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LYAPUNOV-TYPE INEQUALITIES FOR RIEMANN-LIOUVILLE TYPE FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. In this article, we establish Lyapunov-type inequalities for two-point Riemann-Liouville type fractional boundary value problems. To illustrate the applicability of established results, we estimate lower bounds for eigenvalues of the corresponding eigenvalue problems and deduce criteria for the nonexistence of real zeros of certain Mittag-Leffler functions.

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1. Introduction

In 1907, Lyapunov [10] proved a necessary condition for the existence of a nontrivial solution of Hill's equation associated with Dirichlet boundary conditions.

Theorem 1.1. *[10] If the boundary value problem*

$$\begin{cases} y''(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = 0, \quad y(b) = 0, \end{cases} \tag{1.1}$$

has a nontrivial solution, where $q : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b \big|q(s)\big| ds > \frac{4}{(b-a)}. \tag{1.2}$$

The Lyapunov inequality (1.2) has several applications in various problems related to differential equations. Due to its importance, the Lyapunov inequality has been generalized in many forms. For more details on Lyapunov-type inequalities and their applications, we refer [2, 12, 13, 15, 17, 18, 19] and the references therein.

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On the other hand, many researchers have derived Lyapunov-type inequalities for various classes of fractional boundary value problems in the recent years. For the first time, in 2013, Ferreira [4] generalized Theorem 1.1 to the case where the classical second-order derivative in (1.1) is replaced by an α^{th}-order ($1 < \alpha \leq 2$) Riemann-Liouville type derivative.

Theorem 1.2. *[4] If the fractional boundary value problem*

$$\begin{cases} D_a^\alpha y(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = 0, \quad y(b) = 0, \end{cases}$$

has a nontrivial solution, where $q : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)| ds > \Gamma(\alpha)\Big(\frac{4}{b-a}\Big)^{\alpha-1}.$$

Here D_a^α denotes the Riemann-Liouville type α^{th}-order differential operator. In 2014, Ferreira [5] replaced the Riemann-Liouville type derivative in Theorem 1.2 with the Caputo type derivative ${}^C D_a^\alpha$ and obtained the following Lyapunov-type inequality for the resulting problem:

Theorem 1.3. *[5] If the fractional boundary value problem*

$$\begin{cases} {}^C D_a^\alpha y(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = 0, \quad y(b) = 0, \end{cases}$$

has a nontrivial solution, where $q : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |q(s)| ds > \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}.$$

Jleli et al. [6, 7, 8] and Wang et al. [16] obtained Lyapunov-type inequalities for two-point Caputo type fractional boundary value problems associated with Robin, mixed, Sturm-Liouville and general boundary conditions, respectively. Recently, Ntouyas et al. [11] presented a survey of results on Lyapunov-type inequalities for fractional differential equations associated with a variety of boundary conditions. This article shows a gap in the literature on Lyapunov-type inequalities for two-point Riemann-Liouville type fractional boundary value problems associated with mixed, Sturm-Liouville and Robin boundary conditions.

In 2016, Dhar et al. [3] derived Lyapunov-type inequalities for two-point Riemann-Liouville type fractional boundary value problems associated with fractional integral boundary conditions. This article stresses the importance of choosing well-posed boundary conditions for Riemann-Liouville type fractional boundary value problems.

Motivated by these developments, in this article, we establish Lyapunov-type inequalities for two-point Riemann-Liouville type fractional boundary value problems associated with well-posed mixed, Sturm-Liouville, Robin and general boundary conditions.

2. Preliminaries

Throughout, we shall use the following notations, definitions and known results of fractional calculus [9, 14]. Denote the set of all real numbers and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively.

Definition 2.1. [9] Let $\alpha > 0$ and $a \in \mathbb{R}$. The α^{th}-order Riemann-Liouville type fractional integral of a function $y : [a, b] \to \mathbb{R}$ is defined by

$$I_a^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds, \quad a \le t \le b, \tag{2.1}$$

provided the right-hand side exists. For $\alpha = 0$, define I_a^α to be the identity map. Moreover, let n denote a positive integer and assume $n-1 < \alpha \le n$. The α^{th}-order Riemann-Liouville type fractional derivative is defined as

$$D_a^\alpha y(t) = D^n I_a^{n-\alpha} y(t), \quad a \le t \le b, \tag{2.2}$$

where D^n denotes the classical n^{th}-order derivative, if the right-hand side exists.

