ON CHARACTERIZATION OF MINIMAL $k$-BI-IDEALS IN $k$-REGULAR AND COMPLETELY $k$-REGULAR SEMIRINGS

Kalyan Hansda and Tapas Kumar Mondal

Abstract. In this paper, we study $k$-regular and completely $k$-regular semirings. We characterize the minimal $k$-bi-ideals in $k$-regular semirings via principal $k$-bi-ideals and also in completely $k$-regular semirings via $k$-bi-ideals generated by $k$-idempotent elements. Finally we characterize the completely $k$-regular semirings by $k$-bi-ideals generated via $k$-idempotents.

AMS Mathematics Subject Classification (2010): 16Y60
Key words and phrases: $k$-regular semiring; completely $k$-regular semiring; $k$-bi-ideal; $k$-idempotent; $k$-bi-simple semiring

1. Introduction

The notion of a semiring was introduced by Vandiver [15]. In 1951, Bourne defined a regular semiring as a semiring $S$ in which for all $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$. In [1], Adhikari, Sen and Weinert renamed it as a $k$-regular semiring. In [14], Sen and Bhuniya studied $k$-regular semirings with a semilattice additive reduct, and constructed $k$-regular semirings.

If $F$ is any semigroup, then the set $P(F)$ of all subsets of $F$ is a semiring in $SL^{+}$, where addition and multiplication are defined by the set union and the usual product of subsets of a semigroup, respectively. In [14], it is shown that $P(F)$ is a $k$-regular semiring if and only if $F$ is a regular semigroup [Theorem 3.1], and if $(F, \cdot)$ is a regular semigroup, then the $k$-idempotents of $P(F)$ commute if and only if $P(F)$ is a commutative semiring [Theorem 3.4]. Sen and Bhuniya defined $k$-idempotents to characterize the $k$-regular semirings which are distributive lattices of $k$-semifields [13]. Bhuniya and Jana introduced the notion of $k$-bi-ideals in a semiring, characterized the $k$-regular semirings by $k$-bi-ideals, and gave the description of the principal $k$-bi-ideals in a semiring with semilattice additive reduct [2]. In [9], Jana studied quasi $k$-ideals in $k$-regular semirings and characterized the $k$-regular semirings via their quasi $k$-ideals. In [12], Sen and Bhuniya defined completely $k$-regular semirings and presented various interesting properties of classes of such semirings. They characterized

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completely $k$-regular semirings. A semiring $S$ is completely $k$-regular if and only if $S$ is $k$-regular and $a \in a^2S \cap Sa^2$ for all $a \in S$. The structure of such semirings was also given. A semiring is completely $k$-regular if and only if $S$ is a union of $k$-semifields.

In this paper we study the semirings with semilattice additive reduct. Such semirings have been studied by Bhuniya and Mondal\cite{Bhuniya1,Bhuniya2,Bhuniya3} to give the decompositions of underlying semirings through distributive lattice congruence into simpler components. Here we study $k$-regular and completely $k$-regular semirings with semilattice additive reduct and their $k$-bi-ideals. In Section 2, the preliminaries have been provided. In Section 3, we study completely $k$-regular semirings and $k$-regular semirings. We show that in a completely $k$-regular semiring, for any element $a \in S$ there exist two $k$-related $k$-idempotent elements. In Section 4, our main intention is to characterize minimal $k$-bi-ideals in a $k$-regular and completely $k$-regular semiring by principal $k$-bi-ideals generated by $k$-idempotents. We show that a $k$-bi-ideal $B$ in a $k$-regular semiring is minimal if and only if for all $a, b \in B$, the principal $k$-b-ideals generated by $a$ and $b$ are the same, while a $k$-bi-ideal $B$ in a completely $k$-regular semiring $S$ is minimal if and only if the principal $k$-bi-ideals generated by $k$-idempotents in $B$ coincide. We define $k$-bi-simple semirings, and characterize the minimal $k$-bi-ideals by $k$-bi-simplicity of the semirings. Finally, we characterize completely $k$-regular semirings by the principal $k$-bi-ideals generated by $k$-idempotents of $S$.

