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A NOTE ON COMPUTING BAYESIAN TOLERANCE INTERVALS IN EXPONENTIAL DISTRIBUTION BASED ON k-RECORD VALUES

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Abstract. The problem of finding tolerance intervals receives much attention in research and is widely applied in industry. A tolerance interval is a random interval that covers a proportion of the considered products with a specified confidence level. In this paper, we obtain equi-tailed two-sided Bayesian tolerance intervals based on k-record values. We compare the lengths of the proposed Bayesian tolerance intervals for different values of the parameters of the prior distribution in an illustrative example. Finally, some concluding remarks are given.

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Key words and phrases: exponential distribution; k-record data; Bayesian tolerance interval

1. Introduction

Tolerance intervals (TIs) have applications to many applied scientific fields especially quality control when the quality of products must be checked. In a production process, a producer may be interested in finding an interval that contains a specified (usually large) proportion of products with a determined confidence level. The producer knows that a specified proportion of the products must be accepted in the sense that the quality characteristics of the products must conform to the lower and upper specification limits, otherwise a great loss may be made. In fact, if the obtained tolerance limits (TLs) are inside the specification limits, then it can be concluded, with a determined confidence level, that at least a specified proportion of the products conforms to the considered criteria. We note that confidence intervals (CIs) possess information about the unknown parameters of a population while TIs provide information about the population units.

Wilks [20] was among the first papers on TIs. Wald and Wolfowitz [18], Weissberg and Beatty [19], Howe [9] and Krishnamoorthy et al. [11] worked on TIs for the normal distribution. Hall [8] and Tang and Chang [17] focused on finding TIs for logistic and inverse Gaussian distributions, respectively. Krishnamoorthy and Mondal [12] presented a method that improves tolerance

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regions for the multivariate normal distributions. Mbodj and Mathew [13] discussed approximate elliptical tolerance regions for multivariate normal populations. Goodman and Madansky [7] and Engelhardt and Bain [5] worked on TIs and TLs for the exponential distribution. Fernández [6] extended the work of [7] using Bayesian approaches. Jiong and Xiangzhong [4] obtained two-sided TIs based on complete samples for the exponential distribution. MirMostafaee et al. [14] discussed TLs for minimal repairs of a series system with Rayleigh distributed components.

When we deal with sequential experiments, an observation which is larger (or smaller) than its previous observations may be of interest. Such an observation is called an upper (lower) record value. Record data are of importance since in many sequential experiments, only the record values are recorded and the complete sample is not available. Such experiments may include athletic events, geophysics surveys and reliability and quality control experiments. Suppose that $\{X_i : i \geq 1\}$ constitutes a sequence of random variables, then X_j is an upper record if its value is larger than all its previous observations, in other words, if $X_j > \max\{X_1, \cdots X_{j-1}\}, j \geq 2$. The first observation of the sequence is also the first (trivial) record according to the definition of records. Lower records are defined similarly.

However, as [1] pointed out, the record data are rare in practical situations and a sample of size n may yield only $\log n$ records. This problem can be fixed by considering k-records instead, see for example [1]. Upper k-record values are defined as an extension of upper record values and they constitute the k-th largest values yet seen in a sequence of observations.

Let us denote the m-th upper k-record time by T_m^k, then $T_1^k = k$ and for $m \geq 2$

$$T_m^k = \min\{j : j > T_{m-1}^k, X_j > X_{T_{m-1}^k - k+1:T_{m-1}^k}\},$$

where $X_{i:m}$ stands for the i-th order statistic from a sample of size m. Then the m-th upper k-record, denoted by $R_{m(k)}$, is defined as $R_{m(k)} = T_m^k - k + 1 : T_m^k$ for $m = 1, 2\,3, \cdots$. For $k = 1$, the ordinary records are recovered. The lower k-records can be defined similarly.