Definition 2.2. [9] We denote by $L(a, b)$ the space of Lebesgue measurable functions $y : [a, b] \to \mathbb{R}$ for which

$$\|y\|_L = \int_a^b |y(t)| dt < \infty.$$

Definition 2.3. [9] We denote by $C[a, b]$ the space of continuous functions $y : [a, b] \to \mathbb{R}$ with the norm

$$\|y\|_C = \max_{t \in [a,b]} |y(t)|.$$

Definition 2.4. [9] Let $0 \le \gamma < 1$, $y : (a, b] \to \mathbb{R}$ and define $y_\gamma(t) = t^\gamma y(t)$, $t \in [a, b]$. We denote by $C_\gamma[a, b]$ the weighted space of functions y such that $y_\gamma \in C[a, b]$, and

$$\|y\|_{C_\gamma} = \max_{t \in [a,b]} |(t-a)^\gamma y(t)|.$$

Lemma 2.1. *[9] If $\alpha \ge 0$ and $\beta > 0$, then*

$$I_a^\alpha (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1},$$
$$D_a^\alpha (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}.$$

Lemma 2.2. *[9] Let $\alpha > \beta > 0$ and $y \in C[a, b]$. Then,*

$$D_a^\beta I_a^\alpha y(t) = I_a^{\alpha-\beta} y(t), \quad t \in [a, b].$$

Lemma 2.3. *[1] Let $\alpha > 0$ and n be a positive integer such that $n-1 < \alpha \le n$. Then, the fractional differential equation*

$$D_a^{\alpha} y(t) = 0, \quad a < t < b,$$

has a unique solution $y \in C(a,b) \cap L(a,b)$, and is given by

$$y(t) = C_1(t-a)^{\alpha-1} + C_2(t-a)^{\alpha-2} + \cdots + C_n(t-a)^{\alpha-n},$$

where $C_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$.

Lemma 2.4. *[1] Let $\alpha > 0$ and n be a positive integer such that $n-1 < \alpha \le n$. If $y \in C(a,b) \cap L(a,b)$, then*

$$I_a^{\alpha} D_a^{\alpha} y(t) = y(t) + C_1(t-a)^{\alpha-1} + C_2(t-a)^{\alpha-2} + \cdots + C_n(t-a)^{\alpha-n},$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$.

3. Main Results

In this section, we obtain Lyapunov-type inequalities for two-point Riemann-Liouville type fractional boundary value problems associated with well-posed mixed, Sturm-Liouville, Robin and general boundary conditions, using the properties of the corresponding Green's functions.

Theorem 3.1. *Let $1 < \alpha \le 2$ and $h : [a,b] \to \mathbb{R}$. The fractional boundary value problem*

$$\begin{cases} D_a^{\alpha} y(t) + h(t) = 0, \quad a < t < b, \\ l I_a^{2-\alpha} y(a) - m D_a^{\alpha-1} y(a) = 0, \; n y(b) + p D_a^{\alpha-1} y(b) = 0, \end{cases} \tag{3.1}$$

has the unique solution

$$y(t) = \int_a^b G(t,s) h(s) ds, \tag{3.2}$$

where $G(t,s)$ is given by

$$G(t,s) = \begin{cases} G_1(t,s), & a < s \le t \le b, \\ G_2(t,s), & a < t \le s \le b, \end{cases} \tag{3.3}$$

$$G_1(t,s) = G_2(t,s) - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, \tag{3.4}$$

and

$$G_2(t,s) = \Big[\frac{l(t-a)^{\alpha-1} + m(\alpha-1)(t-a)^{\alpha-2}}{A}\Big]\Big[\frac{n(b-s)^{\alpha-1}}{\Gamma(\alpha)} + p\Big]. \tag{3.5}$$

Here l, $p \ge 0$; m, $n > 0$ and $A = ln(b-a)^{\alpha-1} + mn(\alpha-1)(b-a)^{\alpha-2} + lp\Gamma(\alpha)$.

Proof. Applying I_a^α on both sides of (3.1) and using Lemma 2.4, we have

$$y(t) = -I_a^\alpha h(t) + C_1(t-a)^{\alpha-1} + C_2(t-a)^{\alpha-2}, \tag{3.6}$$

for some C_1, $C_2 \in \mathbb{R}$. Applying $I_a^{2-\alpha}$ on both sides of (3.6) and using Lemmas 2.1 - 2.2, we get

$$I_a^{2-\alpha}y(t) = -I_a^2 h(t) + C_1\Gamma(\alpha)(t-a) + C_2\Gamma(\alpha-1). \tag{3.7}$$

Applying $D_a^{\alpha-1}$ on both sides of (3.6) and using Lemmas 2.1 - 2.2, we get

$$D_a^{\alpha-1}y(t) = -I_a^1 h(t) + C_1\Gamma(\alpha). \tag{3.8}$$

Using $lI_a^{2-\alpha}y(a) - mD_a^{\alpha-1}y(a) = 0$ in (3.7) and (3.8), we get

$$-\, mC_1(\alpha-1) + lC_2 = 0. \tag{3.9}$$

Using $ny(b) + pD_a^{\alpha-1}y(b) = 0$ in (3.6) and (3.8), we get

$$C_1\big[n(b-a)^{\alpha-1} + p\Gamma(\alpha)\big] + nC_2(b-a)^{\alpha-2} = nI_a^\alpha h(b) + pI_a^1 h(b). \tag{3.10}$$

