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$C-$CLASS FUNCTIONS ON SOME FIXED POINT RESULTS IN ORDERED PARTIAL METRIC SPACES VIA ADMISSIBLE MAPPINGS[1]

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Abstract. In this paper, we generalized the results presented in the paper W. Long, S. Khaleghizadeh, P. Salimi, S. Radenović and S. Shukla, *Some new fixed point results in partially ordered metric spaces via admissible mappings,* Fixed Point Theory Appl. (2014), 2014:117, in the framework of partial metric spaces by using $C-$class function in ordered structure. Also, we provide an example to support our theoretical results and shows that obtained results are potential generalization of the already existing results in literature.

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1. Introduction and mathematical preliminaries

Fixed point theory has been extensively used as a very powerful tool in the study of nonlinear analysis and related branches of sciences. Particularly, fixed point results has been used in pure and applied analysis, topology, geometry and computer sciences rapidly. A well known and fundamental result of this theory is Banach contraction principle. During last couple of years, the Banach contraction principle has been extensively studied and generalized in many

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settings (see [6]). In 1994, Matthews [9] introduced the notion of partial metric space which generalized the already existing ordinary metric space as a part of the study of denotational semantics of dataflow networks and showed that the Banach contraction principle can be generalized to the partial metric context for applications in program verification. Afterwards, many researchers studied various fixed point results in partial metric spaces. For more details, the reader can see ([4], [5], [8], [9], [13], [14]).

First, we start by recalling some basic definitions and properties of partial metric spaces.

Definition 1.1. [9] A partial metric on a nonempty set X is an operator $p : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$:

(p1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p2) $p(x, x) \leq p(x, y)$,

(p3) $p(x, y) = p(y, x)$,

(p4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is the pair (X, p) such that X is a nonempty set and p is a partial metric on X.

For a partial metric p on X, the operator $p^s : X \times X \to [0, +\infty)$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a (standard) metric on X. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open $p-$balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1.2. ([6]) Let (X, p) be a partial metric space. Then

(a) a sequence $\{x_n\}$ in (X, p) converges to $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$;

(b) a sequence $\{x_n\}$ in (X, p) is called a Cauchy sequence if and only if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$;

(c) a partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n\to\infty} p(x_n, x)$;

(d) a sequence $\{x_n\}$ in (X, p) is called $0-$Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. We say that (X, p) is $0-$complete if every $0-$Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

(e) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_p(x_0, \delta)) \subseteq B_p(x_0, \varepsilon)$.

Proposition 1.3. *[5] Let (X, p) be a partial metric space. Then*

1) *sequence $\{x_n\}$ is Cauchy in a partial metric space (X, p) if and only if $\{x_n\}$ is Cauchy in metric space (X, p^s);*

2) *partial metric space (X, p) is complete if and only if metric space (X, p^s) is complete; moreover, $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if*

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,.m\to\infty} p(x_n, x_m).$$

Remark 1.4. [5]

A limit of a sequence in a partial metric space need not be unique. Moreover, the function $p(.,.)$ need not be continuous in the sense that

$$x_n \xrightarrow{p} x \text{ and } y_n \xrightarrow{p} y \text{ implies } p(x_n, y_n) \to p(x,y).$$

Example 1.5. If $X = [0, +\infty)$ and $p(x,y) = \max\{x,y\}$ for $x, y \in X$, then for $\{x_n\} = 1$, $p(x_n, x) = x = p(x,x)$ for each $x \geq 1$ e.g., $x_n \xrightarrow{p} 2$ and $x_n \xrightarrow{p} 3$ when $n \to \infty$.

2) However, if

$$p(x_n, x) \to p(x,x) = 0$$

then

$$p(x_n, y) \to p(x,y) \quad \text{for all } y \in X.$$

3) It is worth noting that the notions $p-$continuous and p^s-continuous of any function in the context of partial metric spaces are incomparable, in general.

Example 1.6. If $X = [0, +\infty)$, $p(x,y) = \max\{x,y\}$, $p^s(x,y) = |x-y|$, $f(0) = 1$ and $fx = x^2$ for all $x > 0$, $gx = |\sin x|$, then f is $p-$continuous and p^s-discontinuous at the point $x = 0$, while g is $p-$discontinuous and p^s-continuous at $x = \pi$.
In this paper, we say that $f : X \to X$ is continuous if both $f : (X,p) \to (X,p)$ and $f : (X, p^s) \to (X, p^s)$ are continuous.