Let $R_{1(k)}, R_{2(k)}, \cdots, R_{m(k)}$ be a sequence of upper k-records from an arbitrary continuous distribution with probability density function (pdf) f and cumulative distribution function (cdf) F. Then the joint pdf of $R_{1(k)}$, $R_{2(k)}$, $\ldots$, $R_{m(k)}$ is

$$(1.1) \qquad f_{R_{1(k)},R_{2(k)},\cdots,R_{m(k)}}(x_1, \cdots, x_m) = k^m[1 - F(x_m)]^k \prod_{i=1}^{m} \frac{f(x_i)}{1 - F(x_i)},$$

for $x_1 < x_2 < \cdots < x_m$.

The marginal pdf of the m-th upper k-record value, $R_{m(k)}$, is given by

$$(1.2) \qquad f_{R_{m(k)}}(x) = \frac{k^m}{(m-1)!}[1 - F(x_m)]^{k-1}[-\log(1 - F(x))]^{m-1} f(x).$$

Interested readers may consult [2] for more details regarding k-records and their applications. From now on, we will use the word "k-records" instead of "upper k-records" for the sake of convenience.

We say that a random variable X has an exponential distribution with parameter θ if its pdf can be expressed as

$$f(x|\theta) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right), \qquad x > 0,\ \theta > 0. \tag{1.3}$$

Recently, Naghizadeh and Kiapour [15] discussed the problem of finding shortest TIs for the exponential distribution based on record data using an approach suggested by [16]. Kiapour and Naghizadeh [10] extended the work of [15]using Bayesian methods. In this paper, we wish to apply another approach to finding Bayesian TIs. This approach is due to [7] and will be detailed later in Section 2. The rest of the paper is arranged as follows. The main results are given in Section 2. A real data example is presented in Section 3 to illustrate the results developed in Section 2. Finally, the paper ends with some remarks.

2. Main results

Let $\{X_i : i \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables coming from the exponential distribution with parameter θ and the pdf given in (1.3). Moreover, suppose that $\boldsymbol{R} = (R_{1(k)},\ R_{2(k)}, \ldots, R_{m(k)})$ is the set of the first m k-records extracted from the sequence $\{X_i : i \geq 1\}$ and $\boldsymbol{r} = (r_1, r_2, \cdots, r_m)$ is the observed set of $\boldsymbol{R}$. Then from (1.1), the likelihood function for θ, given $\boldsymbol{R} = \boldsymbol{r}$, is given by

$$L(\theta|\boldsymbol{r}) = \left(\frac{k}{\theta}\right)^m \exp\left(-\frac{k\, r_m}{\theta}\right), \qquad 0 < r_1 < r_2 < \cdots < r_m,\ \theta > 0.$$

Upon differentiating the likelihood function and equating the result with zero, the maximum likelihood estimator (MLE) of θ is obtained as $\widehat{\theta} = \dfrac{k\, R_{m(k)}}{m}$. Note that from (1.2), the pdf of $R_{m(k)}$ is given by

$$f_{R_{m(k)}}(r_m) = \frac{1}{\Gamma(m)\, (\theta/k)^m} r_m^{m-1}\, \mathrm{e}^{-\frac{r_m}{\theta/k}}, \qquad r_m > 0.$$

Thus $R_{m(k)} \sim Gamma(m, \theta/k)$ and consequently the quantity $2m\frac{\widehat{\theta}}{\theta} = \frac{2}{\theta/k} R_{m(k)}$ possesses a chi square distribution with $2m$ degrees of freedom.

Now, let us take the conjugate prior inverse gamma distribution, $IGamma(a, b)$, with pdf

$$\pi(\theta) = \frac{b^a\ \exp(-b/\theta)}{\Gamma(a)\theta^{a+1}}, \qquad a, b, \theta > 0, \tag{2.1}$$

as the prior distribution for θ. The hyperparameters a and b are positive numbers that can be determined from prior knowledge about θ. For example, one can derive the hyperparameters from the prior mode, prior mean (provided that $b > 1$) and/or prior standard deviation (provided that $b > 2$) of θ, which are $M(\theta) = \frac{a}{b+1}$, $E(\theta) = \frac{a}{b-1}$ and $D(\theta) = \frac{E(\theta)}{\sqrt{b-1}}$, respectively.

