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# SECOND MULTIPLICATION MODULES

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**Abstract.** In this paper, we introduce second multiplication modules and obtain some related results. Also, we provide a counterexample to a previously published result concerning multiplication modules.

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## 1. Introduction

Throughout this paper, $R$ will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

Let $M$ be an $R$-module. $M$ is said to be a *multiplication module* if for any submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$ [9]. $M$ is said to be a *second module* if $M \neq 0$ and for each $a \in R$, the endomorphism $M \xrightarrow{a} M$ is either surjective or zero. $S$ is said to be a second submodule of $M$ if $S$ is a submodule of $M$ which is a second module [19]. The (*second*) *socle* of $N$ is defined as the sum of all second submodules of $M$ contained in $N$ and it is denoted by $soc(N)$. In case $N$ does not contain any second submodule, the socle of $N$ is defined to be (0). Also, $N \neq 0$ is said to be a *socle submodule* of $M$ if $soc(N) = N$ [5].

Set $Spec^s(M) = \{S : S \textit{ is a second submodule of } M\}$. We call this set the *second spectrum* of $M$ [7].

The purpose of this paper is to introduce the notion of second multiplication modules and provide some information concerning this new class of modules. In Theorem 2.6 of [13], the authors showed that every faithful multiplication module is finitely generated. We show that this is not true in general (see Remark 2.18 and Example 2.19).

## 2. Main Result

**Definition 2.1.** We say that an $R$-module $M$ is a *second multiplication* (*s-multiplication* for short) *module* if $M$ does not have any second submodule or for every second submodule $S$ of $M$, we have $S = IM$, where $I$ is an ideal of $R$.

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**Lemma 2.2.** An $R$-module $M$ is an s-multiplication module if and only if $S = (S :_R M)M$ for each second submodule $S$ of $M$.

*Proof.* Straightforward. □

**Remark 2.3.** It is clear that every multiplication $R$-module is an s-multiplication $R$-module. However, the converse is not true in general. For example, the $\mathbb{Z}$-module $\mathbb{Q}$ (here $\mathbb{Q}$ denotes the field of rational numbers) is an s-multiplication module while, it is not a multiplication $\mathbb{Z}$-module.

A submodule $N$ of an $R$-module $M$ is said to be *pure* if $IN = N \cap IM$ for every ideal $I$ of $R$ [1].

A submodule $N$ of an $R$-module $M$ is said to be *copure* if $(N :_M I) = N + (0 :_M I)$ for every ideal $I$ of $R$ [3].

**Proposition 2.4.** Let $M$ be an s-multiplication $R$-module. Then we have the following.

(a) If $N$ is a pure submodule of $M$, then $N$ is an s-multiplication $R$-module.

(b) Every direct summand of $M$ is an s-multiplication $R$-module.

(c) If $Ann_R(M)$ is a prime ideal of $R$, then every submodule of $M$ is an s-multiplication $R$-module.

(d) If every second submodule of $M$ is copure, then every submodule of $M$ is an s-multiplication module.

(e) If $Ann_R(M)$ is a prime ideal of $R$, then every second submodule of $M$ of the form $(0 :_M I)$ is equal to $M$.

*Proof.* (a) Let $S$ be a second submodule of $N$. Then by hypotheses, $S = IM$ for some ideal $I$ of $R$. As $N$ is pure, $IN = N \cap IM$. Hence $S = N \cap S = IN$.

(b) This follows from part (a) because every direct summand of $M$ is a pure submodule of $M$.

(c) Let $N$ be a submodule of $M$ and $S$ a second submodule of $N$. Then by hypotheses, $S = IM$ for some ideal $I$ of $R$. As $S$ is second, we have $IS = 0$ or $IS = S$. If $IS = 0$, then $0 = IS = I^2M$. This implies that $S = 0$, which is a contradiction. Therefore, $IS = S$. Hence $S = IS \subseteq IN \subseteq IM = S$ and so $S = IN$.