2. Preliminaries

A semiring $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both the additive reduct $(S, +)$ and the multiplicative reduct $(S, \cdot)$ are semigroups and such that the following distributive laws hold:

$$x(y + z) = xy + xz \quad \text{and} \quad (x + y)z = xz + yz.$$ 

Thus the semirings can be regarded as a common generalization of both rings and distributive lattices. By $\mathcal{SL}^+$ we denote the category of all semirings $(S, +, \cdot)$ such that $(S, +)$ is a semilattice, i.e. a commutative and idempotent semigroup. Throughout this paper, unless otherwise stated, $S$ is always a semiring in $\mathcal{SL}^+$.

Let $A$ be a nonempty subset of $S$. The $k$-closure of $A$ is defined by

$$\overline{A} = \{x \in S \mid x + a_1 = a_2 \text{ for some } a_1, a_2 \in A\}.$$ 

We assume that $x + a_1 = a_2$. Hence $x + a_2 = x + x + a_1 = a_2$. So $\overline{A}$ is also described by

$$\overline{A} = \{x \in S \mid x + a = a \text{ for some } a \in A\}.$$ 

Then we have $A \subseteq \overline{A}$ and $\overline{A} = \overline{\overline{A}}$, since $(S, +)$ is a semilattice, $A$ is called a $k$-set if $\overline{A} \subseteq A$. An ideal (left, right) $A$ of $S$ is called a $k$-ideal (left, right) if it is a $k$-set, i.e. $\overline{A} = A$. 
A semiring $S$ is called a $k$-regular semiring [6] if for every $a \in S$, there exists an $s \in S$ such that $a + asa =asa$. A semiring $S$ is called a completely $k$-regular semiring if for every $a \in S$, there exists an $s \in S$ such that $a + asa = asa, as + as^2a = as^2a$ and $sa + as^2a = as^2a$, equivalently, $a + a^2sa^2 = a^2sa^2$ [Theorem 5.1 [12]].

For $a \in S$, the principal left $k$-ideal (resp. principal right $k$-ideal) generated by $a$ is the least left $k$-ideal (resp. least right $k$-ideal) of $S$ containing $a$. Bhuniya and Jana [2] introduced $k$-bi-ideals in a semiring in $SL^+$. A non-empty subset $B$ of $S$ is said to be a $k$-bi-ideal of $S$ if $BSB \subseteq B$ and $B$ is a $k$-subsemiring of $S$. The structures of the principal left $k$-ideal (resp. principal right $k$-ideal and principal $k$-bi-ideal) are given, respectively, by

$$L_k(a) = \{x \in S \mid x + a + sa = a + sa, \text{ for some } s \in S\},$$
$$R_k(a) = \{x \in S \mid x + a + as = a + as, \text{ for some } s \in S\}$$

and

$$B_k(a) = \{x \in S \mid x + a + a^2 + asa = a + a^2 + asa, \text{ for some } s \in S\}.$$

Sen and Bhuniya [12] defined four equivalence relations namely $\overline{L}$, $\overline{R}$, $\overline{J}$ and $\overline{H}$ analogous to the Green’s relations, on a $k$-regular semiring $S$ in $SL^+$. If $a \in S$ be a $k$-regular element, then one has $L_k(a) = S \overline{a}, R_k(a) = a \overline{S}, B_k(a) = a \overline{Sa}$. Bhuniya and Mondal [4], [11] generalized the Green’s relations $\overline{L}$, $\overline{R}$, and $\overline{H}$ on a semiring $S$ in $SL^+$ and they are

$$\overline{L} = \{(x, y) \in S \times S \mid L_k(x) = L_k(y)\},$$
$$\overline{R} = \{(x, y) \in S \times S \mid R_k(x) = R_k(y)\}$$

and

$$\overline{H} = \overline{L} \cap \overline{R}.$$ 

Mondal and Bhuniya also defined an equivalence relation $\overline{B}$ [11] by: for $a, b \in S$,

$$a \overline{B} b \iff B_k(a) = B_k(b).$$

If $S \in SL^+$, then both $\overline{L}$ and $\overline{R}$ are additive congruences on $S$ and $\overline{L}$ is a right congruence and $\overline{R}$ is a left congruence on $S$.

An element $e \in S$ is said to be $k$-idempotent if $e + e^2 = e^2$ [12]. If $A$ is a subsemiring of $S$, then let $E_k(A)$ denote the set of all $k$-idempotents of $A$.

For undefined concepts in semigroup theory we refer to [8], for undefined concepts in semiring theory cf. [7].