The posterior distribution of θ given the k-record values, $\boldsymbol{r}$, becomes

$$\pi(\theta|\boldsymbol{r}) = \frac{(m\widehat{\theta} + b)^{a+m} \exp(-\frac{b+m\widehat{\theta}}{\theta})}{\Gamma(a+m)\theta^{a+m+1}}, \qquad \theta > 0. \tag{2.2}$$

In other words, $\theta|\boldsymbol{R} = \boldsymbol{r}$ has an inverse gamma distribution with parameters $a + m$ and $b + m\widehat{\theta}$. The posterior mode is then equal to $\widehat{\theta}_{\text{mod}} = \frac{b+m\widehat{\theta}}{a+m+1} = \frac{kR_{m(k)}+b}{a+m+1}$ which can be considered as the generalized MLE of θ.

The random interval $(L, U) \equiv (L(\boldsymbol{R}), U(\boldsymbol{R}))$ is a two-sided β-content Bayesian TI with confidence level $1 - \alpha$ if

$$Pr[F(U) - F(L) \geq \beta|\boldsymbol{R}] = 1 - \alpha, \tag{2.3}$$

where F is the considered cdf (here F is the cdf of exponential distribution) and $\alpha, \beta \in (0, 1)$. In fact, (L, U) which satisfies (2.3), contains at least a proportion β of the population with confidence $1 - \alpha$. We consider equi-tailed two-sided Bayesian TIs of the form $(L(\boldsymbol{R}), U(\boldsymbol{R})) = (c_1\widehat{\theta}_{\text{mod}}, c_2\widehat{\theta}_{\text{mod}})$ where $\widehat{\theta}_{\text{mod}}$ is the mode of the posterior distribution and $c_1, c_2 \geq 0$ are called tolerance factors. For finding the factors c_1 and c_2, we take the limit as $a, b \to 0$ in (2.1) which is equivalent to the Jeffrey's prior with the following density

$$\pi(\theta) \propto \sqrt{I(\theta)} \propto \frac{1}{\theta}, \qquad \theta > 0,$$

where $I(\theta)$ is the Fisher information. Then, by imposing the equal-tailedness condition (see [7]), i.e. $Pr_{\widehat{\theta}}(X < L) = Pr_{\widehat{\theta}}(X > U)$, we arrive at the following expression

$$\int_0^{c_1\widehat{\theta}_{\text{mod}}} \frac{1}{\widehat{\theta}} \exp\left(-\frac{x}{\widehat{\theta}}\right) \mathrm{d}x = \int_{c_2\widehat{\theta}_{\text{mod}}}^{\infty} \frac{1}{\widehat{\theta}} \exp\left(-\frac{x}{\widehat{\theta}}\right) \mathrm{d}x,$$

which is equivalent to

$$1 - \exp\left(-c_1\frac{\widehat{\theta}_{\text{mod}}}{\widehat{\theta}}\right) = \exp\left(-c_2\frac{\widehat{\theta}_{\text{mod}}}{\widehat{\theta}}\right). \tag{2.4}$$

From (2.4) and by noting the fact that $\widehat{\theta}_{\text{mod}} = \dfrac{m\widehat{\theta}}{m+1}$ when $a = b = 0$, we obtain

$$c_2 = -\frac{m+1}{m} \ln\left[1 - \exp\left(-\frac{m}{m+1}c_1\right)\right]. \tag{2.5}$$