Solving (3.9) and (3.10) for C_1 and C_2, we have

$$C_1 = \frac{l}{A}\int_a^b \Big[\frac{n(b-s)^{\alpha-1}}{\Gamma(\alpha)} + p\Big]h(s)ds,$$

and

$$C_2 = \frac{m(\alpha-1)}{A}\int_a^b \Big[\frac{n(b-s)^{\alpha-1}}{\Gamma(\alpha)} + p\Big]h(s)ds.$$

Substituting C_1 and C_2 in (3.6), it follows that

$$\begin{aligned} y(t) &= -\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)ds \\ &+ \frac{l(t-a)^{\alpha-1}}{A}\int_a^b \Big[\frac{n(b-s)^{\alpha-1}}{\Gamma(\alpha)} + p\Big]h(s)ds \\ &+ \frac{m(\alpha-1)(t-a)^{\alpha-2}}{A}\int_a^b \Big[\frac{n(b-s)^{\alpha-1}}{\Gamma(\alpha)} + p\Big]h(s)ds \\ &= \int_a^b G(t,s)h(s)ds. \end{aligned}$$

The proof is complete. □

Corollary 1. *Let* $1 < \alpha \le 2$ *and* $h : [a,b] \to \mathbb{R}$. *The fractional boundary value problem*

$$\begin{cases} D_a^\alpha y(t) + h(t) = 0, & a < t < b, \\ y(a) = 0, \ ny(b) + pD_a^{\alpha-1}y(b) = 0, \end{cases} \tag{3.11}$$

has the unique solution

$$y(t) = \int_a^b \bar{G}(t,s)h(s)ds, \tag{3.12}$$

where $\bar{G}(t,s)$ *is given by*

$$\bar{G}(t,s) = \begin{cases} \bar{G}_1(t,s), & a \le s \le t \le b, \\ \bar{G}_2(t,s), & a \le t \le s \le b, \end{cases} \tag{3.13}$$

$$\bar{G}_1(t,s) = \bar{G}_2(t,s) - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, \tag{3.14}$$

and

$$\bar{G}_2(t,s) = \frac{(t-a)^{\alpha-1}}{\bar{A}}\Big[\frac{n(b-s)^{\alpha-1}}{\Gamma(\alpha)} + p\Big]. \tag{3.15}$$

Here $n \ge 0$, $p > 0$ *and* $\bar{A} = n(b-a)^{\alpha-1} + p\Gamma(\alpha)$.

Proof. The proof is similar to the proof of Theorem 3.1. □

Now, we prove that these Green's functions are positive and obtain upper bounds for both the Green's functions and their integrals.

Theorem 3.2. *The Green's function* $G(t,s)$ *for* (3.1) *satisfies* $G(t,s) > 0$ *for* $(t,s) \in (a,b] \times (a,b]$.

Proof. Clearly, for $a < t \le s \le b$,

$$G(t,s) = \Big[\frac{l(t-a)^{\alpha-1} + m(\alpha-1)(t-a)^{\alpha-2}}{A}\Big]\Big[\frac{n(b-s)^{\alpha-1}}{\Gamma(\alpha)} + p\Big] > 0.$$

Now, suppose $a < s \le t \le b$. Consider

$$\begin{aligned} G(t,s) &= \Big[\frac{l(t-a)^{\alpha-1} + m(\alpha-1)(t-a)^{\alpha-2}}{A}\Big]\Big[\frac{n(b-s)^{\alpha-1}}{\Gamma(\alpha)} + p\Big] \\ &\quad - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \frac{1}{A\Gamma(\alpha)}\Big[ln\big[(t-a)^{\alpha-1}(b-s)^{\alpha-1} - (b-a)^{\alpha-1}(t-s)^{\alpha-1}\big] \\ &\quad + mn(\alpha-1)\big[(t-a)^{\alpha-2}(b-s)^{\alpha-1} - (b-a)^{\alpha-2}(t-s)^{\alpha-1}\big] \\ &\quad + lp\Gamma(\alpha)\big[(t-a)^{\alpha-1} - (t-s)^{\alpha-1}\big] + mp(\alpha-1)\Gamma(\alpha)(t-a)^{\alpha-2}\Big] \\ &= \frac{1}{A\Gamma(\alpha)}\big[S_1 + S_2 + S_3 + S_4\big]. \end{aligned} \tag{3.16}$$