3. Completely $k$-regular semirings

In this section we study completely $k$-regular semirings and $k$-regular semirings. In completely $k$-regular semirings we show that for any given element $a \in S$ we can always find two $k$-idempotents, depending on $a$, such that they are $\overline{H}$-related. We also characterize the $k$-regular semirings by the product of a principal right $k$-ideal and a principal left $k$-ideal of the semirings.
Lemma 3.1. Let $S$ be a completely $k$-regular semiring. Then

1. for every $a \in S$, there exists a $z \in S$ such that $a + az = asa$, $a + a^2z = a^2z$ and $a + za^2 = za^2$. 

2. for every $a \in S$, there exists a $u \in S$ such that $a\overline{H} au$ and $a\overline{H} ua$. 

3. for every $a \in S$, there exist $e, f \in E_k(S)$ depending on $a$ such that $e\overline{H} f$.

Proof. (1) Since $S$ is completely $k$-regular, for $a \in S$, there exists a $x \in S$ such that $a + a^2 x a_2 = a^2 x a_2$. Adding $a^3 x a_2 x a_2$ on both sides one gets $a + (a + a^2 x a_2) x a_2 = a(a + a^2 x a_2) x a_2$. This implies $a + a^3 x a_2 x a_2 = a^3 x a_2 x a_2$. Again adding $a^3 x a_2 x a_2 x a_3$ on both sides we get of $a + a^3 x a_2 x (a + a^2 x a_2) a = a^3 x a_2 x (a + a^2 x a_2) a$ so that $a + a^3 x a_2 x a_3 = a^3 x a_2 x a_3$. For $z = a^2 x a_2 x a_2 x a_2$, we get $a + za_2 = za_2$. Again adding $a^4 x a_2 x a_2 x a_2$ on both sides of $a + a^3 x a_2 x a_2 x a_2 = a^3 x a_2 x a_2 x a_2$ we get $a + a^2(a + a^2 x a_2) x a_2 x a_2 = a^2(a + a^2 x a_2) x a_2 x a_2$. This yields $a + a^2 a_2 x a_2 x a_2 x a_2 = a^2 a_2 x a_2 x a_2 x a_2$, i.e. $a + a^2 z = a^2 z$. Similarly one can show that $a + za^2 = za^2$. 

(2) For $a \in S$, there exists a $x \in S$ such that $a + axa = axa$, $ax + ax^2a = ax^2a$ and $xa + ax^2a = ax^2a$. Now we can write $a + axa = axa$ as $a + as = as$ and $a + ta = ta$, where $s = xa, t = ax$. Then we have $a + au = au$ and $a + ua = ua$, where $u = s + t$. Adding $xa^2$ on both sides of $a + axa = axa$, one gets $a + xa^2 + axa = xa^2 + axa$. Now adding $ax^2a$ on both sides we get $a + xa^2 + (ax + ax^2a)a = (ax + ax^2a)a$, giving $a + xaa + axaax = axaaxa$ which yields $a + sa + ax(sa) = sa + ax(sa)$. Similarly adding $a^2 x$ on both sides of $a + axa = axa$ and proceeding as above one gets $a + at + (at)x a = at + (at)x a$. Now adding $ta + atxa$ and $as + asxa$, respectively on $a + sa + ax(sa) = sa + ax(sa)$ and $a + at + (at)x a = at + (at)x a$, we have $a + ua + ax(ua) = ua + ax(ua)$ and $a + au + (au)x a = au + (au)x a$. These two relations yield $a \in L_k(ua) \cap R_k(au)$. Also $ua \in L_k(a)$ and $au \in R_k(a)$. Thus $L_k(a) = L_k(ua)$ and $R_k(a) = R_k(au)$. The relation $xa + ax^2a = ax^2a$ can be written as $s + ts = ts$, and $ax + ax^2a = ax^2a$ as $t + ts = ts$. Then we get $(s + t) + ts = ts$, i.e. $u + ts = ts$. Now $au + ats = ats$, and $ua + tsa = tsa$, i.e. $au + (at)x a = (at)x a$, and $ua + a(xsa) = a(xsa)$. These two yield $au \in S a = L_k(a), ua \in a S = R_k(a)$, since $S$ is $k$-regular. Therefore, $L_k(au) \subseteq L_k(a)$ and $R_k(au) \subseteq R_k(a)$. Also from $a + axa = axa$, we have $a + a(xa + ax) = a(xa + ax), a + (ax + xa)a = (ax + xa)a$, i.e. $a + au = au \in L_k(au), a + ua = ua \in R_k(au)$ so that $a \in L_k(au), a \in R_k(au)$. Therefore, $L_k(a) \subseteq L_k(au)$ and $R_k(a) \subseteq R_k(ua)$. Consequently, $L_k(a) = L_k(au) = L_k(ua)$, and $R_k(a) = R_k(au) = R_k(ua)$. Finally, we get $a\overline{H} au$ and $a\overline{H} ua$. 