(d) Let $N$ be a submodule of $M$ and $S$ a second submodule of $N$. Then $S = IM$ for some ideal $I$ of $R$. Since $S$ is copure, $M = (S :_M I) = S + (0 :_M I)$. This implies that

$$S = IM = I(S + (0 :_M I)) = IS + I(0 :_M I) = IS + 0 = IS.$$

Thus $S = IS \subseteq IN \subseteq IM = S$, as required.

(e) Let $(0 :_M I)$ be a second submodule of $M$. Then $(0 :_M I) \neq 0$ and by assumption, $(0 :_M I) = JM$ for some ideal $J$ of $R$. Thus $JIM = 0$. As $(0 :_M I) \neq 0$ and $Ann_R(M)$ is a prime ideal of $R$, $IM = 0$. Hence $(0 :_M I) = M$. □

It is clear that every homomorphic image of a multiplication $R$-module is a multiplication $R$-module. But in the following example we see that this is not true for s-multiplication $R$-modules in general.

**Example 2.5.** By Remark 2.3, $\mathbb{Q}$ as a $\mathbb{Z}$-module is an s-multiplication $R$-module. But its homomorphic image $\mathbb{Q}/\mathbb{Z}$ is not an s-multiplication $R$-module because for the second submodule $\mathbb{Z}_{p^\infty}$ of $\mathbb{Q}/\mathbb{Z}$, we have $\mathbb{Z}_{p^\infty} \neq I(\mathbb{Q}/\mathbb{Z})$ for each ideal $I$ of $\mathbb{Z}$.

**Lemma 2.6.** Let $M$ be a second $R$-module. Then $M$ is an s-multiplication $R$-module if and only if $Spec^s(M) = \{M\}$.

*Proof.* Let $M$ be an s-multiplication $R$-module and $S$ be a second submodule of $M$. Then by assumption, $S = IM$ for some ideal $I$ of $R$. As $M$ is second, $IM = 0$ or $IM = M$. If $IM = 0$, then $S = 0$, a contradiction. Hence $S = IM = M$, as required. The converse is clear. □

**Example 2.7.** Let $p$ be a prime number and consider $M = \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module. Then $M$ is not an s-multiplication module since $G_1 = \langle 1/p + \mathbb{Z}\rangle$ is a second submodule of $M$, but there does not exist an ideal $I$ of $\mathbb{Z}$ such that $G_1 = IM$.

A proper submodule $N$ of an $R$-module $M$ is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of $M$, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of $M$ is an intersection of completely irreducible submodules of $M$. Thus the intersection of all completely irreducible submodules of $M$ is zero [12].

**Remark 2.8.** Let $N$ and $K$ be two submodules of an $R$-module $M$. To prove $N \subseteq K$, it is enough to show that if $L$ is a completely irreducible submodule of $M$ such that $K \subseteq L$, then $N \subseteq L$.

Recall that an $R$-module $M$ is said to be a *finitely cogenerated* $R$-module if for any family of submodules $\{N_i | i \in I\}$ of $M$, if $\cap_{i\in I} N_i = 0$, then $\cap_{i\in F} N_i = 0$ for a finite subset $F$ of $I$.

Let $P$ be a prime ideal of $R$ and $N$ be a submodule of an $R$-module $M$. *The $P$-interior of $N$ relative to $M$* is defined [5, 2.7] as the set

$$I^M_P(N) = \cap\{L \mid L \textit{ is a completely irreducible submodule of } M \textit{ and}$$

$$rN \subseteq L \textit{ for some } r \in R - P\}.$$

**Lemma 2.9.** (See [5]). Let $P$ be a prime ideal of $R$ and $N$ be a submodule of an $R$-module $M$. If $M/I^M_P(N)$ is a finitely cogenerated $R$-module, then there exists $r \in R - P$ such that $rN \subseteq I^M_P(N)$.

