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AN ITERATIVE METHOD FOR SOLUTION OF FINITE FAMILIES OF SPLIT MINIMIZATION PROBLEMS AND FIXED POINT PROBLEMS

Hammed Anuoluwapo Abass[1], Chinedu Izuchukwu[2], Oluwatosin Temitope Mewomo[34] and Ferdinard Udochukwu Ogbuisi[56]

Abstract

The purpose of this paper is to introduce a proximal iterative algorithm for the approximation of a common solution of finite families of split minimization problem and a fixed point problem in the framework of Hilbert space. Using our iterative algorithm, we prove a strong convergence theorem for approximating a common solution of finite families of split minimization problem and a fixed point problem of nonexpansive mapping. Moreover, our result complements and extends some related results in literature.

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1 Introduction

Let H be a real Hilbert space with inner product and norm as $\langle .,. \rangle$ and $||.||$ respectively. Let C be a nonempty, closed and convex subset of H. A mapping $T : C \to C$ is said to be
(i) a contraction, if there exists a constant $k \in (0, 1)$ such that

$$||Tx - Ty|| \leq k||x - y||, \ \forall \ x, y \in C;$$

(ii) nonexpansive, if

$$||Tx - Ty|| \leq ||x - y||, \ \forall \ x, y \in C.$$

A point $p \in C$ is called a fixed point of T if $Tp = p$. We denote by $F(T)$ the set of all fixed points of T.

[1]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa. e-mail: 216075727@stu.ukzn.ac.za, hammedabass548@gmail.com

[2]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa. e-mail: izuchukwu_c@yahoo.com, izuchukwuc@ukzn.ac.za

[3]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa. e-mail: mewomoo@ukzn.ac.za.

[4]Corresponding author

[5]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa . e-mail: 215082189@stu.ukzn.ac.za, fudochukwu@yahoo.com

[6]Department of Mathematics, University of Nigeria, Nsukka, Nigeria

The iterative approximation of fixed points for nonexpansive mapping have been studied extensively by many authors (see, for example, [5, 8, 9, 12] and the references therein).
For any point $u \in H$, there exists a unique point $P_C u \in C$ such that

$$||u - P_C u|| \leq ||u - y||, \ \forall \ y \in C.$$

P_C is called the metric projection of H onto C. It is also well known that P_C satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq ||P_C x - P_C y||^2.$$

Definition 1.1. A mapping $T : H \to H$ is said to be firmly nonexpansive if and only if $2T - I$ is nonexpansive, where I is the identity mapping, or equivalently

$$\langle x - y, Tx - Ty \rangle \geq ||Tx - Ty||^2, \ \forall \ x, y \in H.$$

Alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive. The metric projection is an example of a firmly nonexpansive mapping.

Definition 1.2. A mapping $T : H \to H$ is said to be an averaged mapping if and only if it can be written as the average of the identity mapping I and a nonexpansive mapping, that is,

$$T = (1 - \alpha)I + \alpha S, \tag{1.1}$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. When (1.1) holds, we say that T is α- averaged.

Definition 1.3. A mapping $T : C \to C$ is said to be
(i) monotone, if

$$\langle Tx - Ty, x - y \rangle \geq 0, \forall \ x, y \in C,$$

(ii) α-inverse strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha ||Tx - Ty||^2, \forall \ x, y \in C.$$

Definition 1.4. Let Q be a convex subset of a vector space X and $f : Q \to \mathbb{R} \cup \{+\infty\}$ be a map. Then,
(i) f is convex if for each $\lambda \in [0, 1]$ and $x, y \in Q$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y);$$

(ii) f is called proper if there exists at least one $x \in Q$ such that

$$f(x) \neq +\infty;$$

(iii) f is lower semi-continuous at $x_0 \in Q$ if

$$f(x_0) \leq \liminf_{x \to x_0} f(x).$$

The Split Feasibility Problem (SFP) was first introduced in [3] by Censor and Elfving. Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2 respectively and $A : H_1 \to H_2$ be a bounded linear operator. The SFP is defined as follows:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q. \tag{1.2}$$

The SFP arises in many fields in the real world, such as signal processing, image reconstruction and intesity-modulated radiation therapy problems. Since its origin, several iterative algorithms have been proposed and analysed to solve it (see [7, 13, 15, 16, 18, 20] and the references therein).

One of the most important problems in optimization theory and non-linear analysis is the problem of approximating solution of Minimization Problem (MP) which is to find $x \in H$ such that

$$f(x) = \min_{y \in H} f(y), \tag{1.3}$$

where $f : H \to (-\infty, \infty]$ is a proper and convex function. We denote by $argmin_{y\in H} f(y)$ the set of all minimizers of f on H.
Recently, Moudafi and Thakur [13] considered the following MP,

$$\min\{g(x) + f_\lambda(Ax) : x \in H_1\}; \tag{1.4}$$

where $g : H_1 \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semi-continuous function, and $f_\lambda(y) := \min_{u\in H_2}\{f(u) + \frac{1}{2\lambda}||u - y||^2\}$ is the Moreau-Yosida approximate of the function f of parameter λ also called the proximal operator of f of order λ and $A : H_1 \to H_2$ is a bounded linear operator. For $\lambda > 0$, the Moreau-Yosida resolvent of f in Hilbert space is defined as follows:

$$J_\lambda^f(x) = Prox_\lambda f(x) = argmin_{y\in H}\{f(y) + \frac{1}{2\lambda}||y - x||^2\}, \ \forall\, x \in H, \tag{1.5}$$

where $argmin\ f := \{\overline{x} \in H : f(\overline{x}) \leq f(x)\ for\ all\ x \in H\}$.
Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2 $g : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $f : H_2 \to \mathbb{R} \cup \{+\infty\}$ be two proper and lower semi-continuous convex functions. Let $A : H_1 \to H_2$ be a bounded linear operator, then the Split Minimization Problem (SMP) is to find

$$x^* \in C \text{ such that } x^* = argmin_{x\in C}\, g(x), \tag{1.6}$$

and such that

$$\text{the point } y^* = Ax^* \in Q \text{ solves } y^* = argmin_{y\in Q}\, f(y). \tag{1.7}$$

In this paper, we consider the finite families of SMP, which is to find

$$x^* \in C \text{ such that } x^* = \cap_{i=1}^N argmin_{x\in C}\, g_i(x), \tag{1.8}$$

and such that

$$\text{the point } y^* = Ax^* \in Q \text{ solves } y^* = \cap_{j=1}^m argmin_{y\in Q}\, f_j(y). \tag{1.9}$$