Definition 1.7. Let $(X, \preceq)$ be a partially ordered set. Then

a) elements $x, y \in X$ are comparable if either $x \preceq y$ or $y \preceq x$,

b) a subset A of X is said to be totally ordered if every two elements of A are comparable,

c) a mapping $f : X \to X$ is nondecreasing with respect to $\preceq$ if $x \preceq y$ implies $fx \preceq fy$.

Definition 1.8. [12] Let $f : X \to X$ and $\gamma : X \times X \to [0, +\infty)$ then f is a $\gamma-$admissible mapping if

$$\gamma(x,y) \geq 1 \text{ implies } \gamma(fx, fy) \geq 1, \text{ for } x, y \in X.$$

Definition 1.9. [11] Let $f : X \to X$ and $\mu : X \times X \to [0, +\infty)$ then f is a $\mu-$subadmissible mapping if

$$\mu(x,y) \leq 1 \text{ implies } \mu(fx, fy) \leq 1, \text{ for } x, y \in X.$$

Assertions similar to the following lemma were used (and proved) in the course of proofs of several fixed point results in various papers [4].

Lemma 1.10. *Let (X,p) be a partial metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a $0-$Cauchy sequence in (X,p), then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following sequences tend to ε^+ when $k \to \infty$;

$$p\left(x_{m(k)}, x_{n(k)}\right),\ p\left(x_{m(k)}, x_{n(k)+1}\right),\ p\left(x_{m(k)-1}, x_{n(k)}\right),$$
$$p\left(x_{m(k)-1}, x_{n(k)+1}\right),\ \ p\left(x_{m(k)+1}, x_{n(k)+1}\right).$$

In the sequel, consider the following classes of functions from $[0,+\infty)$ into itself:

1. $\Psi = \{\psi :\ \psi$ is nondecreasing and lower semicontinuous$\}$,
2. $\Phi_1 = \{\alpha :\ \alpha$ is upper semicontinuous$\}$,
3. $\Phi_2 = \{\beta :\ \beta$ is lower semicontinuous$\}$.

It is well known that ψ is lower semicontinuous if

$$\psi\left(\lim_{n\to\infty} x_n\right) \leq \lim\inf_{n\to\infty} \psi(x_n),$$

while α is upper semicontinuous if $\limsup_{n\to\infty} \alpha(x_n) \leq \alpha(\lim_{n\to\infty} x_n)$.

Definition 1.11. [2] A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called *C-class* function if it is continuous and satisfies the following axioms:

(1) $F(s,t) \leq s$;
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$; for all $s,t \in [0,\infty)$.

We denote the set of all C-class functions as $\mathcal{C}$.

Example 1.12. [2] The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s,t \in [0,\infty)$:

(1) $F(s,t) = s - t$; (2) $F(s,t) = ms$, $0<m<1$;
(3) $F(s,t) = \sqrt[n]{\ln(1+s^n)}$; (4) $F(s,t) = \phi(s)$,here $\phi : [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$,
(5) $F(s,t) = \vartheta(s)$; $\vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a generalized Mizoguchi-Takahashi type function;
(6) $F(s,t) = \frac{s}{\Gamma(1/2)} \int_0^\infty \frac{e^{-x}}{\sqrt{x}+t}\, dx$, where Γ is the Euler Gamma function.

The concept of pair $(\mathcal{F}, h)$, was introduced in [1] which was further reformed in [3]

Definition 1.13. [1], [3] Let $h : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a subclass of type$-II$, if $x,y \geq 1 \Rightarrow h(1,1,z) \leq h(x,y,z)$ for all $z \in \mathbb{R}^+$.

Example 1.14. [1],[3] Define h: $R^+ \times R^+ \times R^+ \to R$ by;

(a) $h(x,y,z) = (z+l)^{xy},\ l > 1$;

(b) $h(x,y,z) = (xy+l)^{z},\ l > 1$;

(c) $h(x,y,z) = z$;

(d) $h(x,y,z) = x^m y^n z^p$, where $m,n,p \in \mathbb{N}$;

(e) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k$, where $m,n,p,q,k \in \mathbb{N}$

for all $x,y,z \in R^+$. Then h is a function of subclass of type$-II$.