**Theorem 2.10.** *Let $R$ be an integral domain (not a field) and let $M$ be an s-multiplication $R$-module such that $M/I^M_0(M)$ is finitely cogenerated and $I^M_0(M) \neq 0$. Then $Spec^s(M) = \{I^M_0(M)\}$.*

*Proof.* By [5, 2.9], $I_0^M(M)$ is a second submodule of $M$ and from the proof of [5, 2.9] we infer $Ann_R((0:_M 0)) = Ann_R(M) = 0$. Thus the conditions of [5, 2.8] are satisfied in the case when $N := M$ and $P := 0$. By the proof of [5, 2.8], we obtain $Ann_R(I_0^M(M)) = 0$. Thus it is suffices to show that if $S$ is a second submodule of $M$, then $S = I_0^M(M)$. By hypothesis, $S = IM$ for some non-zero ideal $I$ of $R$. Let $L$ be a completely irreducible submodule of $M$ such that $S \subseteq L$. Then $IM \subseteq L$. Since $I \neq 0$, it follows that $I_0^M(M) \subseteq L$. Hence $I_0^M(M) \subseteq S$ by Remark 2.8. For the converse, let $L$ be a completely irreducible submodule of $M$ such that $I_0^M(M) \subseteq L$. Then by Lemma 2.9, there exists $0 \neq r \in R$ such that $rM \subseteq L$. Since $I_0^M(M) \subseteq S$, hence $Ann_R(S) \subseteq Ann_R(I_0^M(M)) = 0$. Thus we have $S = rS \subseteq rM \subseteq L$. Hence $S \subseteq I_0^M(M)$ by Remark 2.8. Therefore, $S = I_0^M(M)$, as required. □

**Proposition 2.11.** Let $I$ be an ideal of $R$ such that $I \subseteq Jac(R)$, where $Jac(R)$ denotes the Jacobson radical of $R$, and $M$ be an s-multiplication $R$-module which has a minimal submodule, Then $IM \neq M$.

*Proof.* Assume contrary that $IM = M$. Let $Rm$ for some $m \in M$ be a minimal submodule of $M$. Then $Rm$ is a second submodule of $M$. Since $M$ is an s-multiplication $R$-module, there exists an ideal $J$ of $R$ such that $Rm = JM$. Thus

$$Rm = JM = JIM = IJM = Im.$$

Hence $(1-a)m = 0$ for some $a \in I \subseteq Jac(R)$. This implies that $m = 0$, a contradiction. □

A family $\{N_i\}_{i\in I}$ of submodules of an $R$-module $M$ is said to be an *inverse family of submodules of* $M$ if the intersection of two of its submodules again contains a module in $\{N_i\}_{i\in I}$. Also, $M$ *satisfies the Grothendieck's condition* $AB5^*$ (*the property* $AB5^*$ in short) if for every submodule $K$ of $M$ and every inverse family $\{N_i\}_{i\in I}$ of submodules of $M$, $K + \cap_{i\in I} N_i = \cap_{i\in I}(K + N_i)$. Artinian and uniserial modules are examples of modules which satisfies the property $AB5^*$ [18, p.435].

Recall that an $R$-module $M$ is said to be *fully copure* if every submodule of $M$ is copure [4].

**Theorem 2.12.** *Let $M$ be an s-multiplication $R$-module. Then we have the following.*

*(a) If $M$ is a fully copure $R$-module which satisfies the property $AB5^*$, then $M$ is a multiplication $R$-module.*

*(b) If $M$ is a semisimple $R$-module, then $M$ is a multiplication $R$-module.*

*Proof.* (a) Let $N$ be a non-zero submodule of $M$. By [5, 3.6], $soc(N) = N$. Thus

$$N = soc(N) = \sum_{S\in Spec^s(N)} S = \sum_{S\in Spec^s(N)} (S:M)M = (\sum_{S\in Spec^s(N)} (S:M))M.$$

Set $J = \sum_{S \in Spec^s(N)} (S : M)$. Then $N = JM$, as required.

(b) Since $M$ is an s-multiplication $R$-module, $Ann_R(M) \neq Ann_R(M/S)$ for each second submodule $S$ of $M$. Thus the result follows from [2, 3.6]. $\square$

Let $M$ be an $R$-module. A prime ideal $P$ of $R$ is said to be *associated with* $M$ if there exists $x \in M$ such that $P$ is equal to the annihilator of $x$. The set of prime ideals associated with $M$ is denoted by $Ass_R(M)$, or simply $Ass(M)$ [10].