We denote the solution set of problem (1.8)-(1.9) by Θ.
For $\lambda > 0$, $x \in H_1$, we define

$$h(x) := \frac{1}{2}||(I - Prox_{\lambda f})Ax||^2; \tag{1.10}$$

$$l(x) := \frac{1}{2}||(I - Prox_{\lambda g})x||^2; \tag{1.11}$$

$$\theta(x) := \sqrt{||\nabla h(x)||^2 + ||\nabla l(x)||^2}; \tag{1.12}$$

and

$$\gamma_n := \rho_n \frac{h(x_n) + l(x_n)}{\theta^2(x_n)}, \ n \geq 1, \tag{1.13}$$

where $0 < \rho_n < 4$. Then the gradients ∇h and ∇l of h and l, respectively, are

$$\nabla h(x) := A^*(I - Prox_{\lambda f})Ax; \tag{1.14}$$

and

$$\nabla l(x) := (I - Prox_{\lambda g})x. \tag{1.15}$$

Using (1.10)-(1.13), Moudafi and Thakur [13] studied the following proximal point algorithm and proved a weak convergence theorem for the sequence generated by their algorithm to a solution of SMP (1.6)-(1.7):
Given an initial point $x_1 \in H_1$, assume that x_n has been constructed and $\theta(x_n) \neq 0$, then compute x_{n+1} as follows:

$$x_{n+1} = Prox_\lambda g(x_n - \mu_n A^*(I - Prox_{\lambda f})Ax_n), \ \forall \ n \geq 1. \tag{1.16}$$

If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of MP (1.4) and the iterative process stops, otherwise, we set n:= n+1 and go to (1.16).

Very Recently, Abbas et. al. [1] proposed two iterative algorithms which generate sequences that converge strongly to a solution of SMP (1.6)-(1.7). Using (1.10)-(1.13), they proposed the following modified split proximal point algorithm and proved that the sequence generated by their iterative scheme converges strongly to a solution of SMP (1.6)-(1.7):
Given an initial point $x_1 \in H_1$, assume that x_n has been constructed and $\theta(x_n) \neq 0$, then compute x_{n+1} by the following iterative scheme;

$$x_{n+1} = Prox_{\lambda g}((1 - \epsilon_n)x_n - \gamma_n A^*(I - Prox_{\lambda f})Ax_n); \text{ for } n \geq 1; \tag{1.17}$$

where stepsize $\gamma_n := \rho_n \frac{h(x_n)+l(x_n)}{\theta^2(x_n)}$ with $0 < \rho_n < 4$. If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of SMP (1.6)-(1.7) and the iterative process stops, otherwise, we set $n := n + 1$ and go to (1.17).
They proved the following theorem.

Theorem 1.5. *[1] Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator $A^* : H_2 \to H_1$. Assume that $g : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $f : H_2 \to \mathbb{R} \cup \{+\infty\}$ are proper, convex and lower semi-continuous functions, and that SMP (1.6)-(1.7) is consistent. Let $\{\epsilon_n\}$ be a sequence in $(0,1)$ such that the following conditions hold:*
(a) $\lim_{n\to\infty} \epsilon_n = 0$;
(b) $\sum_{n=1}^{\infty} \epsilon_n = \infty$;
(c) $a \leq \rho_n \leq \frac{4(1-\epsilon_n)h(x_n)}{h(x_n)+l(x_n)} - a$ *for some* $a > 0$.
Then the sequence $\{x_n\}$ generated by (1.17) converges strongly to a solution x^* of SMP (1.6)-(1.7).

The proximal point algorithm have been used extensively by many authors to solve the SFP and MP (see [13, 15] and the references therein).

In 2017, Shehu and Iyiola [15] proposed the following modified proximal point split feasibility iterative scheme:

Algorithm (1.1)
(1) Given the initial points $x_1, u \in H_1$;
(2) Set $n := 1$ and compute;
(3) $y_n = \alpha_n u + (1-\alpha_n)x_n$;
(4) $\theta(y_n) = ||A^*(I - prox_{\lambda f})Ay_n + (I - prox_{\lambda g})y_n||$;
(5) $z_n = y_n - \rho_n \frac{h(y_n)+l(y_n)}{\theta^2(y_n)}$;
(6) $x_{n+1} = (1-\beta_n)y_n + \beta_n z_n$;
(7) If $A^*(I - prox_{\lambda f})Ay_n = 0 = (I - prox_{\lambda g})y_n$ and $x_{n+1} = x_n$, then stop, otherwise;
(8) Set $n := n+1$ and repeat step(3)-(6).
They proved that Algorithm (1.1) converges strongly to a solution of (1.4).

Remark 1.6. We observe that the method of proof used in [1] is divided into two cases, but we were able to prove our strong convergence theorem without dividing our method of proof into two cases. The method of proof in this paper looks shorter and easier to read.
In Theorem 1.5, they imposed $a \leq \rho_n \leq \frac{4(1-\epsilon_n)h(x_n)}{h(x_n)+l(x_n)} - a$ for some $a > 0$ on their iterative scheme to prove a strong convergence theorem. We were able to prove a strong convergence result without imposing this condition.
Most authors working in this direction have considered either the SMP or MP (see [1, 2, 10] and the references therein), but in this paper, we considered the finite families of SMP.

Motivated by the works of Shehu and Iyiola [14, 15], Yao et. al. [20] and other researchers working in this direction, we introduce a proximal point iterative algorithm for approximating the common solution of finite family of SMP and fixed point problem in the framework of Hilbert space. Using our iterative algorithm, we prove a strong convergence theorem for approximating the common solution of split minimization problem and fixed point problem of nonexpansive mapping. Our method of proof is quite different from others working in this direction, see([13, 20] and others therein).

2 Preliminaries

In this section, we state some well known results which will be used in the sequel. Throughout this paper, we denote the weak and strong convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. We also denote by $\prod_{j=1}^{n} J_\mu^j x = J_\mu^1 \circ J_\mu^2 \circ \cdots J_\mu^n$ the composition of our resolvents.

Lemma 2.1. *[4] Let H be a real Hilbert space. Then the following identities hold:*
(I) $2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x-y||^2 = ||x+y||^2 - ||x||^2 - ||y||^2, \forall\ x, y \in H.$
(II) $||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2.$

Lemma 2.2. *[19] Let H be a real Hilbert space and $T : H \to H$ be a nonlinear mapping, then the following hold:*
(i) f is nonexpansive if and only if the complement $I - f$ is $\frac{1}{2}$-ism.
(ii) If f is ν-ism and $\gamma > 0$, then γf is $\frac{\nu}{\gamma}$-ism.
(iii) f is averaged if and only if the complement $I - f$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\beta \in (0, 1)$, f is β-averaged if and only if $I - f$ is $\frac{1}{2\beta}$-ism.
(iv) If f_1 is β_1-averaged and f_2 is β_2-averaged, where $\beta_1, \beta_2 \in (0, 1)$, then the composite $f_1 f_2$ is β-averaged, where $\beta = \beta_1 + \beta_2 - \beta_1\beta_2$.

Lemma 2.3. *[16] Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator with $A \neq 0$, and $S : H_2 \to H_2$ be a nonexpansive mapping. Then $A^*(I - S)A$ is $\frac{1}{2||A||^2}$-ism.*

Lemma 2.4. *[17] Let H_1 and H_2 be real Hilbert spaces. Let C be a nonempty, closed and convex subset of H_1. Let $S : H_2 \to H_2$ be a nonexpansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $C \cap A^{-1}F(S) \neq \emptyset$. Let $\gamma > 0$ and $x^* \in H_1$. Then the following are equivalent.*

(i) $x^* = P_C(I - \gamma A^*(I - S)A)x^*$;

(ii) $0 \in A^*(I - S)Ax^* + N_C x^*$;

(iii) $x^* \in C \cap A^{-1}F(S)$.

