $g_1 = f^m$.

Suppose $h(z)$ is not constant. Then from (3.5) we have,

(3.6) $$\{(n-m)(n-2m)(h^n-1)g_1 - n(n-2m)(h^{n-m}-1)\}^2 = -n(n-2m)\Phi(h),$$

where $\Phi(h) = (n-m)^2(h^n - 1)(h^{n-2m} - 1) - n(n-2m)(h^{n-m} - 1)^2$ is a polynomial of degree $2n - 2m$. Therefore, in view of Lemma 2.16, (3.6) can be written as

$$\{(n-m)(n-2m)(h^n - 1)g_1 - n(n-2m)(h^{n-m} - 1)\}^2 = -n(n-2m)\prod_{j=1}^{m}(h - \omega_j)^4 \prod_{i=1}^{2n-6m}(h - \eta_i),$$

where $\omega_j = \cos\frac{2j\pi}{m} + i\sin\frac{2j\pi}{m}$, $j = 0, 1, 2, \ldots, m-1$ and $\eta_1, \eta_2, \ldots, \eta_{2n-6m}$ are the simple zeros of $\Phi(h)$.

It can easily be seen from the above equation that all the zeros of $h - \eta_j$ have order at least 2. Since $f(z)$, $f(z+c)$ share $(0, \infty)$ and (∞, ∞), it follows that h omits the value 0 and ∞.

Therefore, applying the Second Fundamental Theorem to h, we have

$$\begin{aligned}(2n-6m)T(r,h) &\leq \sum_{j=1}^{2n-6m}\overline{N}(r,\eta_j;h) + \overline{N}(r,0;h) + \overline{N}(r,\infty;h) + S(r,h) \\ &\leq \frac{1}{2}\sum_{j=1}^{2n-6m}N(r,\eta_j;h) + S(r,h) \\ &\leq (n-3m)T(r,h) + S(r,h).\end{aligned}$$

i.e.,

$$(n-3m)T(r,h) \leq S(r,h),$$

which is impossible.

So, h is constant. Hence, from (3.5), we have $h^n - 1 = 0$, $h^{n-m} - 1 = 0$ and $h^{n-2m} - 1 = 0$. Since $gcd(n,m) = 1$, we must have $h \equiv 1$.

i.e.,

$$f(z+c) \equiv f(z).$$

Subcase 2.2.2: Suppose $A = 0$ and $C \neq 0$.

Then (3.1) becomes

$$F \equiv \frac{1}{\gamma G + \delta},$$

where $\gamma = \frac{C}{B}$ and $\delta = \frac{D}{B}$.

If F has no 1-point, the case can be treated in the same way as done in Subcase 2.2.1.

So let F has some 1-point. Then $\gamma + \delta = 1$.

Now, γ can not be equal to 1. For otherwise $FG \equiv 1$ which is not possible by Lemma 2.15.

Therefore,

$$F \equiv \frac{1}{\gamma G + 1 - \gamma}.$$

Since $C \neq 0$, $\gamma \neq 0$, G omits the value $-\frac{1-\gamma}{\gamma}$.

By the Second Fundamental Theorem, we have

$$T(r,G) \quad \leq \quad \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,-\frac{1-\gamma}{\gamma};G) + S(r,G).$$

i.e.,

$$(n-2m-1)T(r,f(z+c)) \leq S(r,f(z+c)),$$

which is a contradiction. This completes the proof of the theorem. □

Proof of Theorem 1.13. Since $f(z)$ and $f(z+c)$ share $(0,\infty)$, $\overline{N}_*(r,0;f(z),f(z+c)) = 0$. Therefore, the proof of the theorem can be carried out along the lines of the proof of Theorem 1.11. So we omit the details. □

Proof of Theorem 1.9. Let F and G be two functions defined in (2.1) and (2.2).

Since $E_{f(z)}(S,2) = E_{f(z+c)}(S,2)$ and $E_{f(z)}(\{\infty\},2) = E_{f(z+c)}(\{\infty\},2)$, it follows that F, G share $(1,2)$ and $(\infty, 3n-1)$.

Case 1: Suppose $H_1 \not\equiv 0$. Then $F \not\equiv G$. So, it follows from Lemma 2.17 that $V_1 \not\equiv 0$.

Hence using (3.1), (3.2) and Lemma 2.11, for $t = 2$ and Lemma 2.14 for $k = 0$ and $k = 2$, we obtain

$$\begin{aligned}
& n\left(m+\frac{1}{2}\right)\{T(r,f)+T(r,f(z+c))\} \\
\leq\ & (2m+2)\{T(r,f(z)+T(r,f(z+c))\}+(mn-m)\{T(r,f(z+c) \\
& +T(r,f(z))\}+\frac{4m+2}{n-1}\{T(r,f(z)+T(r,f(z+c))\}+\frac{2m+1}{3n-1}T(r,f(z)) \\
& +\frac{2m+1}{3n-1}T(r,f(z+c))+S(r,f(z))+S(r,f(z+c)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \left\{\frac{n}{2}-2-m-\frac{4m+2}{n-1}-\frac{2m+1}{3n-1}\right\}\{T(r,f)+T(r,f(z+c))\} \\
& \leq S(r,f)+S(r,f(z+c)),
\end{aligned}$$

i.e.,

$$(n-b)\{T(r,f)+T(r,f(z+c))\}\leq S(r,f)+S(r,f(z+c)),$$

which is a contradiction since $n > max\{\chi_n a, b\}$.