An $R$-module $M$ is said to be a *weak comultiplication module* if $M$ does not have any second submodule or for every second submodule $S$ of $M$ there exists an ideal $I$ of $R$ such that $S = (0 :_M I)$, equivalently $M$ is a weak comultiplication module if and only if $S = (0 :_M Ann_R(S))$ for every second submodule $S$ of $M$ or $Spec^s(M) = \emptyset$ [6].

**Theorem 2.13.** *Let $R$ be a Noetherian ring and $M$ be a weak comultiplication s-multiplication faithful $R$-module. Then $M$ has at most a finite number of second submodules.*

*Proof.* Let $S$ be a second submodule of $M$. Then there exists an ideal $I$ of $R$ such that $S = IM$. Thus as $M$ is faithful,

$$Ann_R(S) = Ann_R(IM) = Ann_R(I) = Ann_R(\sum_{a \in I} Ra) = \cap_{a \in I} Ann_R(Ra).$$

Since $S$ is second, $Ann_R(S)$ is a prime ideal of $R$. Therefore, $Ann_R(S) = Ann_R(Ra)$ for some $a \in I$ because $I$ is a finitely generated ideal of $R$ and so $\{Ra\}_{a \in I}$ is a finite set. Hence $Ann_R(S) \in Ass_R(R)$. As $R$ is Noetherian, $Ass_R(R)$ is a finite set [14, 7.G]. Since $M$ is a weak comultiplication module, there is a bijective correspondence between the set of second submodules of $M$ and the set of their annihilators. So the number of second submodules of $M$ is finite. $\square$

Let $M$ be an $R$-module and $c$ the function from $M$ to the set of ideals of $R$ defined by

$$c(x) = \cap\{I : I \textit{ is an ideal of } R \textit{ and } x \in IM\}.$$

$M$ is said to be a *content $R$-module* if $x \in c(x)M$, for all $x \in M$ [15].

**Lemma 2.14.** (See [15]). Let $M$ be an $R$-module. The following statements are equivalent:

(1) $M$ is a content $R$-module.

(2) For any non-empty family of ideals $\{I_i\}_{i \in I}$ of $R$, $\cap_{i \in I} I_i M = (\cap_{i \in I} I_i)M$.

Recall that a topological space $X$ is *Noetherian* provided that the open (respectively, closed) subsets of $X$ satisfy the ascending (respectively, descending) chain condition, or the maximal (respectively, minimal) condition [8] and [10].

Let $N$ be a submodule of an $R$-module $M$, $V^s(N) = \{S \in Spec^s(M) : Ann_R(N) \subseteq Ann_R(S)\}$ and Set $\zeta^s(M) = \{V^s(N) : N \leq M\}$. Then there exists a topology, $\tau^s$ say, on $Spec^s(M)$ having $\zeta^s(M)$ as the family of all closed sets. We call the topology $\tau^s$ the *Zariski topology on* $Spec^s(M)$[7].

The *Zariski socle* of a submodule $N$ of an $R$-module $M$, denoted by $Z.soc(N)$ is the sum of all members of $V^s(N)$, that is,

$$Z.soc(N) = \sum\{S : S \in V^s(N)\} = \sum\{S : Ann_R(N) \subseteq Ann_R(S), S \in Spec^s(M)\}.$$

If $V^s(N) = \emptyset$, then $Z.soc(N) = 0$. We say that a submodule $N$ of $M$ is a *Zariski socle submodule* if $N = Z.soc(N)$ [11].

**Theorem 2.15.** *Let $R$ be an Artinian ring and $M$ be a content s-multiplication $R$-module. Then we have the following.*

*(a) $M$ satisfies dcc on socle submodules.*

*(b) $M$ has only a finite number of maximal second submodules.*

*(c) $\tau^s$ is a Noetherian space.*

*Proof.* (a) By the proof of Theorem 2.12 (a), for each socle submodule $N$ of $M$, $N = soc(N) = IM$ for some ideal $I$ of $R$. Now let

$$I_1M \supseteq I_2M \supseteq I_3M \supseteq ....$$

be a descending chain of socle submodules of $M$. Then since $M$ is a content module, $\cap_{i=1}^{\infty} I_iM = (\cap_{i=1}^{\infty} I_i)M$. As $R$ is an Artinian ring, there exists a positive integer $n$ such that $\cap_{i=1}^{\infty} I_iM = (\cap_{i=1}^{n} I_i)M = I_nM$, as required.