Clearly, $A\Gamma(\alpha) > 0$. Consider

$$(t-a)(b-s) - (b-a)(t-s) = (s-a)(b-t) \geq 0,$$

implies

$$S_1 = ln\big[(t-a)^{\alpha-1}(b-s)^{\alpha-1} - (b-a)^{\alpha-1}(t-s)^{\alpha-1}\big] \geq 0. \tag{3.17}$$

Since

$$a < s \leq t \leq b,$$

we have

$$(t-a)^{\alpha-2} \geq (b-a)^{\alpha-2},\ (b-s)^{\alpha-1} \geq (t-s)^{\alpha-1} \text{ and } (t-a)^{\alpha-1} > (t-s)^{\alpha-1},$$

implying that

$$\begin{aligned} S_2 &= mn(\alpha-1)\big[(t-a)^{\alpha-2}(b-s)^{\alpha-1} - (b-a)^{\alpha-2}(t-s)^{\alpha-1}\big] \\ &\geq mn(\alpha-1)(b-a)^{\alpha-2}\big[(b-s)^{\alpha-1} - (t-s)^{\alpha-1}\big] \geq 0, \end{aligned} \tag{3.18}$$

and

$$S_3 = lp\Gamma(\alpha)\big[(t-a)^{\alpha-1} - (t-s)^{\alpha-1}\big] > 0. \tag{3.19}$$

Clearly,

$$S_4 = mp(\alpha-1)\Gamma(\alpha)(t-a)^{\alpha-2} > 0. \tag{3.20}$$

Using (3.17) - (3.20) in (3.16), we have $G(t,s) > 0$. The proof is complete. □

Corollary 2. *The Green's function $\bar{G}(t,s)$ for* (3.11) *satisfies $G(t,s) \geq 0$ for $(t,s) \in [a,b] \times [a,b]$.*

Proof. The proof is similar to the proof of Theorem 3.2. □

Theorem 3.3. *For the Green's function $G(t,s)$ defined in* (3.3)*,*

$$\max_{s\in(a,b]} G(t,s) = G(t,t), \quad t \in (a,b],$$

and

$$(t-a)^{2-\alpha}G(t,t) < \Big[\frac{l(b-a)+m(\alpha-1)}{A}\Big]\Big[\frac{n(b-a)^{\alpha-1}}{\Gamma(\alpha)} + p\Big], \quad t \in [a,b].$$

Proof. For the first part, we show that for any fixed $t \in (a,b]$, $G(t,s)$ increases in s for s from a to t, and then decreases in s for s from t to b. Let $a < t < s < b$. Consider

$$\frac{\partial}{\partial s}G(t,s) = -\frac{n(\alpha-1)(b-s)^{\alpha-2}}{\Gamma(\alpha)}\Big[\frac{l(t-a)^{\alpha-1} + m(\alpha-1)(t-a)^{\alpha-2}}{A}\Big] < 0,$$

implying that $G(t,s)$ is a decreasing function of s. Now, suppose $a < s < t \leq b$. Consider

$$\begin{aligned}\frac{\partial}{\partial s}G(t,s) &= -\frac{n(\alpha-1)(b-s)^{\alpha-2}}{\Gamma(\alpha)}\Big[\frac{l(t-a)^{\alpha-1}+m(\alpha-1)(t-a)^{\alpha-2}}{A}\Big]\\ &+ \frac{(\alpha-1)(t-s)^{\alpha-2}}{\Gamma(\alpha)}\\ &= \frac{(\alpha-1)}{A\Gamma(\alpha)}\Big[ln\big[-(t-a)^{\alpha-1}(b-s)^{\alpha-2}+(b-a)^{\alpha-1}(t-s)^{\alpha-2}\big]\\ &+ mn(\alpha-1)\big[-(t-a)^{\alpha-2}(b-s)^{\alpha-2}+(b-a)^{\alpha-2}(t-s)^{\alpha-2}\big]\\ &+ lp\Gamma(\alpha)(t-s)^{\alpha-2}\Big]\\ &= \frac{(\alpha-1)}{A\Gamma(\alpha)}\big[S_5+S_6+S_7\big]. \end{aligned} \tag{3.21}$$

Clearly, $\frac{(\alpha-1)}{A\Gamma(\alpha)} > 0$. Since

$$(t-a)(b-s)-(b-a)(t-s) = (s-a)(b-t) \geq 0,$$

we have that

$$S_6 = mn(\alpha-1)\big[-(t-a)^{\alpha-2}(b-s)^{\alpha-2}+(b-a)^{\alpha-2}(t-s)^{\alpha-2}\big] \geq 0. \tag{3.22}$$