Case 2: Suppose $H_1 \equiv 0$. We omit the rest of the proof as by using Lemma 2.18, the same can be carried out along the lines of the proof of Theorem 1.7. □

Proof of Theorem 1.7. Let F and G be two functions defined in (2.1) and (2.2). Since $E_{f(z)}(S,2) = E_{f(z+c)}(S,2)$, it follows that F, G share $(1,2)$.
Case 1: Suppose $H_1 \not\equiv 0$. Hence using Lemma 2.12 for $t = 2$ and equations (3.1), (3.2), we have

$$\begin{aligned}
& n\left(m+\frac{1}{2}\right)\{T(r,f)+T(r,f(z+c))\} \\
\leq\ & (2m+2)\{T(r,f(z)+T(r,f(z+c))\}+(mn-m)\{T(r,f(z) \\
& +T(r,f(z+c))\}+2\{T(r,f(z)+T(r,f(z+c))\}+S(r,f(z)) \\
& +S(r,f(z+c)).
\end{aligned}$$

i.e.,

$$(n-2m-8)\{T(r,f(z))+T(r,f(z+c))\}\leq S(r,f(z))+S(r,f(z+c)),$$

which is a contradiction since $n > max\{3m+2, 2m+8\}$.

Case 2: Suppose $H_1 \equiv 0$. We omit the rest of the proof as it can be carried out along the lines of the proof of Theorem 1.7. □

4. Some relevant issues

Putting $m = 1$, $n = 5$ and $d = \frac{1}{12}$ in (1.1), we have $P(z) = z^5 - \frac{5}{2}z^4 + \frac{5}{3}z^3 - \frac{1}{12}$. Suppose f and g are two non-constant meromorphic functions defined on $\mathbb{C}$ satisfying $f + g = 1$. We claim that f and g share the set

$$S=\{z: z^5-\frac{5}{2}z^4+\frac{5}{3}z^3-\frac{1}{12}=0\}. \tag{4.1}$$

Proof of claim:

$$\begin{aligned}
P(f) &= f^5 - \frac{5}{2}f^4 + \frac{5}{3}f^3 - \frac{1}{6} \\
&= (1-g)^5 - \frac{5}{2}(1-g)^4 + \frac{5}{3}(1-g)^3 - \frac{1}{12} \\
&= (1-g)^3\Big\{(1-g)^2 - \frac{5}{2}(1-g) + \frac{5}{3}\Big\} - \frac{1}{12} \\
&= (1-3g+3g^2-g^3)(g^2 + \frac{1}{2}g + \frac{1}{6}) - \frac{1}{12} \\
&= -(g^5 - \frac{5}{2}g^4 + \frac{5}{3}g^3 - \frac{1}{12}) \\
&= -P(g).
\end{aligned}$$

Therefore, $E_f(S,\infty) = E_g(S,\infty)$. But $f \not\equiv g$.

Now putting $d_1 = \frac{1}{2}$ and $n = 5$ in Corollary 1.12, we obtain

$$S_* = \{z : 6z^5 - 15z^4 + 10z^3 - \frac{1}{2} = 0\}. \tag{4.2}$$

Then by the same procedure as above we can show that for two non-constant meromorphic functions f and g in $\mathbb{C}$ with $f + g = 1$, $E_f(S_*,\infty) = E_g(S_*,\infty)$.

Now we note that if the function $g(z) \not\equiv f(z+c)$, the following counter example can be produced corresponding to Theorem 1.11 and Corollary 1.12 for $d = \frac{1}{12}$ and $d_1 = \frac{1}{2}$, respectively.

Example 4.1. Suppose $f(z) = \frac{e^z}{1+e^z}$, $g(z) = \frac{1}{1+e^z}$. Then f and g are of finite order sharing $(0,\infty)$, (∞,∞). Also f and g share the set S as well as S_* CM. But $f \not\equiv g$.

However, we were not able to find the case when $g(z) = f(z+c)$, c is a non-zero complex constant.

Next we consider the case when f is of infinite order. It is interesting to investigate whether in Theorem 1.11 and Corollary 1.12, respectively for the case $d = \frac{1}{12}$ and $d_1 = \frac{1}{2}$ such counter example exists at all.

The following example shows that such situation is feasible.

Example 4.2. Let $f(z) = \frac{1}{e^{e^z}+1}$. Then $f(z+c) = \frac{e^{e^z}}{e^{e^z}+1}$, where c is chosen such that $e^c = -1$. Clearly $f(z)$, $f(z+c)$ share $(0,\infty)$, (∞,∞) and the sets S and S_* CM, but $f(z) \not\equiv f(z+c)$.

However, unfortunately when $d \neq \frac{1}{12}$ or $d_1 \neq \frac{1}{2}$, we were again unsuccessful to find out the counter example in this case.

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