Lemma 2.5. *[21] Let C be a nonempty, closed and convex subset of a real Hilbert space H and $S : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0 (i.e., if $\{x_n\}$ converges weakly to $x \in C$ and $\{x_n - Tx_n\}$ converges strongly to 0, then $x = Tx$).*

Lemma 2.6. *[6]. Let H be a real Hilbert space and $f : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $x, y \in H$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda}||J_\lambda x - y||^2 - \frac{1}{2\lambda}||x - y||^2 + \frac{1}{2\lambda}||x - J_\lambda x||^2 + f(J_\lambda x) \leq f(y).$$

Lemma 2.7. *[11] Let $\{a_n\}$ be a sequence of non-negative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n r_n,$$

where $\{r_n\}$ is a sequence of real numbers bounded from above and $\{\alpha_n\} \subset [0,1]$ satisfies $\sum \alpha_n = \infty$. Then

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} r_n.$$

3 Main Result

Throughout this paper, we shall denote by J_λ^f, $\lambda > 0$, the resolvent of a proper convex and lower semi-continuous function f.

Lemma 3.1. *Let H be a real Hilbert space and $f : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for $0 < \lambda \leq \mu$ and $x \in H$, we have $||J_\lambda x - x|| \leq ||J_\mu x - x||$.*

Proof. For $x, y \in H$, we obtain from the definition of the resolvent of f that

$$f(J_\mu x) + \frac{1}{2\mu}||J_\mu x - x||^2 \leq f(y) + \frac{1}{2\mu}||y - x||^2.$$

In particular, we have that

$$f(J_\mu x) + \frac{1}{2\mu}||J_\mu x - x||^2 \leq f(J_\lambda x) + \frac{1}{2\mu}||J_\lambda x - x||^2. \tag{3.1}$$

Similarly, we obtain

$$f(J_\lambda x) + \frac{1}{2\lambda}||J_\lambda x - x||^2 \leq f(J_\mu x) + \frac{1}{2\lambda}||J_\mu x - x||^2. \tag{3.2}$$

Adding (3.1) and (3.2), we obtain that

$$||J_\lambda x - x||^2 - \frac{\lambda}{\mu}||J_\lambda x - x||^2 \leq ||J_\mu x - x||^2 - \frac{\lambda}{\mu}||J_\mu x - x||^2.$$

That is,

$$\left(1 - \frac{\lambda}{\mu}\right)||J_\lambda x - x||^2 \leq \left(1 - \frac{\lambda}{\mu}\right)||J_\mu x - x||^2.$$

Since $0 < \lambda \leq \mu$, we obtain that

$$||J_\lambda x - x|| \leq ||J_\mu x - x||.$$

□

Lemma 3.2. *Let H be a real Hilbert space and $f_j : H \to (-\infty, \infty]$, $j = 1, 2, \ldots, m$ be proper convex and lower semi-continuous functions. Let $T : H \to H$ be a nonexpansive mapping, then for $0 < \lambda \leq \mu$, we have that*

$$F\left(T\prod_{j=1}^{m} J_{\mu}^{(j)}\right) \subseteq \left(F(T) \cap \left(\cap_{j=1}^{m} F\left(J_{\lambda}^{(j)}\right)\right)\right).$$

Proof. For $x \in F\left(T\prod_{j=1}^{m} J_{\mu}^{(j)}\right)$ and $y \in \left(F(T) \cap \left(\cap_{j=1}^{m} F\left(J_{\mu}^{(j)}\right)\right)\right)$, we have that

$$\begin{aligned} ||x-y||^2 &= ||T\prod_{j=1}^{m} J_{\mu}^{(j)} x - T\prod_{j=1}^{m} J_{\mu}^{(j)} y||^2 \\ &\leq ||\prod_{j=1}^{m} J_{\mu}^{(j)} x - \prod_{j=1}^{m} J_{\mu}^{(j)} y||^2 \\ &= ||\prod_{j=1}^{m} J_{\mu}^{(j)} x - y||^2. \end{aligned} \tag{3.3}$$

Furthermore, we obtain from Lemma 2.6 that

$$\frac{1}{2\mu}||\prod_{j=1}^{m} J_{\mu}^{(j)} x - y||^2 - \frac{1}{2\mu}||\prod_{j=2}^{m} J_{\mu}^{(j)} x - y||^2 + \frac{1}{2\mu}||\prod_{j=2}^{m} J_{\mu}^{(j)} x - \prod_{j=1}^{m} J_{\mu}^{(j)} x||^2$$
$$+ f(\prod_{j=1}^{m} J_{\mu}^{(j)} x) \leq f(y).$$

Since $f(y) \leq f(\prod_{j=1}^{m} J_{\mu}^{(j)} x)$, we obtain from (3.3) that

$$\begin{aligned} ||\prod_{j=2}^{m} J_{\mu}^{(j)} x - \prod_{j=1}^{m} J_{\mu}^{(j)} x||^2 &\leq ||\prod_{j=2}^{m} J_{\mu}^{(j)} x - y||^2 - ||\prod_{j=1}^{m} J_{\mu}^{(j)} x - y||^2 \\ &\vdots \\ &\leq ||x-y||^2 - ||\prod_{j=1}^{m} J_{\mu}^{(j)} x - y||^2 \\ &\leq ||\prod_{j=1}^{m} J_{\mu}^{(j)} x - y||^2 - ||\prod_{j=1}^{m} J_{\mu}^{(j)} x - y||^2, \end{aligned}$$

which implies

$$\prod_{j=1}^{m} J_{\mu}^{(j)} x = \prod_{j=2}^{m} J_{\mu}^{(j)} x. \tag{3.4}$$

Similarly, we obtain from Lemma 2.6 and (3.3) that

$$\begin{aligned}
||\prod_{j=3}^{m} J_\mu^{(j)}x - \prod_{j=2}^{m} J_\mu^{(j)}x||^2 &\leq ||\prod_{j=3}^{m} J_\mu^{(j)}x - y||^2 - ||\prod_{j=2}^{m} J_\mu^{(j)}x - y||^2 \\
&\vdots \\
&\leq ||x - y||^2 - ||\prod_{j=2}^{m} J_\mu^{(j)}x - y||^2 \\
&\leq ||\prod_{j=1}^{m} J_\mu^{(j)}x - y||^2 - ||\prod_{j=1}^{m} J_\mu^{(j)}x - y||^2,
\end{aligned}$$

which implies

$$\prod_{j=2}^{m} J_\mu^{(j)}x = \prod_{j=3}^{m} J_\mu^{(j)}x. \tag{3.5}$$

Continuing in this manner, we obtain that

$$\prod_{j=3}^{m} J_\mu^{(j)}x = \prod_{j=4}^{m} J_\mu^{(j)}x = \cdots = \prod_{j=m-1}^{m} J_\mu^{(j)}x = J_\mu^{(m)}x = x. \tag{3.6}$$