It is clear that c_2 is a decreasing function of c_1. From (2.3), we have

$$Pr\left[F(c_2\widehat{\theta}_{\text{mod}}) - F(c_1\widehat{\theta}_{\text{mod}}) \geq \beta\Big|\boldsymbol{R}\right] = 1 - \alpha,$$

where F is the cdf of the exponential distribution. Therefore, we have

$$Pr\left[\exp\left(-c_1\frac{\widehat{\theta}_{\text{mod}}}{\theta}\right)-\exp\left(-c_2\frac{\widehat{\theta}_{\text{mod}}}{\theta}\right)\geq\beta\middle|\boldsymbol{R}\right]=1-\alpha.$$

By considering $Z=\dfrac{\widehat{\theta}_{\text{mod}}}{\theta}$ and from (2.5), we get

$$Pr\left[\exp(-c_1 Z)-\left\{1-\exp\left(-\frac{mc_1}{m+1}\right)\right\}^{-\frac{(m+1)Z}{m}}\geq\beta\middle|\boldsymbol{R}\right]=1-\alpha. \tag{2.6}$$

Let

$$g(c_1,z)=\exp(-c_1 z)-\left\{1-\exp\left(-\frac{mc_1}{m+1}\right)\right\}^{-\frac{(m+1)z}{m}},$$

then $g(c_1,z)$ is a unimodal positive function with respect to z and attains its unique maximum value when $z=z_0$ where

$$z_0=\frac{\ln c_1-\ln\left[-\ln\left(1-\exp\left(-\frac{mc_1}{m+1}\right)\right)\right]}{c_1+\frac{m}{m+1}\ln\left(1-\exp\left(-\frac{mc_1}{m+1}\right)\right)}$$

Therefore from (2.6), we have

$$Pr\left[z_1<Z<z_2|\boldsymbol{R}\right]=1-\alpha, \tag{2.7}$$

where

$$g(c_1,z_i)=\beta,\qquad i=1,2. \tag{2.8}$$

Note that

$$\begin{aligned}2(a+m+1)Z &= 2(a+m+1)\frac{\widehat{\theta}_{\text{mod}}}{\theta}\\ &= 2(a+m+1)\frac{(m\widehat{\theta}+b)/(a+m+1)}{\theta}\\ &= \frac{2(m\widehat{\theta}+b)}{\theta}.\end{aligned}$$

As $\theta|\boldsymbol{R}\sim IGamma(a+m,m\widehat{\theta}+b)$, we simply find out that $2(a+m+1)Z|\boldsymbol{R}\sim\chi^2(2(a+m))$ where $\chi^2(v)$ stands for the chi square distribution with v degrees of freedom. Thus equation (2.7) becomes equivalent to the following equation

$$\int_{z_1}^{z_2}\frac{(a+m+1)^{a+m}}{\Gamma(a+m)}\,z^{a+m-1}\exp\{-(a+m+1)z\}\mathrm{d}z=1-\alpha. \tag{2.9}$$

The tolerance factor c_1 is obtained by solving the non-linear equations (2.8) and (2.9) through choosing suitable initial values for c_1, z_1 and z_2. The tolerance factor c_2 is then obtained from (2.4).

Table 1 contains the Bayesian tolerance factors c_1 and c_2 for selected values of m,β,a and $1-\alpha=0.95$. From Table 1, we find out that

- For a fixed value of m, as a increases, c_1 increases and c_2 decreases.
- For $a = 0$, as m increases, c_1 increases and c_2 decreases, whereas for $a \neq 0$, as m increases, both c_1 and c_2 decrease.

Table 1: The Bayesian tolerance factors c_1 and c_2 for $1 - \alpha = 0.95$ and selected values of m, β, a.