Since $a < s < t \leq b$, we have

$$(t-s)^{\alpha-2} \geq (b-s)^{\alpha-2} \text{ and } (b-a)^{\alpha-1} \geq (t-a)^{\alpha-1},$$

implying that

$$\begin{aligned} S_5 &= ln\left[-(t-a)^{\alpha-1}(b-s)^{\alpha-2}+(b-a)^{\alpha-1}(t-s)^{\alpha-2}\right]\\ &\geq ln(b-s)^{\alpha-2}\big[-(t-a)^{\alpha-1}+(b-a)^{\alpha-1}\big] \geq 0. \end{aligned} \tag{3.23}$$

Clearly,

$$S_7 = lp\Gamma(\alpha)(t-s)^{\alpha-2} > 0. \tag{3.24}$$

Using (3.22) - (3.24) in (3.21), we have $G(t,s) > 0$, implying that $G(t,s)$ is an increasing function of s. Then, it follows that

$$\max_{s\in(a,b]} G(t,s) = G(t,t), \quad t \in (a,b].$$

To prove the second part, for $t \in [a,b]$, consider

$$\begin{aligned}(t-a)^{2-\alpha}G(t,t) &= \Big[\frac{l(t-a)+m(\alpha-1)}{A}\Big]\Big[\frac{n(b-t)^{\alpha-1}}{\Gamma(\alpha)}+p\Big]\\ &< \Big[\frac{l(b-a)+m(\alpha-1)}{A}\Big]\Big[\frac{n(b-a)^{\alpha-1}}{\Gamma(\alpha)}+p\Big]. \end{aligned}$$

The proof is complete. □

Corollary 3. *For the Green's function $\bar{G}(t,s)$ defined in* (3.13),

$$\max_{s\in[a,b]} \bar{G}(t,s) = \bar{G}(t,t), \quad t \in [a,b],$$

and

$$\bar{G}(t,t) \le \Big[\frac{(b-a)^{\alpha-1}}{\bar{A}}\Big]\Big[\frac{n(b-a)^{\alpha-1}}{\Gamma(\alpha)} + p\Big], \quad t \in [a,b].$$

Proof. The first part of the proof is similar to the proof of Theorem 3.3. To prove the second part, for $t \in [a,b]$, consider

$$\begin{aligned}\bar{G}(t,t) &= \Big[\frac{(t-a)^{\alpha-1}}{\bar{A}}\Big]\Big[\frac{n(b-t)^{\alpha-1}}{\Gamma(\alpha)} + p\Big] \\ &\le \Big[\frac{(b-a)^{\alpha-1}}{\bar{A}}\Big]\Big[\frac{n(b-a)^{\alpha-1}}{\Gamma(\alpha)} + p\Big].\end{aligned}$$

The proof is complete. □

Theorem 3.4. *For the Green's function $G(t,s)$ defined in* (3.3),

$$\int_a^b (t-a)^{2-\alpha} G(t,s)ds \le \Big[\frac{l(b-a)+m(\alpha-1)}{A}\Big]\Big[\frac{n(b-a)^{\alpha}}{\Gamma(\alpha+1)} + p(b-a)\Big], \ t \in [a,b].$$

Proof. Consider

$$\begin{aligned}&\int_a^b (t-a)^{2-\alpha} G(t,s)ds \\ &= \int_a^t (t-a)^{2-\alpha} G_1(t,s)ds + \int_t^b (t-a)^{2-\alpha} G_2(t,s)ds \\ &= \Big[\frac{l(t-a)+m(\alpha-1)}{A}\Big]\Big[\frac{n(b-a)^{\alpha}}{\Gamma(\alpha+1)} + p(b-a)\Big] - \frac{(t-a)^2}{\Gamma(\alpha+1)} \\ &\le \Big[\frac{l(b-a)+m(\alpha-1)}{A}\Big]\Big[\frac{n(b-a)^{\alpha}}{\Gamma(\alpha+1)} + p(b-a)\Big].\end{aligned}$$

The proof is complete. □

Corollary 4. *For the Green's function $\bar{G}(t,s)$ defined in* (3.13),

$$\int_a^b \bar{G}(t,s)ds \le \frac{(b-a)^{\alpha}\big[n(b-a)^{\alpha-1} + p\Gamma(\alpha+1)\big]}{\bar{A}\Gamma(\alpha+1)}, \quad t \in [a,b].$$

Proof. Consider

$$\begin{aligned}\int_a^b \bar{G}(t,s)ds &= \int_a^t \bar{G}_1(t,s)ds + \int_t^b \bar{G}_2(t,s)ds \\ &= \frac{(t-a)^{\alpha-1}}{\bar{A}}\Big[\frac{n(b-a)^{\alpha}}{\Gamma(\alpha+1)} + p(b-a)\Big] - \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} \\ &\le \frac{(b-a)^{\alpha}\big[n(b-a)^{\alpha-1} + p\Gamma(\alpha+1)\big]}{\bar{A}\Gamma(\alpha+1)}.\end{aligned}$$

The proof is complete. □

We are now able to formulate Lyapunov-type inequalities for the fractional boundary value problems (3.1) and (3.11).