From (3.6), we have

$$x = J_\mu^{(m)}x. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$x = \prod_{j=m-1}^{m} J_\mu^{(j)}x = J_\mu^{(m-1)}J_\mu^{(m)}x = J_\mu^{(m-1)}x. \tag{3.8}$$

Continuing in this manner, we obtain from (3.4)-(3.8) that

$$x = J_\mu^{(m-2)}x = \cdots = J_\mu^{(2)}x = J_\mu^{(1)}x. \tag{3.9}$$

That is,

$$J_\mu^{(1)}x = J_\mu^{(2)}x = \cdots = J_\mu^{(m-1)}x = J_\mu^{(m)}x = x. \tag{3.10}$$

Furthermore, we get from (3.4)-(3.6) that

$$x = T\prod_{j=1}^{m} J_\mu^{(j)} = Tx. \tag{3.11}$$

Now, since $0 < \lambda \leq \mu$, we obtain from Lemma 3.1 and (3.10) that

$$||x - J_\lambda^{(j)}x|| \leq ||x - J_\mu^{(j)}x|| = 0, \ j = 1, 2, \ldots, m,$$

which implies that $x \in F(J_\lambda^{(j)})$, $j = 1, 2, \ldots, m$. This together with (3.11) implies that $x \in \left(F(T) \cap \left(\cap_{j=1}^m F\left(J_\lambda^{(j)}\right)\right)\right)$. Therefore, we conclude that

$$F\left(T\prod_{j=1}^{m} J_\mu^{(j)}\right) \subseteq \left(F(T) \cap \left(\cap_{j=1}^m F\left(J_\lambda^{(j)}\right)\right)\right).$$

□

Theorem 3.3. *Let H_1 and H_2 be two real Hilbert spaces and C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and θ a contraction mapping with coefficient $\tau \in (0, 1)$. Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be two nonexpansive mappings. For $i = 1, 2, ...n$ and $j = 1, 2, ...m$, let $g_i : H_1 \to (-\infty, +\infty]$ and $f_j : H_2 \to (-\infty, +\infty]$ be two families of proper, convex and lower semi continuous functions. Assume that $\Gamma := \{z \in F(T) : z \in \cap_{i=1}^N argmin_{y\in H} g_i(y)$ and $Az \in F(S)$ such that $Az \in \cap_{j=1}^m argmin_{y\in H} f_j(y)\} \neq \emptyset$ and the sequence $\{x_n\}$ is generated for arbitrary $x_1, u \in H_1$ by*

$$\begin{cases} x_{n+1} = (1-\beta_n)y_n + \beta_n z_n; \\ z_n = (1-t_n)y_n + t_n T\prod_{i=1}^n J_{\lambda_n}^{(i)}(y_n); \\ y_n = P_C\left(u_n - \gamma_n A^*\left(I - S\prod_{j=1}^m J_{\lambda_n}^{(j)}\right)Au_n\right); \\ u_n = (1-\alpha_n)x_n + \alpha_n\theta(x_n),\ n \geq 1; \end{cases} \tag{3.12}$$

where $\{\gamma_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{||A||^2}\right), 0 < \lambda \leq \lambda_n$ and $\{\alpha_n\}, \{\beta_n\}, \{t_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n < 1$.
Then, the sequence $\{x_n\}$ converges strongly to $z \in \Gamma$, where $z = P_\Gamma\theta(z)$.

Proof. Let $z = P_\Gamma\theta(z)$, then $z = T\prod_{i=1}^N J_{\lambda_n}^{(i)}(z)$ and $Az = S\prod_{j=1}^m J_{\lambda_n}^{(j)}(Az)$. Also, we know that the composition $S\prod_{j=1}^m J_{\lambda_n}^{(j)}$ is nonexpansive. Thus, it follows from Lemma 2.2 (ii), (iii), (iv) and Lemma 2.3 that

$$P_C\left(I - \gamma_n A^*\left(I - S\prod_{j=1}^m J_{\lambda_n}^{(j)}\right)A\right)$$

is $\frac{1+\gamma_n||A||^2}{2}$-averaged. Hence y_n can be written as

$$y_n = (1-\mu_n)u_n + \mu_n T_n u_n, \tag{3.13}$$

where T_n is nonexpansive and $\mu_n = \frac{1+\gamma_n||A||^2}{2}$. Thus, from (3.13) and Lemma

2.1 (II), we obtain that

$$
\begin{aligned}
||y_n - z||^2 &= ||(1-\mu_n)u_n + \mu_n T_n u_n - z||^2 \\
&= (1-\mu_n)||u_n - z||^2 + \mu_n||T_n u_n - z||^2 \\
&- \mu_n(1-\mu_n)||u_n - T_n u_n||^2 \\
&= ||u_n - z||^2 - \mu_n(1-\mu_n)||u_n - T_n u_n||^2. \qquad (3.14)
\end{aligned}
$$

From (3.12) and Lemma 2.1 (II), we obtain that

$$
\begin{aligned}
||x_{n+1} - z||^2 &= ||(1-\beta_n)y_n + \beta_n z_n - z||^2 \\
&= (1-\beta_n)||y_n - z||^2 + \beta_n||z_n - z||^2 \\
&- \beta_n(1-\beta_n)||y_n - z_n||^2. \qquad (3.15)
\end{aligned}
$$

Also, we obtain from (3.12) that

$$
\begin{aligned}
||z_n - z||^2 =&||(1-t_n)y_n + t_n T\Pi_{i=1}^N J_{\lambda_n}^{(i)}(y_n) - z||^2 \\
=&(1-t_n)||y_n - z||^2 + t_n||T\Pi_{i=1}^N J_{\lambda_n}^{(i)}(y_n) - z||^2 \\
&- t_n(1-t_n)||y_n - T\Pi_{i=1}^N J_{\lambda_n}^{(i)}(y_n)||^2 \\
\leq&(1-t_n)||y_n - z||^2 \\
&+ t_n||y_n - z||^2 - t_n(1-t_n)||y_n - T\Pi_{i=1}^N J_{\lambda_n}^{(i)}(y_n)||^2 \\
\leq&||y_n - z||^2. \qquad (3.16)
\end{aligned}
$$