		$\beta = 0.7$		$\beta = 0.8$		$\beta = 0.9$		$\beta = 0.95$	
m	a	c_1	c_2	c_1	c_2	c_1	c_2	c_1	c_2
2	0	0.0017	10.16	1.7×10^{-4}	13.58	3.5×10^{-6}	19.43	7.1×10^{-8}	25.29
	10	0.203	3.09	0.12	3.79	0.05	4.98	0.02	6.19
	20	0.23	2.90	0.14	3.53	0.07	4.60	0.03	5.67
	50	0.25	2.79	0.16	3.39	0.08	4.41	0.04	5.43
3	0	0.015	5.94	0.0035	7.89	2.8×10^{-4}	11.26	2.2×10^{-5}	14.65
	10	0.16	2.83	0.1003	3.49	0.04	4.65	0.016	5.82
	20	0.19	2.63	0.12	3.22	0.05	4.22	0.026	5.22
	50	0.21	2.51	0.14	3.04	0.07	3.96	0.034	4.87
4	0	0.034	4.52	0.0108	3.34	0.0014	8.44	1.9×10^{-4}	10.96
	10	0.15	2.70	0.088	3.08	0.035	4.47	0.014	5.61
	20	0.17	2.52	0.11	2.90	0.049	4.06	0.022	5.04
	50	0.19	2.39	0.12	4.99	0.06	3.79	0.03	4.67
5	0	0.05	3.83	0.018	4.99	0.003	7.04	0.0005	9.13
	10	0.14	2.61	0.083	3.24	0.032	4.34	0.0127	5.45
	20	0.16	2.44	0.102	3.006	0.044	3.96	0.019	4.93
	50	0.18	2.32	0.119	2.82	0.056	3.69	0.027	4.56
6	0	0.06	3.44	0.02	4.44	0.005	6.22	0.001	8.04
	10	0.13	2.55	0.07	3.16	0.031	4.24	0.0121	5.33
	20	0.15	2.4	0.09	2.94	0.042	3.89	0.018	4.85
	50	0.17	2.27	0.113	2.77	0.053	3.63	0.025	4.48