Theorem 3.5. *If the following fractional boundary value problem*

$$\begin{cases} D_a^\alpha y(t) + q(t)y(t) = 0, \quad a < t < b, \\ lI_a^{2-\alpha} y(a) - mD_a^{\alpha-1} y(a) = 0, \ ny(b) + pD_a^{\alpha-1} y(b) = 0, \end{cases} \tag{3.25}$$

has a nontrivial solution, then

$$\int_a^b (s-a)^{\alpha-2} \big|q(s)\big| ds > \frac{A\Gamma(\alpha)}{\big[n(b-a)^{\alpha-1} + p\Gamma(\alpha)\big]\big[l(b-a) + m(\alpha-1)\big]}. \tag{3.26}$$

Proof. Let $\mathfrak{B} = C_{2-\alpha}[a,b]$ be the Banach space of functions y endowed with norm

$$\|y\|_{C_{2-\alpha}} = \max_{t\in[a,b]} \big|(t-a)^{2-\alpha} y(t)\big|.$$

It follows from Theorem 3.1 that a solution to (3.25) satisfies the equation

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds.$$

Hence,

$$\begin{aligned} \|y\|_{C_{2-\alpha}} &= \max_{t\in[a,b]} \left|(t-a)^{2-\alpha} \int_a^b G(t,s)q(s)y(s)ds\right| \\ &\le \max_{t\in[a,b]} \left[\int_a^b (t-a)^{2-\alpha} G(t,s)\big|q(s)\big|\big|y(s)\big| ds\right] \\ &\le \|y\|_{C_{2-\alpha}} \left[\max_{t\in[a,b]} \int_a^b (t-a)^{2-\alpha} G(t,s)(s-a)^{\alpha-2}\big|q(s)\big| ds\right] \\ &\le \|y\|_{C_{2-\alpha}} \left[\max_{t\in[a,b]} (t-a)^{2-\alpha} G(t,t)\right] \int_a^b (s-a)^{\alpha-2}\big|q(s)\big| ds, \end{aligned}$$

or, equivalently,

$$1 < \left[\max_{t\in[a,b]} (t-a)^{2-\alpha} G(t,t)\right] \int_a^b (s-a)^{\alpha-2}\big|q(s)\big| ds.$$

An application of Theorem 3.3 yields the result. □

Corollary 5. *If the following fractional boundary value problem*

$$\begin{cases} D_a^\alpha y(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = 0, \ ny(b) + pD_a^{\alpha-1} y(b) = 0, \end{cases} \tag{3.27}$$

has a nontrivial solution, then

$$\int_a^b \big|q(s)\big| ds > \frac{\bar{A}\Gamma(\alpha)}{\big[n(b-a)^{2\alpha-2} + p(b-a)^{\alpha-1}\Gamma(\alpha)\big]}. \tag{3.28}$$

Proof. Let $\mathfrak{B} = C[a,b]$ be the Banach space of functions y endowed with norm

$$\|y\| = \max_{t\in[a,b]} |y(t)|.$$

It follows from Corollary 1 that a solution to (3.27) satisfies the equation

$$y(t) = \int_a^b \bar{G}(t,s)q(s)y(s)ds.$$

Hence,

$$\begin{aligned}\|y\| = \max_{t\in[a,b]} \left| \int_a^b \bar{G}(t,s)q(s)y(s)ds \right| &\le \max_{t\in[a,b]} \left[\int_a^b \bar{G}(t,s)\big|q(s)\big|\big|y(s)\big|ds \right] \\ &\le \|y\| \left[\max_{t\in[a,b]} \int_a^b \bar{G}(t,s)\big|q(s)\big|ds \right] \\ &\le \|y\| \left[\max_{t\in[a,b]} \bar{G}(t,t) \right] \int_a^b \big|q(s)\big|ds,\end{aligned}$$

or, equivalently,

$$1 < \left[\max_{t\in[a,b]} \bar{G}(t,t) \right] \int_a^b \big|q(s)\big|ds.$$

An application of Corollary 3 yields the result. □

Take $l = p = 0$ in Theorem 3.5. Then, we obtain the following Lyapunov-type inequality for the left-focal fractional boundary value problem.