Again from (3.12), we have that

$$
z_n - y_n = \frac{1}{\beta_n}(x_{n+1} - y_n). \qquad (3.17)
$$

Similarly, we have from (3.13) that

$$
T_n u_n - u_n = \frac{1}{\mu_n}(y_n - u_n). \qquad (3.18)
$$

Now, from (3.14), (3.4), (3.17) and (3.18), we have that

$$
\begin{aligned}
||x_{n+1} - z||^2 &\leq (1-\beta_n)||y_n - z||^2 + \beta_n||y_n - z||^2 \\
&- \beta_n(1-\beta_n)||y_n - z_n||^2 \\
&= ||y_n - z||^2 - \frac{1}{\beta_n}(1-\beta_n)||x_{n+1} - y_n||^2 \\
&\leq ||u_n - z||^2 - \frac{1}{\mu_n}(1-\mu_n)||y_n - u_n||^2 \\
&- \frac{1}{\beta_n}(1-\beta_n)||x_{n+1} - y_n||^2, \qquad (3.19)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
||x_{n+1} - z|| &\leq ||u_n - z|| \\
&= ||(1-\alpha_n)x_n + \alpha_n\theta(x_n) - z|| \\
&\leq \alpha_n||\theta(x_n) - z|| + (1-\alpha_n)||x_n - z|| \\
&\leq \alpha_n\tau||x_n - z|| + \alpha_n||\theta(z) - z|| + (1-\alpha_n)||x_n - z|| \\
&= (1-\alpha_n(1-\tau))||x_n - z|| + \alpha_n||\theta(z) - z|| \\
&\leq \max\Big\{||x_n - z||, \frac{||\theta(z) - z||}{1-\tau}\Big\} \\
&\vdots \\
&\leq \max\Big\{||x_1 - z||, \frac{||\theta(z) - z||}{1-\tau}\Big\}.
\end{aligned}
$$

Therefore, $\{x_n\}$ is bounded and consequently, $\{u_n\}, \{y_n\}$ and $\{z_n\}$ are also bounded.
From (3.17), we have that

$$
\begin{aligned}
||z_n - y_n||^2 &= ||\frac{1}{\beta_n}(x_{n+1} - y_n)||^2 \\
&= \frac{1}{\beta_n^2}||x_{n+1} - y_n||^2 \\
&= \frac{\alpha_n}{\beta_n}\left(\frac{||x_{n+1} - y_n||^2}{\alpha_n\beta_n}\right). \qquad (3.20)
\end{aligned}
$$

Also, from (3.18), we obtain

$$
||u_n - T_nu_n||^2 = \frac{\alpha_n}{\mu_n}\left(\frac{||y_n - u_n||^2}{\alpha_n\mu_n}\right). \qquad (3.21)
$$

From Lemma 2.1 (I) and (3.12), we have

$$
\begin{aligned}
||u_n - z||^2 &= ||\alpha_n(\theta(x_n) - z) + (1-\alpha_n)(x_n - z)||^2 \\
&= \alpha_n^2||\theta(x_n) - z||^2 + 2\alpha_n(1-\alpha_n)\langle\theta(x_n) - z, x_n - z\rangle \\
&+ (1-\alpha_n)^2||x_n - z||^2 \\
&\leq \alpha_n^2||\theta(x_n) - z||^2 - 2\alpha_n(1-\alpha_n)\langle\theta(x_n) - z, z - x_n\rangle \\
&+ (1-\alpha_n)||x_n - z||^2. \qquad (3.22)
\end{aligned}
$$

Substituting (3.22) into (3.19), we obtain

$$\begin{aligned}
||x_{n+1} - z||^2 \le{}& \alpha_n^2||\theta(x_n) - z||^2 - 2\alpha_n(1-\alpha_n)\langle\theta(x_n) - z, z - x_n\rangle \\
&+ (1-\alpha_n)||x_n - z||^2 - \frac{1}{\beta_n}(1-\beta_n)||x_{n+1} - y_n||^2 \\
&- \frac{1}{\mu_n}(1-\mu_n)||y_n - u_n||^2 \\
={}& (1-\alpha_n)||x_n - z||^2 - \alpha_n\big(-\alpha_n||\theta(x_n) - z||^2 \\
&+ 2(1-\alpha_n)\langle\theta(x_n) - z, z - x_n\rangle \\
&+ \frac{1}{\alpha_n\beta_n}(1-\beta_n)||x_{n+1} - y_n||^2 \\
&+ \frac{1}{\alpha_n\mu_n}(1-\mu_n)||y_n - u_n||^2\big). \qquad (3.23)
\end{aligned}$$

Let

$$\begin{aligned}
\Upsilon_n :={}& -\alpha_n||\theta(x_n) - z||^2 + 2(1-\alpha_n)\langle\theta(x_n) - z, z - x_n\rangle \\
&+ \frac{1}{\alpha_n\beta_n}(1-\beta_n)||x_{n+1} - y_n||^2 + \frac{1}{\alpha_n\mu_n}(1-\mu_n)||y_n - u_n||^2. \qquad (3.24)
\end{aligned}$$

Thus, (3.23) becomes

$$||x_{n+1} - z||^2 \le (1-\alpha_n)||x_n - z||^2 - \alpha_n\Upsilon_n.$$

Since $\{x_n\}$ is bounded, it is bounded below. Hence, $\{\Upsilon_n\}$ is bounded below. Furthermore, using Lemma 2.7 and condition (i) in (3.12), we obtain

$$\begin{aligned}
\limsup_{n\to\infty} ||x_n - z||^2 &\le \limsup_{n\to}(-\Upsilon_n) \\
&= -\liminf_{n\to}\Upsilon_n. \qquad (3.25)
\end{aligned}$$

Therefore, $\liminf_{n\to\infty}\Upsilon_n$ is finite. We have from (3.24) that

$$\begin{aligned}
\liminf_{n\to\infty}\Upsilon_n = \liminf_{n\to\infty}&\big(2\langle\theta(x_n) - z, z - x_n\rangle \\
&+ \frac{1}{\alpha_n\beta_n}(1-\beta_n)||x_{n+1} - y_n||^2 \\
&+ \frac{1}{\alpha_n\mu_n}(1-\mu_n)||y_n - u_n||^2\big).
\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q \in H$ and

$$\begin{aligned}
\liminf_{n\to\infty}\Upsilon_n = \liminf_{k\to\infty}&\big(2\langle\theta(x_{n_k}) - z, z - x_{n_k}\rangle \\
&+ \frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})||x_{n_k+1} - y_{n_k}||^2 \\
&+ \frac{1}{\alpha_{n_k}\mu_{n_k}}(1-\mu_{n_k})||y_{n_k} - u_{n_k}||^2\big). \qquad (3.26)
\end{aligned}$$

Since $\{x_n\}$ is bounded and $\liminf_{n\to\infty} \Upsilon_n$ is finite, we have that $\frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})||x_{n_k+1}-y_{n_k}||^2$ and $\frac{1}{\alpha_{n_k}\mu_{n_k}}(1-\mu_{n_k})||y_{n_k}-u_{n_k}||^2$ are bounded. Also, by assumption (ii), we have that there exists $b\in(0,1)$ such that $\beta_n \le b < 1$ and this implies that $\frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k}) \ge \frac{1}{\alpha_{n_k}\beta_{n_k}}(1-b) > 0$ and we have that $\{\frac{1}{\alpha_{n_k}\beta_{n_k}}||x_{n_k+1}-y_{n_k}||^2\}$ is bounded. Now, observe from assumptions (i) and (ii) that there exists $a\in(0,1)$ such that

$$0 < \frac{\alpha_{n_k}}{\beta_{n_k}} \le \frac{\alpha_{n_k}}{a} \to 0, k\to\infty.$$