(3) Let $a \in S$. Then from the proof of (1), one has $a + az = az$. Then $az + (az)^2 = (az)^2$ and $za + (za)^2 = (za)^2$ yield $e(= za), f(= az) \in E_k(S)$. Now $a + az = az$, i.e. $a + ae = ae \in S a \subseteq L_k(e)$, whence $a \in L_k(e)$. Also $e = za \in L_k(a)$. Therefore, $a\overline{H} e$. Now $a + za^2 = za^2$, i.e. $a + ea = ea \in R_k(e)$ so that $a \in R_k(e)$. Again $za + zaza = zaza$, i.e. $e + a^2 x a^2 x a^2 x a^2 = a^2 x a^2 x a^2 x a^2 \in R_k(a)$ so that $e \in R_k(a)$. Thus $a\overline{H} e$. Consequently, $a\overline{H} e$. Similarly, one can get $a\overline{H} f$, whence $e\overline{H} f$, since $\overline{H}$ is an equivalence relation on $S$. 

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Lemma 3.2. If $A, B$ are two subsemirings of a semiring $S$, then

1. for $x \in S, a_1, a_2 \in A, b_1, b_2 \in B$ with $x + a_1b_1 = a_2b_2$, there exist $u \in A, v \in B$ such that $x + uv = uv$.

2. if $a, b, u, v, s, t \in S$ satisfying $u + as = as$ and $v + ta = ta$, then there exists a $w \in S$ such that $u + aw = aw$ and $v + wa = wa$.

Proof. (1) Follows if we take $u = a_1 + a_2, v = b_1 + b_2$.

(2) $w = s + t$ serves our purpose.

Lemma 3.3. [Theorem 3.2 [9]] A semiring $S$ is $k$-regular if and only if for every right $k$-ideal $R$ and left $k$-ideal $L$ of $S, RL = R \cap L$.

Lemma 3.4. Let $S$ be a $k$-regular semiring. Then

1. for every $a \in S, B_k(a) = \overline{R_k(a)L_k(a)}$.

2. for any subset $A$ of $S$, $\overline{SA \cap AS} = \overline{SA \cap AS}$.

Proof. (1) Let $a \in S$ and $x \in B_k(a)$. Then there exists an $s \in S$ such that $x + as = sa$. Since $S$ is a $k$-regular, there exists $u \in S$ such that $a + uau = uau$. Adding $uasa + auasa + auasa$ on both sides of $x + as = sa$, we get $x + (a + uau)s(a + uau) = (a + uau)s(a + uau)$, i.e., $x + (auas)(aua) = (auas)(aua) \in R_k(a)L_k(a)$. This implies that $x \in \overline{R_k(a)L_k(a)}$. Therefore $B_k(a) \subseteq \overline{R_k(a)L_k(a)}$. Conversely, suppose that $x \in \overline{R_k(a)L_k(a)}$. Then by Lemma 3.2, there are $u \in R_k(a), v \in L_k(a)$ such that $x + uv = uv$. Also there is a $w \in S$ such that $u + aw = aw, v + wa = wa$. Adding $uwa + awv + awva$ on both sides of $x + uv = uv$, one gets $x + (u + aw)(v + wa) = (u + aw)(v + wa)$ so that $x + awva = awva = a\overline{S}a = B_k(a)$. This implies $x \in B_k(a)$. Thus $\overline{R_k(a)L_k(a)} \subseteq B_k(a)$. Consequently, $B_k(a) = \overline{R_k(a)L_k(a)}$.