(b) Since by part (a), $M$ satisfies dcc on socle submodules, this follows from [5, 2.5].

(c) This follows from [11, 4.1], part (a), and the fact that every Zariski socle submodule of $M$ is a socle submodule of $M$. □

Recall that a commutative ring $R$ satisfies the *double annihilator property* if for each ideal $I$ of $R$ we have $I = Ann_R Ann_R(I)$ [1].

**Proposition 2.16.** Suppose $R$ satisfies the double annihilator property and $M$ is a faithful s-multiplication $R$-module. Then the natural map $\psi^s : Spec^s(M) \to Spec(R/Ann_R(M))$ defined by $S \mapsto Ann_R(S)/Ann_R(M)$ is injective.

*Proof.* Let $S_1$ and $S_2$ be two second submodules of $M$ such that $\psi^s(S_1) = \psi^s(S_2)$. Then by assumption, there exist ideals $I$ and $J$ of $R$ such that $S_1 = IM$ and $S_2 = JM$. Now since $M$ is faithful,

$$Ann_R(I) = Ann_R(IM) = Ann_R(S_1) = Ann_R(S_2) = Ann_R(JM) = Ann_R(J).$$

Thus as $R$ satisfies the double annihilator property, $I = J$, as required. □

**Lemma 2.17.** Let $M$ be an $R$-module. Then $M$ is a finitely generated $R$-module if and only if $M$ is a finitely generated $R/Ann_R(M)$-module.

*Proof.* This is clear. □

**Remark 2.18.** In Theorem 2.6 of [13], the authors showed that every faithful multiplication module $M$ is finitely generated. This implies that every multiplication $R$-module is finitely generated by Lemma 2.17. But the following example shows that this is not true in general (this mistake in the proof of Theorem 2.6 occurs because the authors accepted that the product of $Pq$, where $P$ and $q$ are maximal ideals of $R$, is a prime ideal without any justification. Clearly, the product of two maximal ideals need not be a prime ideal in general. Hence they can not use the Theorem 2.3 to complete the proof).

**Example 2.19.** (See [17, 4.30]) Let $K$ be a field, $R$ be the polynomial ring $K[x_1, x_2, x_3, ...]$ in a countably infinite set of indeterminates $x_1, x_2, x_3, ..$, $A = x_1R + x_2R + x_3R + ...$, $B = (x_1 - x_1^2)R + (x_2 - x_2^2)R + (x_3 - x_3^2)R + ...$, and $M = A/B$. Then $M$ is a multiplication $R$-module. But $M$ is not finitely generated.

The following lemma is known, but we write it here for the sake of references.

**Lemma 2.20.** Let $M$ be a finitely generated second $R$-module. Then $M$ is a semisimple $R$-module.

*Proof.* Since $M$ is second, $M \neq 0$. Since $M$ is finitely generated, $M$ has a maximal submodule, $U$ say. Thus $M/U$ is a simple $R$-module and hence $P := Ann_R(M/U)$ is a maximal ideal of $R$. Clearly, $Ann_R(M) \subseteq P$. Now let $r \in P$. Then $r(M + U)/U = 0$. As $M$ is second, $rM = M$ or $rM = 0$. If $rM = M$, then $M = U$ which is a contradiction. Thus $r \in Ann_R(M)$. Hence $Ann_R(M) = P$. Thus by [16, 3.7], $M$ is a semisimple $R/P$-module. Hence $M$ is a semisimple $R$-module. $\square$

**Theorem 2.21.** *Let $M$ be a finitely generated second s-multiplication $R$-module. Then $M$ is a simple $R$-module.*

*Proof.* By Lemma 2.20, $M$ is a semisimple $R$-module. Thus by Theorem 2.12 (b), $M$ is a multiplication $R$-module. Now let $N$ be a submodule of $M$. Then $N = IM$ for some ideal $I$ of $R$. As $M$ is second, $IM = 0$ or $IM = M$, as desired. $\square$

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