Following the same argument as in above, and using assumption (i) and the definition of γ_n, we obtain that

$$0 < \frac{\alpha_{n_k}}{\mu_{n_k}} \le \frac{\alpha_{n_k}}{a} \to 0, k\to\infty.$$

Therefore, we obtain from (3.20) that

$$\lim_{k\to\infty} ||z_{n_k}-y_{n_k}|| = 0. \tag{3.27}$$

Similarly, we obtain that

$$\lim_{k\to\infty} ||T_{n_k}-u_{n_k}|| = 0. \tag{3.28}$$

From (3.17) and (3.27), we obtain that

$$||x_{n+1}-y_{n_k}|| = \beta_{n_k}||z_{n_k}-y_{n_k}|| \to 0, k\to\infty. \tag{3.29}$$

Also, from (3.18) and (3.28), we obtain

$$||y_{n_k}-u_{n_k}|| = \mu_{n_k}||T_{n_k}u_{n_k}-u_{n_k}|| \to 0, k\to\infty. \tag{3.30}$$

Furthermore, from (3.12) and condition (i), we obtain that

$$||u_{n_k}-x_{n_k}|| = \alpha_{n_k}||\theta(x_{n_k})-x_{n_k}|| \to 0, k\to\infty. \tag{3.31}$$

Hence,

$$||y_{n_k}-x_{n_k}|| \le ||y_{n_k}-u_{n_k}|| + ||u_{n_k}-x_{n_k}|| \to 0, k\to\infty. \tag{3.32}$$

Now, set $v_n^{(i)} = J_{\lambda_n}^{(i)} v_n^{(i+1)}, i = 1,2,....N$, where $v_n^{(N+1)} = y_n, \forall n \ge 1$. Then, $v_n^{(N)} = J_{\lambda_n}^{(N)}(y_n)$, $v_n^{(N-1)} = J_{\lambda_n}^{(N-1)} J_{\lambda_n}^{(N)}(y_n), \ldots, v_n^{(2)} = \prod_{i=2}^N J_{\lambda_n}^{(i)}(y_n)$, $v_n^{(1)} = \prod_{i=1}^N J_{\lambda_n}^{(i)}(y_n)$. Thus, from (3.16) and (3.27), we obtain that

$$\begin{aligned} t_{n_k}(1-t_{n_k})||y_{n_k} - T\prod_{i=1}^N J_{\lambda_{n_k}}^{(i)}(y_{n_k})||^2 &\le ||y_{n_k}-z||^2 - ||z_{n_k}-z||^2 \\ &\le ||y_{n_k}-z_{n_k}||^2 \\ &\quad +2||y_{n_k}-z_{n_k}||||z_{n_k}-z||. \end{aligned}$$

Thus, by condition (iii), we obtain that

$$\lim_{k\to\infty} ||y_{n_k} - Tv_{n_k}^{(1)}|| = 0. \tag{3.33}$$

Now, using Lemma 2.6, we obtain for $i = 1$ that

$$\begin{aligned} &\frac{1}{2\lambda_{n_k}}||z - v_{n_k}^{(1)}||^2 - \frac{1}{2\lambda_{n_k}}||z - v_{n_k}^{(2)}||^2 \\ &+\frac{1}{2\lambda_{n_k}}||v_{n_k}^{(2)} - v_{n_k}^{(1)}||^2 + f(v_{n_k}^{(1)}) \leq f(z). \end{aligned}$$

Since $f(z) \leq f(v_{n_k}^{(1)})$, we obtain from (3.33) that

$$\begin{aligned} ||v_{n_k}^{(1)} - v_{n_k}^{(2)}||^2 &\leq ||z - v_{n_k}^{(2)}||^2 - ||z - v_{n_k}^{(1)}||^2 \\ &\leq ||z - y_{n_k}||^2 - ||z - v_{n_k}^{(1)}||^2 \\ &\leq ||z - y_{n_k}||^2 - ||z - Tv_{n_k}^{(1)}||^2 \\ &\leq ||y_{n_k} - Tv_{n_k}^{(1)}||^2 + 2||z - Tv_{n_k}^{(1)}||||y_{n_k} - Tv_{n_k}^{(1)}|| \to 0, \end{aligned} \tag{3.34}$$

as $k \to \infty$.
Following similar argument as above, we obtain that

$$\begin{aligned} ||v_{n_k}^{(2)} - v_{n_k}^{(3)}||^2 &\leq ||z - v_{n_k}^{(3)}||^2 - ||z - v_{n_k}^{(2)}||^2 \\ &\leq ||z - y_{n_k}||^2 - ||z - v_{n_k}^{(1)}||^2 \\ &\leq ||z - y_{n_k}||^2 - ||z - Tv_{n_k}^{(1)}||^2 \to 0,\ k \to \infty. \end{aligned} \tag{3.35}$$

Continuing in the same manner, we can show that

$$\begin{aligned} \lim_{k\to\infty} ||v_{n_k}^{(3)} - v_{n_k}^{(4)}|| &= \cdots = \lim_{k\to\infty} ||v_{n_k}^{(N-1)} - v_{n_k}^{(N)}|| = \\ &\lim_{k\to\infty} ||v_{n_k}^{(N)} - v_{n_k}^{(N+1)}|| = 0. \end{aligned} \tag{3.36}$$

From (3.33)-(3.36), we obtain that

$$\begin{aligned} ||v_{n_k}^{(1)} - y_{n_k}|| &\leq ||v_{n_k}^{(1)} - v_{n_k}^{(2)}|| + ||v_{n_k}^{(2)} - v_{n_k}^{(3)}|| + \cdots + ||v_{n_k}^{(N)} - y_n|| \\ &= ||v_{n_k}^{(1)} - v_{n_k}^{(2)}|| + ||v_{n_k}^{(2)} - v_{n_k}^{(3)}|| + \cdots + ||v_{n_k}^{(N)} - v_{n_k}^{(N+1)}|| \to 0, \end{aligned} \tag{3.37}$$

as $k \to \infty$.
Also, using Lemma 2.6, we obtain for each $i = 1, 2, ...N$ that

$$\frac{1}{2\lambda_{n_k}}||z - v_{n_k}^{(i)}||^2 - \frac{1}{2\lambda_{n_k}}||z - v_{n_k}^{(i+1)}||^2 + \frac{1}{2\lambda_{n_k}}||v_{n_k}^{(i+1)} - v_{n_k}^{(i)}||^2 + f(v_{n_k}^{(i)}) \leq f(z).$$

Since $f(z) \leq f(v_{n_k}^{(i)})$, we obtain that

$$||v_{n_k}^{(i)} - v_{n_k}^{(i+1)}||^2 \leq ||z - v_{n_k}^{(i+1)}||^2 - ||z - v_{n_k}^{(i)}||^2.$$

Taking sum in the above inequality from $i = 1$ to $i = N$, we obtain from (3.37) that

$$\begin{aligned}\sum_{i=1}^{N} ||v_{n_k}^{(i)} - v_{n_k}^{(i+1)}||^2 &\leq ||z - v_{n_k}^{(N+1)}||^2 - ||z - v_{n_k}^{(1)}||^2 \\ &= ||z - y_{n_k}||^2 - ||z - v_{n_k}^{(1)}||^2 \to 0 \text{ as } n \to \infty.\end{aligned}$$