Definition 1.15. [1], [3] Let $h\colon \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, then the pair $(\mathcal{F}, h)$ is an upper class of type$-II$, if h is a subclass of type$-II$ and also the following conditions hold,

(i) $0 \le s \le 1 \Rightarrow \mathcal{F}(s,t) \le \mathcal{F}(1,t)$,
(ii) $h(1,1,z) \le \mathcal{F}(s,t) \Rightarrow z \le st$ for all $s,t,z \in \mathbb{R}^+$.

Example 1.16. [1], [3] Define $h\colon R^+ \times R^+ \times R^+ \to R$ and $F\colon R^+ \times R^+ \to R$ by:

(a) $h(x,y,z) = (z+l)^{xy}, l > 1, F(s,t) = st + l$;

(b) $h(x,y,z) = (xy+l)^z, l > 1, F(s,t) = (1+l)^{st}$;

(c) $h(x,y,z) = z, F(s,t) = st$;

(d) $h(x,y,z) = x^m y^n z^p$, where $m,n,p \in \mathbb{N},\ F(s,t) = s^p t^p$

(e) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k$, where $m,n,p,q,k \in \mathbb{N},\ F(s,t) = s^k t^k$

for all $x,y,z,s,t \in R^+$. Then the pair $(\mathcal{F}, h)$ is an upper class of type$-II$.

Denote by Φ_1 the family of continuous functions $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) = 0$ if and only if $t = 0$ and let Φ_2 be the family of continuous functions $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) \ge 0$.Note that $\Phi_1 \subset \Phi_u$.

2. Main Results

In this section, we state and prove some fixed point results in the framework of partially ordered $0-$complete partial metric spaces, which generalize already existing results in [7], [8] and [10].

Our first main result is as follows:

Theorem 2.1. *Let $(X, p, \preceq)$ be a partially ordered partial metric space such that (X,p) is $0-$complete. Assume $f : X \to X$ and let γ, $\mu : X \times X \to [0,\infty)$ are two mappings such that f is a nondecreasing and γ-admissible mapping and μ-subadmissible mapping. Also assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that*

$$(2.1) \qquad t > 0 \text{ and } (s = t \quad \text{or} \quad s = 0) \text{ implies } \psi(t) - F(\alpha(s), \beta(s)) > 0,$$

for all $t, s \geq 0$, *and*

$$\begin{aligned} \gamma(x,fx)\gamma(y,fy) &\geq 1 \text{ and } \mu(x,fx)\mu(y,fy) \leq 1 \\ &\Rightarrow \\ h(\gamma(x,x),\gamma(y,y),\psi(p(fx,fy))) &\leq \mathcal{F}(\mu(x,x)\mu(y,y),F(\alpha(p(x,y)),\beta(p(x,y))) \end{aligned} \tag{2.2}$$

for all comparable $x, y \in X$, *where pair* $(\mathcal{F}, h)$ *is an upclass of type-II and* $F \in \mathcal{C}$. *Suppose that either*

(i) f *is continuous, or*

(ii) If there exists a nondecreasing sequence $\{x_n\}$ *such that*

$$x_n \xrightarrow{p} x \text{ as } n \to \infty, \gamma(x_n, x_n) \geq 1, \mu(x_n, x_n) \leq 1 \text{ for all } n,$$

then

$$\gamma(x,x) \geq 1,\ \mu(x_n,x_n) \leq 1 \text{ and } x_n \preceq x \text{ for all } n \in \mathbb{N}.$$

If there exists $x_0 \in X$ *such that*

$$\gamma(x_0,x_0) \geq 1,\ \gamma(x_0,fx_0) \geq 1,\ \mu(x_0,x_0) \leq 1,\ \mu(x_0,fx_0) \leq 1 \text{ and } x_0 \preceq fx_0,$$

then f *has a fixed point.*

Proof. Let $x_0 \in X$ be such that

$$\gamma(x_0,x_0) \geq 1,\ \gamma(x_0,fx_0) \geq 1 \text{ and } \mu(x_0,x_0) \leq 1,\ \mu(x_0,fx_0) \leq 1.$$