(2) Let $x \in \overline{SA \cap AS}$. Then using Lemma 3.2, we get $x + sa = sa, x + as = as$ for some $s \in S$. Since $S$ is $k$-regular, there exists a $z \in S$ such that $x + xz = xz$. Adding $xzs + asxz + aszsa$ on both sides we get $x + (x + as)z(x + sa) = (x + as)z(x + sa)$, i.e., $x + azs = azsa \in \overline{SA \cap AS}$ yielding $x \in \overline{SA \cap AS}$. Conversely, for $x \in \overline{SA \cap AS}$, there are $u, v \in \overline{SA \cap AS}$ such that $x + u = v$. Now there are $s_1, s_2, s_3, s_4 \in S, a_1, a_2, a_3, a_4 \in A$ such that $u = s_1a_1 = a_2s_2, v = s_3a_3 = a_4s_4$. Then one gets $x + s_1a_1 = s_3a_3, x + a_2s_2 = a_4s_4$ so that $x \in \overline{SA \cap AS}$. Therefore $\overline{SA \cap AS} \subseteq \overline{SA \cap AS}$. Consequently, $\overline{SA \cap AS} = \overline{SA \cap AS}$.

4. Characterization of minimal $k$-bi-ideals

In this section we find a necessary and sufficient condition for a $k$-bi-ideal to be minimal in a semiring, $k$-regular semiring as well as in completely $k$-regular semiring.
**Lemma 4.1.** Let $S$ be a $k$-regular semiring. A $k$-bi-ideal $B$ of $S$ is minimal if and only if $B_k(a) = B_k(b)$ for all $a, b \in B$.

*Proof.* Let $B$ be a minimal $k$-bi-ideal. Then for $a, b \in B$ one has $B_k(a) \subseteq B, B_k(b) \subseteq B$ so that $B_k(a) = B = B_k(b)$. Conversely, suppose that the given condition holds. Let $C$ be a $k$-bi-ideal of $S$ with $C \subseteq B$. For $x \in C, y \in B$, we have $x, y \in B$. This implies $B_k(x) = B_k(y)$ so that $y \in B_k(x) \subseteq C$, i.e., $B \subseteq C$. Consequently, $B$ is minimal. □

In the following lemma we find that in a $k$-regular semiring the relation $\mathcal{B}$ coincides with the relation $\mathcal{H}$.

**Lemma 4.2.** The following results hold in a semiring $S$:

1. $\mathcal{B} \subseteq \mathcal{H}$.

2. If $S$ is $k$-regular, then $\mathcal{B} = \mathcal{H}$.

*Proof.* (1) Let $a, b \in S$ with $a\mathcal{B}b$. Then there are $s, t \in S$ such that $a + b + b^2 + bsb = b + b^2 + bsb$ and $b + a + a^2 + ata = a + a^2 + ata$. Then we have $a + b + (b + bs)b = b + (b + bs)b$ and $b + a + (a + at)a = a + (a + at)a$ yielding $a \in L_k(b), b \in L_k(a)$ so that $L_k(a) = L_k(b)$, i.e., $a\mathcal{B}b$. Again we can write $a + b + b(b + sb) = b + b(b + sb)$ and $b + a + a(a + ta) = a + a(a + ta)$ yielding $a \in R_k(b), b \in R_k(a)$ so that $R_k(a) = R_k(b)$. Thus $a\mathcal{B}b$. Consequently, $\mathcal{B} \subseteq \mathcal{H}$.

(2) Let $S$ be $k$-regular, and $x\mathcal{H}y$. Then one has $L_k(a) = L_k(b), R_k(a) = R_k(b)$. Since $S$ is $k$-regular, by Lemmas 3.3 and 3.4 we get $B_k(a) = R_k(a) \cap L_k(a) = R_k(a) \cap L_k(a) = R_k(b) \cap L_k(b) = B_k(b)$ yielding $a\mathcal{B}b$. Consequently, $\mathcal{B} = \mathcal{H}$. □

In the following theorem we characterize the minimal $k$-bi-ideals in a semiring via the relation $\mathcal{B}$.