This implies that

$$\lim_{k\to\infty} ||v_{n_k}^{(i)} - v_{n_k}^{(i+1)}|| = 0, i = 1, 2, \dots, N. \tag{3.38}$$

From (3.38), and applying triangle inequality, we obtain for each $i = 1, 2, ..., N$ that

$$\lim_{k\to\infty} ||v_{n_k}^{(i)} - y_{n_k}|| = \lim_{k\to\infty} ||v_{n_k}^{(i)} - v_{n_k}^{(N+1)}|| = 0. \tag{3.39}$$

Also from (3.38) and (3.39), we obtain

$$\begin{aligned}||J_\lambda^{(i)} y_{n_k} - J_\lambda^{(i)} v_{n_k}^{(i+1)}|| &\leq ||y_{n_k} - v_{n_k}^{(i+1)}|| \\ &\leq ||y_{n_k} - v_{n_k}^{(i)}|| + ||v_{n_k}^{(i)} - v_{n_k}^{(i+1)}|| \to 0,\ k \to \infty.\end{aligned} \tag{3.40}$$

Furthermore, since $\lambda_n \geq \lambda > 0$ for all $n \geq 1$, we obtain from Lemma 3.1 and (3.38) that

$$||v_{n_k}^{(i+1)} - J_\lambda^{(i)} v_{n_k}^{(i+1)}|| \leq ||v_{n_k}^{(i+1)} - J_{\lambda_{n_k}}^{(i)} v_{n_k}^{(i+1)}|| \to 0,\ k \to \infty,\ i = 1, 2, \dots, N. \tag{3.41}$$

From (3.38), (3.39), (3.40) and (3.41), we obtain

$$\begin{aligned}||J_\lambda^{(i)} y_{n_k} - y_{n_k}|| &\leq ||J_\lambda^{(i)} y_{n_k} - J_\lambda^{(i)} v_{n_k}^{(i+1)}|| + ||J_\lambda^{(i)} v_{n_k}^{(i+1)} - v_{n_k}^{(i+1)}|| \\ &\quad + ||v_{n_k}^{(i+1)} - v_{n_k}^{(i)}|| + ||v_{n_k}^{(i)} - y_{n_k}|| \to 0,\ n \to \infty,\ i = 1, 2, \dots, N.\end{aligned} \tag{3.42}$$

Again, we obtain from (3.33) and (3.37) that

$$\begin{aligned}||y_{n_k} - Ty_{n_k}|| &\leq ||y_{n_k} - Tv_{n_k}^{(1)}|| + ||Tv_{n_k}^{(1)} - Ty_{n_k}|| \\ &\leq ||y_{n_k} - Tv_{n_k}^{(1)}|| + ||v_{n_k}^{(1)} - y_{n_k}|| \to 0,\ k \to \infty.\end{aligned} \tag{3.43}$$

Moreover, since $\{x_{n_k}\}$ converges weakly to $q \in H_1$, it follows from (3.32) that the subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converges weakly to $q \in H_1$. Hence, by Lemma 2.5 and (3.42), we obtain that $q \in F(J_\lambda^{(i)} y_{n_k})$ for each $i = 1, 2, \dots, N$. Similarly, we obtain from (3.43) that $q \in F(T)$. Thus, $q \in \left(F(T) \cap \left(\cap_{i=1}^N F\left(J_\lambda^{(i)}\right)\right)\right)$. Furthermore, we may assume without loss of generality that the subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$ converges to a point $\bar{\gamma} \in \left(0, \frac{1}{\|A\|^2}\right)$. By Lemma 2.3, $A^*(I -$

$S\prod_{j=1}^{m} J_{\lambda_{n_k}}^{(j)})A$ is inverse strongly monotone, thus $\left\{A^*(I - S\prod_{j=1}^{m} J_{\lambda_{n_k}}^{(j)})Au_{n_k}\right\}$ is bounded. It then follows from the nonexpansivity of P_C that

$$\begin{aligned}
&\|P_C(I - \gamma_{n_k} A^*(I - S\prod_{j=1}^{m} J_{\lambda_{n_k}}^{(j)})A)u_{n_k} - P_C(I - \bar{\gamma} A^*(I - S\prod_{j=1}^{m} J_{\lambda_{n_k}}^{(j)})A)u_{n_k}\| \\
\leq \quad & |\gamma_{n_k} - \bar{\gamma}|\|A^*(I - S\prod_{j=1}^{m} J_{\lambda_{n_k}}^{(j)})Au_{n_k}\| \to 0, \text{ as } k \to \infty.
\end{aligned}$$

That is,

$$\lim_{k\to\infty} \|y_{n_k} - P_C(I - \bar{\gamma} A^*(I - S\prod_{j=1}^{m} J_{\lambda_{n_k}}^{(j)})A)u_{n_k}\| = 0,$$

which implies from (3.30) that

$$\lim_{k\to\infty} ||u_{n_k} - P_C(I - \bar{\gamma} A^*(I - S\prod_{j=1}^{m} J_{\lambda_{n_k}}^{(j)})A)u_{n_k}|| = 0. \tag{3.44}$$

It then follows from Lemma 2.5 that $q \in F(P_C(I - \bar{\gamma} A^*(I - J_{\lambda_{n_k}}^{M_2}(I - \lambda_{n_k} f_2))A))$. Thus, from Lemma 2.4, we obtain that $q \in C \cap A^{-1}F\left(S\prod_{j=1}^{m} J_{\lambda_{n_k}}^{(j)}\right)$. Hence, from Lemma 3.2, we obtain that

$$Aq \in F\left(S\prod_{j=1}^{m} J_{\lambda_{n_k}}^{(j)}\right) \subseteq \left(F(S) \cap \left(\cap_{j=1}^{m} F\left(J_{\lambda}^{(j)}\right)\right)\right).$$

Therefore, we conclude that $q \in \Gamma$.
Finally, we show that $\{x_n\}$ converges strongly to z, where $z = P_\Gamma u$.
Now, from (3.26), (3.29), (3.30) and the characteristic property of the metric projection, we obtain that

$$\begin{aligned}
\liminf_{n\to\infty} \Upsilon_n &= \liminf_{k\to\infty} \Big(2\langle u - z, z - x_{n_k}\rangle + \frac{1}{\alpha_{n_k}\beta_{n_k}}(1 - \beta_{n_k})||x_{n_k+1} - y_{n_k}||^2 \\
&\quad + \frac{1}{\alpha_{n_k}\mu_{n_k}}(1 - \mu_{n_k})||y_{n_k} - u_{n_k}||^2\Big) \\
&= 2\langle u - z, z - q\rangle \\
&\geq 0, \qquad (3.45)
\end{aligned}$$

which implies from (3.25) that $\limsup_{n\to\infty} ||x_n - z||^2 \leq 0$. Hence, we conclude that $\{x_n\}$ converges strongly to z. $\square$

Setting $i, j = 1$ in Theorem (3.3), we have the following result.