Then there exists $x_1 = fx_0$ such that $\gamma(x_0,x_1) \geq 1$, $\mu(x_0,x_1) \leq 1$. Since the mapping f is a $\gamma-$admissible and μ-subadmissible mapping, we have

$$\gamma(fx_{n-1},fx_n) \geq 1,\ \mu(fx_{n-1},fx_n) \leq 1.$$

So, we can define a sequence $\{x_n\}$ such that $x_n = fx_{n-1}$, and

$$\gamma(x_n,x_{n+1}) \geq 1,\ \mu(x_n,x_{n+1}) \leq 1, \text{ for all } n \in \mathbb{N}.$$

Since f is nondecreasing and $x_0 \preceq fx_0$, we have

$$x_0 \preceq x_1 \preceq ... \preceq x_n \preceq x_{n+1} \preceq \dots.$$

Since f is γ-admissible and μ-subadmissible, we conclude that

$$\gamma(x_n,x_n) \geq 1, \gamma(x_n,fx_n) = \gamma(x_n,x_{n+1}) \geq 1$$

and

$$\mu(x_n,x_n) \leq 1, \mu(x_n,fx_n) = \mu(x_n,x_{n+1}) \leq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, a fixed point of f is found. Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, that is $p(x_n,x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Hence

$$x_0 \prec x_1 \prec ... \prec x_n \prec x_{n+1} \prec \dots$$

for all $n \in \mathbb{N} \cup \{0\}$.

Now, we will show that $\{p(x_n, x_{n+1})\}$ is nonincreasing sequence. Indeed, if

$$p(x_{k-1}, x_k) < p(x_k, x_{k+1}),$$

for some $k \in \mathbb{N}$ then

$$\psi(p(x_{k-1}, x_k)) \leq \psi(p(x_k, x_{k+1})),$$

and taking $x = x_{k-1}$ and $y = x_k$ in (2.2), we obtain

$$\begin{aligned}
&h(1,1,\psi(p(x_k, x_{k+1}))) \\
&\quad = \ h(1,1,\psi(p(fx_{k-1}, fx_k))) \\
&\quad \leq \ h(\gamma(x_{k-1}, x_{k-1}), \gamma(x_k, x_k)(\psi(p(fx_{k-1}, fx_k)) + l) \\
&\quad \leq \ \mathcal{F}(\mu(x_{k-1}, x_{k-1})\mu(x_k, x_k), F(\alpha(p(x_{k-1}, x_k)), \beta(p(x_{k-1}, x_k))) \\
&\quad \leq \ \mathcal{F}(1, F(\alpha(p(x_{k-1}, x_k)), \beta(p(x_{k-1}, x_k))),
\end{aligned}$$

that is

$$\psi(p(x_{k-1}, x_k)) \leq \psi(p(x_k, x_{k+1})) \leq F(\alpha(p(x_{k-1}, x_k)), \beta(p(x_{k-1}, x_k))), \tag{2.3}$$

a contradiction. Therefore, we proved that $\{p(x_n, x_{n+1})\}$ is a nonincreasing sequence of positive real numbers. Hence, there exists $p^* \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = p^*.$$

If $p^* > 0$ then by the following condition

$$\psi(p(x_{n+1}, x_n)) \leq F(\alpha(p(x_n, x_{n-1})), \beta(p(x_n, x_{n-1}))),$$

together with the properties of ψ, α, β, we have

$$\begin{aligned}
\psi(p^*) \ &\leq \ \liminf \psi(p(x_n, x_{n+1})) \\
&\leq \ \limsup \psi(p(x_n, x_{n+1})) \\
&\leq \ \limsup F(\alpha(p(x_{n-1}, x_n)), \beta(p(x_{n-1}, x_n))) \\
&= \ \limsup F(\alpha(p(x_{n-1}, x_n)), \liminf \beta(p(x_{n-1}, x_n))) \\
&\leq \ F(\alpha(p^*), \beta(p^*)),
\end{aligned}$$

which is a contradiction to (2.1), where $t = s = p^* > 0$. Hence,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