**Theorem 4.3.** A $k$-bi-ideal $B$ of a semiring $S$ is minimal if and only if it is a $\mathcal{B}$-class.

*Proof.* Let $B$ be a minimal $k$-bi-ideal of a semiring $S$, and $a, b \in B$. Then by Lemma 4.1 one gets $B_k(a) = B_k(b)$. This implies that $a\mathcal{B}b$. Thus $B$ is a $\mathcal{B}$-class. Conversely, suppose that $B$ is a $\mathcal{B}$-class, and $K$ a $k$-bi-ideal of $S$ such that $K \subseteq B$. Let $x \in B, y \in K$. Then $x, y \in B$ giving that $x\mathcal{B}y$, i.e., $B_k(x) = B_k(y)$. Then $x \in B_k(x) = B_k(y) \subseteq K$. Therefore $x \in K$. Thus $B \subseteq K$. Consequently, $B$ is minimal. □

**Theorem 4.4.** A $k$-bi-ideal $B$ in a $k$-regular semiring $S$ is minimal if and only if $B$ is an $\mathcal{H}$-class of $S$.

*Proof.* Let $B$ be a minimal $k$-ideal of $S$. Then by Lemma 4.1 and Theorem 4.3 we find that $B$ is an $\mathcal{H}$-class of $S$. Converse part follows from the Lemma 4.2 and Theorem 4.3. □

In the following theorem we characterize the minimal $k$-bi-ideals in a completely $k$-regular semiring via $k$-bi-ideals generated by $k$-idempotent elements.
Theorem 4.5. A $k$-bi-ideal $B$ of a completely $k$-regular semiring $S$ is minimal if and only if $B_k(a) = B_k(e)$ for all $a \in B$ and for all $e \in E_k(B)$.

Proof. Let $B$ be a minimal $k$-bi-ideal of $S$, and $a \in B, e \in E_k(B)$. Then $a, e \in B$, and so by Lemma 4.1 one gets $B_k(a) = B_k(e)$. Conversely suppose that the given conditions hold, and let $K$ be a $k$-bi-ideal of $S$ such that $K \subseteq B$. Let $x \in K, b \in B$. Since $S$ is completely $k$-regular, there exists an $s \in S$ such that $x + xsx = xsx, xs + xs^2x = xs^2x, sx + xs^2x = xs^2x$. Then $xs, sx \in E_k(S)$.

Now $xs + xs^2x = xs^2x \in xSx = B_k(x)$ implies $xs \in B_k(x) \subseteq B$ so that $xs \in E_k(B)$. By hypothesis, $b, xs \in B$ implies $B_k(b) = B_k(xs)$. Similarly, $B_k(b) = B_k(x)$. Now $b \in B_k(xs)$ ensures the existence of some $w \in S$ such that $b + xswxs = xswxs$. Adding $xswxs^2x$ on both sides we get $b + xsw(xs + xs^2x) = xsw(xs + xs^2x)$. This implies $b + xswxs^2x = xswxs^2x \in xSx = B_k(x)$ so that $b \in B_k(x) \subseteq K$, i.e., $b \in K$. Therefore $B \subseteq K$. Consequently, $B$ is minimal. \hfill \Box

Corollary 4.6. A $k$-bi-ideal $B$ of a completely $k$-regular semiring $S$ is minimal if and only if $B_k(e) = B_k(f)$ for all $e, f \in E_k(B)$.

Proof. Let $B$ be a minimal $k$-bi-ideal of $S$, and $e, f \in E_k(B)$. Then by Lemma 4.1 one gets $B_k(e) = B_k(f)$. Conversely, suppose that the given conditions hold, and $a \in B, e \in E_k(B)$. Then as in the proof of (1) of Lemma 3.1, for this $a \in B$, there is $z = a^2xa^2xa^2xa^2$ such that $a + az = az, a + a^2z = a^2z$ and $a + za^2 = za^2$. Since $B$ is a $k$-bi-ideal of $S$, $z = a(axa^2xa^2xa)a \in BSB \subseteq B$. Consequently, $az \in E_k(B)$. Since $e, az \in E_k(B)$, one gets $B_k(az) = B_k(e)$. Now adding $aza^2z$ on both sides of $a + az = az$, we get $a + az(a + a^2z) = az(a + a^2z)$, that is, $a + az(a)az = az(a)az$. This yields $a \in B_k(az)$. Also $az + azaz = azaz$ gives $az + a(zaa^2xa^2xa^2xa)a = a(zaa^2xa^2xa^2xa)a$, and so $az \in B_k(a)$. Now $B_k(a) = B_k(az) = B_k(e)$, and then by Theorem 4.5, $B$ is a minimal $k$-bi-ideal of $S$. \hfill \Box

In [11], Mondal and Bhuniya defined $\overline{B}$-simple semirings. In this paper we rename it by $k$-bi-simple semirings. Then a semiring $S$ is called $k$-bi-simple if it has no non-trivial proper $k$-bi-ideal.