Corollary 6. *If the following fractional boundary value problem*

$$\begin{cases} D_a^\alpha y(t) + q(t)y(t) = 0, & a < t < b, \\ D_a^{\alpha-1} y(a) = 0,\ y(b) = 0, \end{cases} \tag{3.29}$$

has a nontrivial solution, then

$$\int_a^b (s-a)^{\alpha-2}\big|q(s)\big|ds > \frac{\Gamma(\alpha)}{(b-a)}. \tag{3.30}$$

Take $n = 0$ in Corollary 5. Then, we obtain the following Lyapunov-type inequality for the right-focal fractional boundary value problem.

Corollary 7. *If the following fractional boundary value problem*

$$\begin{cases} D_a^\alpha y(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = 0,\ D_a^{\alpha-1} y(b) = 0, \end{cases} \tag{3.31}$$

has a nontrivial solution, then

$$\int_a^b \big|q(s)\big|ds > \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}. \tag{3.32}$$

Take $l = m = n = p = 1$ in Theorem 3.5. Then, we obtain the following Lyapunov-type inequality for the fractional boundary value problem with Robin boundary conditions.

Corollary 8. *If the following fractional boundary value problem*

$$\begin{cases} D_a^\alpha y(t) + q(t)y(t) = 0, \quad a < t < b, \\ I_a^{2-\alpha} y(a) - D_a^{\alpha-1} y(a) = 0, \ y(b) + D_a^{\alpha-1} y(b) = 0, \end{cases} \tag{3.33}$$

has a nontrivial solution, then

$$\int_a^b (s-a)^{\alpha-2} \big|q(s)\big| ds > \frac{\big[(b-a)^{\alpha-1} + (\alpha-1)(b-a)^{\alpha-2} + \Gamma(\alpha)\big]\Gamma(\alpha)}{\big[(b-a)^{\alpha-1} + \Gamma(\alpha)\big](b-a+\alpha-1)}. \tag{3.34}$$

Take $l > 0$ and $p = 0$ in Theorem 3.5. Then, we obtain the following Lyapunov-type inequality for the fractional boundary value problem with Sturm-Liouville boundary conditions.

Corollary 9. *If the following fractional boundary value problem*

$$\begin{cases} D_a^\alpha y(t) + q(t)y(t) = 0, \quad a < t < b, \\ l I_a^{2-\alpha} y(a) - m D_a^{\alpha-1} y(a) = 0, \ y(b) = 0, \end{cases} \tag{3.35}$$

has a nontrivial solution, then

$$\int_a^b (s-a)^{\alpha-2} \big|q(s)\big| ds > \frac{\Gamma(\alpha)}{(b-a)}. \tag{3.36}$$

4. Applications

In this section, we discuss two applications of Theorem 3.5 and Corollary 5. First, we estimate lower bounds for the eigenvalues of the Riemann-Liouville type fractional eigenvalue problems corresponding to (3.25) and (3.27).

Theorem 4.1. *Assume that y is a nontrivial solution of the Riemann-Liouville type fractional eigenvalue problem*

$$\begin{cases} D_a^\alpha y(t) + p(t)y(t) = 0, \quad a < t < b, \\ l I_a^{2-\alpha} y(a) - m D_a^{\alpha-1} y(a) = 0, \ n y(b) + p D_a^{\alpha-1} y(b) = 0, \end{cases}$$

where $y(t) \neq 0$ for each $t \in (a,b)$. Then,

$$|\lambda| > \frac{(\alpha-1)A\Gamma(\alpha)}{(b-a)^{\alpha-1}\big[n(b-a)^{\alpha-1} + p\Gamma(\alpha)\big]\big[l(b-a) + m(\alpha-1)\big]}. \tag{4.1}$$

Corollary 10. *Assume that y is a nontrivial solution of the Riemann-Liouville type fractional eigenvalue problem*

$$\begin{cases} D_a^\alpha y(t) + p(t)y(t) = 0, \quad a < t < b, \\ y(a) = 0, \ n y(b) + p D_a^{\alpha-1} y(b) = 0, \end{cases}$$

where $y(t) \neq 0$ *for each* $t \in (a, b)$. *Then,*

$$|\lambda| > \frac{\bar{A}\Gamma(\alpha)}{(b-a)^\alpha \big[n(b-a)^{\alpha-1} + p\Gamma(\alpha)\big]}. \tag{4.2}$$

Consider the one and two-parameter Mittag-Leffler functions [9]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{4.3}$$

and

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{4.4}$$

where z, $\beta \in \mathbb{C}$ and $\Re(\alpha) > 0$.

As the second application, we use Theorem 3.5 and Corollary 5 to obtain an interval in which some functions of Mittag-Leffler functions (4.3) and (4.4) have no real zeros.