3. A real data example

Consider the following data concerning the times (in minutes) between 24 consecutive telephone calls to a company's switchboard:

```
1.34 0.14 0.33 1.68 1.86 1.31 0.83 0.33  2.20 0.62 3.20 1.38
0.96 0.28 0.44 0.59 0.25 0.51 1.61 1.85 0.47 0.41 1.46  0.09.
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These data are presented by [3]. The Kolmogorov-Smirnov (K-S) test is used for checking the adequacy of the exponential distribution with the mean $\theta = 1.0059$ for the above data. The K-S test statistic is obtained to be $D = 0.1489$ with a corresponding significant p-value 0.6618. This implies that the exponential distribution fits the data well. The observed k-records extracted from the data are presented in Table 2. The Bayesian tolerance intervals for the case $1-\alpha = 0.95$, $\beta \in \{0.9, 0.95\}$ and selected values of a, b, k and m are computed and summarized in Tables 3 and 4. For example, the $(0.95, 0.90)$-Bayesian tolerance interval based on the observed 2-record values and by considering the Jeffrey's prior, is obtained to be $(0.0005, 0.65)$. Then, we can state with 95% confidence that at least 90% of the times between consecutive telephone calls to this company's switchboard will be between 0.0005 and 0.65.

From Tables 3 and 4, we observe that the lengths of Bayesian TIs for a fixed value of m and $a = b = 0$ are larger than the other cases. Also, for a fixed value of fixed m, the lengths of the TIs decrease as a increases.

Table 2: The k-record values extracted from the real data of the example.

	m					
k	1	2	3	4	5	6
1	1.34	1.68	1.86	2.20	3.20	
2	0.14	0.33	1.34	1.68	1.86	2.2
3	0.14	0.33	1.34	1.68	1.86	

Table 3: Bayesian tolerance intervals for $1-\alpha=0.95$, $\beta=0.9$ and selected values of $a,\ b,\ k$ and m.

		m				
k	(a,b)	2	3	4	5	6
1	(0,0)	$(1.9\times10^{-6}, 10.88)$	$(1.2\times10^{-4}, 5.13)$	(0.0006,3.71)	(0.001,3.73)	
	(10,1)	(0.01,1.03)	(0.008,0.95)	(0.007,0.95)	(0.008,1.14)	
	(20,1)	(0.008,0.53)	(0.006,0.5)	(0.006,0.52)	(0.007,0.64)	
	(50,1)	(0.004,0.22)	(0.003,0.21)	(0.004,0.22)	(0.004,0.28)	
2	(0,0)	$(7\times10^{-6}, 4.27)$	(0.0001,7.52)	(0.0009,5.67)	(0.002,4.34)	(0.0005,0.65)
	(10,1)	(0.06,0.64)	(0.01,1.22)	(0.01,1.3)	(0.009,1.28)	(0.01,1.34)
	(20,1)	(0.005,0.33)	(0.008,0.64)	(0.008,0.71)	(0.008,0.72)	(0.008,0.77)
	(50,1)	(0.003,0.14)	(0.005,0.27)	(0.005,0.3)	(0.004,0.31)	(0.005,0.34)
3	(0,0)	$(10^{-5}, 6.41)$	(0.0003,11.32)	(0.001,8.5)	(0.005,10.91)	
	(10,1)	(0.008,0.76)	(0.01,1.67)	(0.01,1.8)	(0.02,2.79)	
	(20,1)	(0.006,0.4)	(0.01,0.88)	(0.01,0.98)	(0.02,1.79)	
	(50,1)	(0.003,0.16)	(0.006,0.37)	(0.007,0.42)	(0.009,0.68)	

Table 4: Bayesian tolerance intervals for $1-\alpha=0.95$, $\beta=0.95$ and selected values of $a,\ b,\ k$ and m.

		m				
k	(a,b)	2	3	4	5	6
1	(0,0)	$(4\times10^{-8}, 14.62)$	$(10^{-5}, 6.68)$	(0.0008,4.82)	(0.002,4.84)	
	(10,1)	(0.004,1.27)	(0.003,1.19)	(0.003,1.2)	(0.003,1.43)	
	(20,1)	(0.003,0.66)	(0.003,0.62)	(0.002,0.64)	(0.003,0.8)	
	(50,1)	(0.002,0.27)	(0.002,0.26)	(0.001,0.27)	(0.002,0.34)	
2	(0,0)	$(10^{-7}, 5.56)$	(0.0004,9.79)	(0.0001,7.36)	(0.003,5.63)	(0.0001,0.84)
	(10,1)	(0.003,0.79)	(0.004,1.53)	(0.004,1.63)	(0.003,1.6)	(0.004,1.69)
	(20,1)	(0.002,0.41)	(0.004,1.8)	(0.004,0.88)	(0.003,0.89)	(0.004,0.97)
	(50,1)	(0.001,0.17)	(0.002,0.33)	(0.003,0.37)	(0.002,0.38)	(0.002,0.42)
3	(0,0)	$(2\times10^{-8}, 8.34)$	$(2\times10^{-5}, 14.72)$	(0.0001,11.05)	(0.008,14.51)	
	(10,1)	(0.003,0.95)	(0.006,2.09)	(0.005,2.26)	(0.008,3.51)	
	(20,1)	(0.003,0.49)	(0.005,1.09)	(0.005,1.22)	(0.007,1.95)	
	(50,1)	(0.002,0.2)	(0.003,0.45)	(0.003,0.51)	(0.005,0.83)	

4. Discussion

In the present paper, we compute two-sided equal-tailed Bayesian tolerance intervals for exponential distribution based on k-record values. Tolerance factors are computed by solving a system of three nonlinear equations using the Newton's method via Mathematica version 7. The lengths of the Bayesian tolerance intervals are compared for different values of the parameter of the prior distribution in a real data example. The results show that the Bayesian tolerance intervals with a large value of a have small lengths when m is kept fixed. Also, the Bayesian tolerance intervals constructed based on Jeffrey's priors have larger lengths than those based on informative gamma priors.

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