Example 4.7. Let $\mathbb{R}^+$ denote the set of all positive real numbers, and consider the group $(\mathbb{R}^+, \cdot)$. Let $P_f(\mathbb{R}^+)$ be the set of all finite subsets of $\mathbb{R}^+$. Define $+$ and $\cdot$ on $P_f(\mathbb{R}^+)$ by: $A + B = A \cup B$ and $A \cdot B = \{ab \mid a \in A, b \in B\}$ for all $A, B \in P_f(\mathbb{R}^+)$. Then $(P_f(\mathbb{R}^+), +, \cdot)$ is a $k$-bi-simple semiring.

Then we have the following lemma:

Lemma 4.8. [Lemma 3.1 [11]] In a semiring $S$ the following conditions are equivalent:

1. $S$ is a $t$-$k$-simple semiring;
2. $S$ is a $k$-bi-simple ($\overline{B}$-simple) semiring;
3. $S$ is a $\overline{H}$-simple semiring.


Now as in Remark 2.6 [1], we find that that every $k$-bi-simple semiring is a $k$-regular semiring. Now we are in a position to characterize the $k$-bi-simple semirings via $k$-bi-ideals generated by $k$-idempotent elements.

**Theorem 4.9.** A semiring $S$ is $k$-bi-simple if and only if $S$ is $k$-regular and for all $e, f \in E_k(S), B_k(e) = B_k(f)$.

**Proof.** Let $S$ be a $k$-bi-simple semiring and $e, f \in E_k(S)$. Then by hypothesis, we have $B_k(e) = B_k(f)$. Also, every $k$-bi-simple semiring is $k$-regular. Conversely, suppose that the given conditions hold, and let $B$ be a $k$-bi-ideal of $S$. We are interested to show $B = S$. For, let $s \in S, b \in B$. Since $S$ is $k$-regular, there are $x, y \in S$ such that $s + sx = xs$ and $b + by = yb$. Then $sx, xs, by, yb$ are all in $E_k(S)$. By hypothesis, we get $B_k(sx) = B_k(by)$ and $B_k(xs) = B_k(yb)$. This implies $sxsb = xsbyb$. Then by Lemma 4.2, one gets $sxsb = xsbyb$. Now there exist $u, v \in S$ such that $sx + byu = byu$ and $xs + vyb = vyb$. Now adding $sxvx$ on both sides of $s + sx = xs$ we have $s + sx(s + sx) = sx(s + sx)$. This implies $s + sxvx = sxvx$. Again adding $sxvx + vyb + byvx + byvx$ on both sides one can write $s + (sx + byu)s(xs + vyb) = (sx + byu)s(xs + vyb)$, i.e., $s + byvx + byvx = byvx + byvx \in B$ so that $s \in B_k(b) \subseteq B$ yielding $s \in B$. Thus $S \subseteq B$ so that $S = B$, whence $B$ is $k$-bi-simple.

In the following theorem we characterize the minimal $k$-bi-ideals by its $k$-bi-simplicity. Before that we have the lemma:

**Lemma 4.10.** If $B$ is a $k$-bi-ideal of a semiring $S$, then for $a \in S$, $\overline{aBa}$ is a $k$-bi-ideal of $S$.

**Proof.** Let $x, y \in \overline{aBa}$. Then there exist $b_1, b_2 \in B$ such that $x + ab_1a = ab_1a, y + ab_2a = ab_2a$. These yield $x + ab_1a = ab_1a, y + ab_2a = ab_2a$, where $b = b_1 + b_2 \in B$. Then $(x + y) + ab_1a = ab_1a \in \overline{aBa}$ implies that $x + y \in \overline{aBa}$. If $s \in S, b \in \overline{aBa}$, then there exists a $z \in B$ such that $b + az = az$. Now $zasaz \in B$, since $B$ is a $k$-bi-ideal of $S$. Multiplying both sides of $b + az = az$ by $sb$ on the right, we have $bsb + azasb = azasb$. Adding $zasazb$ on both sides we get $bsb + azasb = azasb$. This implies $bsb + azasb = azasb \in \overline{aBa}$ so that $bsb \in \overline{aBa}$. Thus $\overline{aBaSaBa} \subseteq \overline{aBa}$. Now to show that $\overline{aBa}$ is a $k$-set, suppose that $x \in S, y \in \overline{aBa}$ satisfying $x + y = y$. Now there exists a $b \in B$ such that $y + ab = ab$. This implies $x + y + ab = ab$, that is, $x + ab = ab \in \overline{aBa}$ yielding $x \in \overline{aBa}$. Consequently, $\overline{aBa}$ is a $k$-bi-ideal of $S$.