Corollary 3.4. *Let H_1 and H_2 be two real Hilbert spaces and C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and θ be contraction mapping with coefficient $\tau \in (0,1)$. Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be two nonexpansive mappings, and $g : H_1 \to (-\infty, +\infty]$, $f : H_2 \to (-\infty, +\infty]$ be two proper, convex and lower semi continuous functions. Assume that $\Gamma := \{z \in F(T) : z \in argmin_{y \in H} g(y)$ and $Az \in F(S)$ such that $Az \in argmin_{y \in H} f(y)\} \neq \emptyset$ and the sequence $\{x_n\}$ is generated for arbitrary $x_1, u \in H_1$ by*

$$
(3.46) \qquad \begin{cases} x_{n+1} = (1-\beta_n)y_n + \beta_n z_n; \\ z_n = (1-t_n)y_n + t_n T J_{\lambda_n}(y_n); \\ y_n = P_C\left(u_n - \gamma_n A^*\left(I - SJ_{\lambda_n}\right)Au_n\right); \\ u_n = (1-\alpha_n)x_n + \alpha_n \theta(x_n), \; n \geq 1; \end{cases}
$$

where $\{\gamma_n\} \subset [a,b]$ for some $a, b \in \left(0, \frac{1}{||A||^2}\right), 0 < \lambda \leq \lambda_n$ and $\{\alpha_n\}, \{\beta_n\}, \{t_n\}$ are sequences in $(0,1)$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n < 1$.
Then, the sequence $\{x_n\}$ converges strongly to $z \in \Gamma$, where $z = P_\Gamma \theta(z)$.

Remark 3.5. The problem solved in Corollary 3.4 is problem (1.6)-(1.7), while Theorem 3.3 solves problem (1.8)- (1.9).

4 Application to SFP

Let the solution set of problem (1.2) be denoted by Ω. If $g \equiv \delta_C(x)$ [defined as $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise] and $f \equiv \delta_Q$, the indicator functions of nonempty, closed and convex subsets of H_1 and H_2 respectively. Then SMP (1.6) reduces to (1.2). Thus, applying Corollary 3.4, we have the following result.

Theorem 4.1. *Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and θ be contraction mapping with coefficient $\tau \in (0,1)$. Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be two nonexpansive mappings, assume that $\Gamma := \{F(T) \cap \Omega\} \neq \emptyset$ and the sequence $\{x_n\}$ is generated for arbitrary $x_1, u \in H_1$ by*

$$
(4.1) \qquad \begin{cases} x_{n+1} = (1-\beta_n)y_n + \beta_n z_n; \\ z_n = (1-t_n)y_n + t_n T P_C(y_n); \\ y_n = P_C(u_n - \gamma_n A^*(I - SP_Q)Au_n); \\ u_n = (1-\alpha_n)x_n + \alpha_n \theta(x_n), \; n \geq 1; \end{cases}
$$

where $\{\gamma_n\} \subset [a,b]$ for some $a, b \in \left(0, \frac{1}{||A||^2}\right), 0 < \lambda \leq \lambda_n$ and $\{\alpha_n\}, \{\beta_n\}, \{t_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n < 1$.
Then the sequence $\{x_n\}$ *converges strongly to* $z \in \Gamma$, *where* $z = P_\Gamma \theta(z)$.

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References

[1] Abbas, M., Alshahrani, M., Ansari, Q. H., Iyiola, O. S., and Shehu, Y. Iterative methods for solving proximal split minimization problems. *Numer. Algorithms 78*, 1 (2018), 193–215.

[2] Cai, G., and Shehu, Y. An iterative algorithm for fixed point problem and convex minimization problem with applications. *Fixed Point Theory Appl.*, 7 (2015), 17pp.

[3] Censor, Y., and Elfving, T. A multiprojection algorithm using Bregman projections in a product space. *Numer. Algorithms 8*, 2-4 (1994), 221–239.

[4] Chidume, C. E. Geometric properties of banach spaces and nonlinear iterations, lecture notes in mathematics, 1965. *Springer-Verlag London* (2009), xviii+326 pp.

[5] Kazmi, K. R., and Rizvi, S. H. Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem. *J. Egyptian Math. Soc. 21*, 1 (2013), 44–51.

[6] Laowang, W., and Panyanak, B. Strong and δ-convergence theorems for multivalued mappings in cat(0) space. *J. Inequal. Appl.*, 2 (2009), 16pp.

[7] Latif, A., Vahidi, J., and Eslamian, M. Strong convergence for generalized multiple-set split feasibility problem. *Filomat 30*, 2 (2016), 459–467.

[8] Lemoto, S., and Takahashi, W. Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a hilbert space. *Nonlinear Anal. 71*, 12 (2009), 2080–2089.

[9] Liu, H. B. Convergence theorems for a finite family of nonspreading and nonexpansive multivalued mappings and equilibrium problems with application. *Math. Appl. 3* (2013), 49–61.

[10] Maingé, P. E. Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. *Set-Valued Anal. 16*, 7-8 (2008), 899–912.

[11] Maingé, P.-E., and Măruşter, c. Convergence in norm of modified krasnoselki-mann iteration for fixed points of demicontractive mapping. *Appl. Math. Comput. 217*, 24 (2011), 9864–9874.

[12] Marino, G., and Xu, H. A general iterative method for nonexpansive mappings in hilbert spaces. *J. Math. Anal. Appl. 318*, 1 (2006), 43–52.

[13] Moudafi, A., and Thakur, B. S. Solving proximal split feasibility problems without prior knowledge of the operator norm. *Optim. Lett. 8*, 7 (2014), 2099–2110.

[14] Shehu, Y., and Iyiola, O. S. Convergence analysis for the proximal split feasibility problem using an inertial extrapolation term method. *J. Fixed point Theory Appl. 19*, 4 (2017), 2483–2510.

[15] Shehu, Y., and Iyiola, O. S. Strong convergence result for proximal split feasibility problem in hilbert spaces. *Optimization 66*, 12 (2017), 2275–2290.

[16] Takahashi, W., Xu, H. K., and Yao, J. C. Iterative methods for generalised split feasibility problems in hilbert spaces. *Set-Valued Var. Anal. 23*, 2 (2015), 205–221.

[17] Tian, M., and Jiang, B.-N. Weak convergence theorem for a class of split variational inequality problems and applications in hilbert space. *Journal of Inequalities and Applications*, 123 (2017), 17pp.

[18] Xu, H.-K. An iterative approach to quadratic optimization. *J. Optim. Theory Appl. 116*, 3 (2003), 659–678.

[19] Xu, H.-K. Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces. *Inverse Problems 26*, 10 (2010), 105018, 17.

[20] Yao, Z., Cho, S. Y., and Kang, S. M. A regularized algorithm for proximal split feasibility problem. *Abstr. Appl. Anal.* (2014), 6pp.

[21] Zhou, H. Convergence theorems of fixed points for strict pseudo-contractions in hilbert spaces. *Nonlinear Anal. 69*, 2 (2008), 456–462.

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