Theorem 4.2. *Let* $1 < \alpha \leq 2$. *Then, the function*

$$lpE_\alpha(x) + (ln - mpx)E_{\alpha,\alpha}(x) + mnE_{\alpha,\alpha-1}(x)$$

has no real zeros for

$$|x| \leq \frac{(\alpha-1)\big[ln + mn(\alpha-1) + lp\Gamma(\alpha)\big]\Gamma(\alpha)}{(l + m(\alpha-1))(n + p\Gamma(\alpha))}.$$

Proof. Let $a = 0$, $b = 1$ and consider the fractional boundary value problem

$$\begin{cases} D_0^\alpha y(t) + \lambda y(t) = 0, \quad 0 < t < 1, \\ lI_0^{2-\alpha} y(0) - mD_0^{\alpha-1} y(0) = 0, \ ny(1) + pD_0^{\alpha-1} y(1) = 0. \end{cases} \tag{4.5}$$

By Corollary 5.1 of [9], the general solution of the fractional differential equation

$$D_0^\alpha y(t) + \lambda y(t) = 0$$

is given by

$$y(t) = c_1 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) + c_2 t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha), \quad t \in (0,1]. \tag{4.6}$$

Denote by

$$g(t) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) = t^{\alpha-1} \sum_{n=0}^{\infty} \frac{(-\lambda)^n t^{\alpha n}}{\Gamma(\alpha n + \alpha)}.$$

Then

$$g'(t) = t^{\alpha-2} \sum_{n=0}^{\infty} \frac{(-\lambda)^n t^{\alpha n}}{\Gamma(\alpha n + \alpha - 1)} = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha).$$

Note that

$$I_0^{2-\alpha}g(t) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n+\alpha)} I_0^{2-\alpha} t^{\alpha n+\alpha-1} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n+\alpha)} \frac{\Gamma(\alpha n+\alpha)}{\Gamma(\alpha n+2)} t^{\alpha n+1} = t\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n+2)} t^{\alpha n} = tE_{\alpha,2}(-\lambda t^{\alpha}), \tag{4.7}$$

$$D_0^{\alpha-1}g(t) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n+\alpha)} D_0^{\alpha-1} t^{\alpha n+\alpha-1} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n+\alpha)} \frac{\Gamma(\alpha n+\alpha)}{\Gamma(\alpha n+1)} t^{\alpha n} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n+1)} t^{\alpha n} = E_{\alpha}(-\lambda t^{\alpha}), \tag{4.8}$$

$$I_0^{2-\alpha}g'(t) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n+\alpha-1)} I_0^{2-\alpha} t^{\alpha n+\alpha-2} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n+\alpha-1)} \frac{\Gamma(\alpha n+\alpha-1)}{\Gamma(\alpha n+1)} t^{\alpha n} = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n+1)} t^{\alpha n} = E_{\alpha}(-\lambda t^{\alpha}), \tag{4.9}$$

and

$$D_0^{\alpha-1}g'(t) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n+\alpha-1)} D_0^{\alpha-1} t^{\alpha n+\alpha-2} = \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n+\alpha-1)} \frac{\Gamma(\alpha n+\alpha-1)}{\Gamma(\alpha n)} t^{\alpha n-1} = -\lambda\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{\Gamma(\alpha n+\alpha)} t^{\alpha n+\alpha-1} = -\lambda g(t). \tag{4.10}$$

Also, note that

$$I_0^{2-\alpha}g(0) = D_0^{\alpha-1}g'(0) = 0,\ D_0^{\alpha-1}g(0) = I_0^{2-\alpha}g'(0) = 1. \tag{4.11}$$

Using $lI_0^{2-\alpha}y(0) - mD_0^{\alpha-1}y(0) = 0$, we get

$$mc_1 = lc_2.$$

Using $ny(1) + pD_0^{\alpha-1}y(1) = 0$, we get that the eigenvalues $\lambda \in \mathbb{R}$ of (4.5) are the solutions of

$$lpE_\alpha(-\lambda) + (ln - mp\lambda)E_{\alpha,\alpha}(-\lambda) + mnE_{\alpha,\alpha-1}(-\lambda) = 0, \tag{4.12}$$

and the corresponding eigenfunctions are given by

$$y(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) + \frac{m}{l}t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha), \quad t \in (0,1]. \tag{4.13}$$

By Theorem 3.5, if a real eigenvalue λ of (4.5) exists, i.e. λ is a zero of (4.12), then

$$|\lambda| > \frac{(\alpha-1)\big[ln + mn(\alpha-1) + lp\Gamma(\alpha)\big]\Gamma(\alpha)}{(l + m(\alpha-1))(n + p\Gamma(\alpha))}.$$

The proof is complete. □

Corollary 11. *Let $1 < \alpha \leq 2$. Then, the function $pE_\alpha(x) + nE_{\alpha,\alpha}(x)$ has no real zeros for*

$$|x| \leq \Gamma(\alpha).$$

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