**Theorem 4.11.** Let $S$ be a semiring. Then a $k$-bi-ideal $B$ is minimal if and only if it is $k$-bi-simple.

**Proof.** Let $B$ be a minimal $k$-bi-ideal of $S$, and $T$ a $k$-bi-ideal of $B$. Let $t \in T$, then one gets $tTt \subseteq TBT \subseteq T = T \subseteq B$. By Lemma 4.10 $tTt$ is a $k$-bi-ideal of $S$, and $B$ is minimal in $S$, we get $tTt = B$. This implies $T = B$, whence $B$ is $k$-bi-simple. Conversely, suppose that $B$ is a $k$-bi-simple, and $C$ a $k$-bi-ideal of $S$ with $C \subseteq B$. Let $c \in C$. Then $cBc$ is a $k$-bi-ideal of $B$. Since $B$ is a
Lemma 3.1, one has

\( B = cBc \subseteq \overline{C} = \overline{C} \subseteq \overline{C} = C \)

yielding \( B = C \). Consequently, \( B \) is minimal.

Finally we characterize the completely \( k \)-regular semirings via \( k \)-bi-ideals generated by \( k \)-idempotent elements.

**Theorem 4.12.** A semiring \( S \) is completely \( k \)-regular semiring if and only if

1. for every \( k \)-bi-ideal \( B \) of \( S \), there is some \( e \in E_k(S) \) such that \( B = B_k(e) \), and
2. for every \( x \in B \), \( B_k(x^2) = B_k(e) \).

**Proof.** Let \( S \) be a completely \( k \)-regular semiring.

(1): Let \( B \) be a \( k \)-bi-ideal of \( S \), and \( a \in B \). Now by the proof of (1) of Lemma 3.1, one has \( a + aza = aza, a + a^2z = a^2z \) and \( a + za^2 = za^2 \), where \( z = a^2xa^2xa^2xa^2 \). Now adding \( za^2z \) on both sides of \( a + aza = aza \) one gets \( a + az(a + a^2z) = az(a + a^2z) \), i.e. \( a + azaz = azaz = eae \in eSe = B_k(e) \) so that \( a \in B_k(e) \) yielding \( B \subseteq B_k(e) \). Now suppose that \( y \in B_k(e) = eSe \). Then there exists an \( u \in S \) such that \( y = azuaz = azuaz \), i.e. \( y + azuaz = azuaz = azua^3xa^2xa^2xa^2 = azua^3xa^2xa^2xa^2 \subseteq aSa \subseteq B_k(a) \subseteq B \) yielding \( B_k(a) = B = B_k(e) \).

(2): Let \( x \in B \). Then \( x^2 \in B \), whence by (1), there exists an \( f \in E_k(S) \) such that \( B = B_k(x^2) = B_k(f) \). Consequently, \( B_k(x^2) = B_k(f) = B = B_k(e) \), by (1).

Conversely, suppose that the conditions hold, and \( a \in S \). Consider the \( k \)-bi-ideal \( B_k(a) \) of \( S \). Then there exists an \( e \in E_k(S) \) such that \( B_k(a) = B_k(e) \).

Since \( a^2 \in B_k(a) \), by condition (2), it follows that \( B_k(e) = B_k(a^2) \). Then \( a + a^2sa^2 = a^2sa^2 \) yielding \( a \) is completely \( k \)-regular element. Consequently, \( S \) is completely \( k \)-regular.

**Acknowledgement**

We express our deepest gratitude to the assigned journal editor Prof. Petar Marković for communicating the paper and to the referees of the paper for their important valuable comments.

**References**


Received by the editors June 2, 2017
First published online June 5, 2018