Now, suppose, on contrary that $\{x_n\}$ is not a $0-$Cauchy sequence. Then according to Lemma 1.10, we obtain the contradiction. Indeed, taking $x = x_{m(k)}, y = x_{n(k)}$ in (2.2), we have

$$\psi\left(p\left(x_{m(k)+1}, x_{n(k)+1}\right)\right) \leq F(\alpha\left(p\left(x_{m(k)}, x_{n(k)}\right)\right), \beta\left(p\left(x_{m(k)}, x_{n(k)}\right)\right)). \tag{2.4}$$

Then by (2.4) together with the properties of ψ, α, β, we have

$$\begin{aligned}
\psi(\varepsilon) &\leq \liminf \psi\left(p\left(x_{m(k)+1}, x_{n(k)+1}\right)\right) \\
&\leq \limsup \psi\left(p\left(x_{m(k)+1}, x_{n(k)+1}\right)\right) \\
&\leq \limsup F(\alpha\left(p\left(x_{m(k)}, x_{n(k)}\right)\right), \beta\left(p\left(x_{m(k)}, x_{n(k)}\right)\right)) \\
&= \limsup F(\alpha\left(p\left(x_{m(k)}, x_{n(k)}\right)\right), \liminf \beta\left(\left(p\left(x_{m(k)}, x_{n(k)}\right)\right)\right)) \\
&\leq F(\alpha(\varepsilon), \beta(\varepsilon)).
\end{aligned}$$

Hence, $\psi(\varepsilon) \leq F(\alpha(\varepsilon), \beta(\varepsilon))$ a contradiction with (2.1), where $t = s = \varepsilon > 0$.

Hence the sequence $\{x_n\}$ is a $0-$Cauchy sequence in the $0-$complete partial metric space (X, p). Then there exists $u \in X$ such that $x_n \xrightarrow{p} u$.

Suppose that (**i**) holds. Then

$$u = \lim_{n\to\infty} x_{n+1} = f\left(\lim_{n\to\infty} x_n\right) = f(u),$$

implies that u is a fixed point of f in X and $p(u,u) = 0$ because $p(u,u) \leq p(u, fu) = 0$.

Now, suppose that (**ii**) holds.

In this case

$$\gamma(u,u) \geq 1, \gamma(u, fu) \geq 1, \mu(u,u)\mu(u,fu) \leq 1 \text{ and } x_n \preceq u,$$

for all $\mathbb{N} \cup \{0\}$. We claim that u is a fixed point of f, that is, $p(u, fu) = 0$. Let $p(u, fu) > 0$. However, according to Remark 1.4 (2) we have

$$p(u, fu) = \lim_{n\to\infty} p(x_{n+1}, fu).$$

Due to the condition (2.2), we have

$$(\gamma(u, fu)\gamma(x_n, fx_n) \geq 1, \mu(u, fu)\mu(x_n, fx_n) \leq 1);$$

So,

$$\begin{aligned}
&h(1, 1, \psi(p(x_{n+1}, fu))) \\
&\quad= h(1, 1, \psi(p(fx_n, fu))) \\
&\quad\leq h(\gamma(u,u), \gamma(x_n, x_n), \psi(p(fx_n, fu))) \\
&\quad\leq \mathcal{F}(\mu(u,u)\mu(x_n, x_n), F(\alpha(p(x_{k-1}, x_k)), \beta(p(x_{k-1}, x_k))) \\
&\quad\leq \mathcal{F}(1, F(\alpha(p(x_{k-1}, x_k)), \beta(p(x_{k-1}, x_k))),
\end{aligned}$$

$$\Longrightarrow$$

$$\psi(p(x_{n+1}, fu)) \leq F(\alpha(p(u, x_n)), \beta(p(u, x_n))).$$

Again using the properties of the functions ψ, α, β in the above inequality, we obtain

$$\begin{aligned}
\psi(p(u, fu)) &\leq \liminf \psi(p(x_{n+1}, fu)) \\
&= \liminf \psi(p(fx_n, fu)) \\
&\leq \limsup \psi(p(fx_n, fu)) \\
&\leq \limsup F(\alpha(p(u, x_n)), \beta(p(u, x_n))) \\
&\leq \limsup F(\alpha(p(u, x_n)), \liminf \beta(p(u, x_n))) \\
&\leq F(\alpha(0), \beta(0)),
\end{aligned}$$

a contradiction with (2.1), where $t = p(u, fu)$, $s = 0$. Hence,

$$\lim_{n\to\infty} p(x_{n+1}, fu) = p(u, fu) = 0 \text{ as } u = fu.$$

It is clear that $p(u, u) = 0$. The proof of the theorem is complete. □

Choosing $F(s,t) = s - t$ and $h(x, y, z) = (z + l)^{xy}, h(x, y, z) = (xy + l)^z, h(x, y, z) = xyz, h(x, y, z) = z$ in Theorem 2.1 we have the following corollaries:

Corollary 2.2. *Let $(X, p, \preceq)$ be a partially ordered partial metric space such that (X, p) is $0-$complete. Assume that $f : X \to X$ and $\gamma : X \times X \to [0, \infty)$ are two mappings such that f is a nondecreasing and γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that*

$$t > 0 \text{ and } (s = t \text{ or } s = 0) \text{ implies } \psi(t) - \alpha(s) + \beta(s) > 0$$

for all $t, s \geq 0$, and

$$\begin{aligned}
&\gamma(x, fx)\gamma(y, fy) \geq 1 \\
&\quad \Rightarrow \quad (\psi(p(fx, fy)) + \ell)^{\gamma(x,x)\gamma(y,y)} \leq \alpha(p(x, y)) - \beta(p(x, y)) + \ell
\end{aligned}$$

for all comparable $x, y \in X$, where $l \geq 1$. Suppose that either
(i) f is continuous, or
(ii) if there exists a nondecreasing sequence $\{x_n\}$ such that

$$x_n \xrightarrow{p} x \text{ as } n \to \infty, \gamma(x_n, fx_n) \geq 1 \text{ and } \gamma(x_n, x_n) \geq 1,$$

for all n, then

$$\gamma(x, x) \geq 1, \gamma(x, fx) \geq 1 \text{ and } x_n \preceq x,$$

for all $n \in \mathbb{N}$.
If there exists $x_0 \in X$ such that

$$\gamma(x_0, x_0) \geq 1, \gamma(x_0, fx_0) \geq 1 \text{ and } x_0 \preceq fx_0,$$

then f has a fixed point u in X and $p(u, u) = 0$.

Corollary 2.3. *Let $(X,p,\preceq)$ be a partially ordered partial metric space such that (X,p) is $0-$complete. Assume that $f: X \to X$ and $\gamma : X \times X \to [0,\infty)$ are two mappings such that f is a nondecreasing and γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that for all $t, s \geq 0$,*

$$t > 0 \text{ and } (s = t \text{ or } s = 0) \text{ implies } \psi(t) - \alpha(s) + \beta(s) > 0$$

and

$$\gamma(x, fx)\gamma(y, fy) \geq 1 \Rightarrow (\psi(p(fx, fy)) + 1)^{\gamma(x,x)\gamma(y,y)} \leq 2^{\alpha(p(x,y))-\beta(p(x,y))}$$

for all comparable $x, y \in X$. Suppose that either
(i) f is continuous, or
(ii) if there exists a nondecreasing sequence $\{x_n\}$ such that

$$x_n \xrightarrow{p} x \text{ as } n \to \infty, \gamma(x_n, fx_n) \geq 1 \text{ and } \gamma(x_n, x_n) \geq 1,$$

for all n, then $\gamma(x,x) \geq 1$, $\gamma(x, fx) \geq 1$ and $x_n \preceq x$ for all $n \in \mathbb{N}$.
If there exists $x_0 \in X$ such that

$$\gamma(x_0, x_0) \geq 1, \gamma(x_0, fx_0) \geq 1 \text{ and } x_0 \preceq fx_0,$$

then f has a fixed point u in X and $p(u,u) = 0$.

Corollary 2.4. *Let $(X,p,\preceq)$ be a partially ordered partial metric space such that (X,p) is $0-$complete. Assume $f: X \to X$ and $\gamma : X \times X \to [0,\infty)$ are two mappings such that f is a nondecreasing and γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that for all $t, s \geq 0$,*

$$t > 0 \text{ and } (s = t \text{ or } s = 0) \text{ implies } \psi(t) - \alpha(s) + \beta(s) > 0$$

and

$$\gamma(x, fx)\gamma(y, fy) \geq 1 \Rightarrow \gamma(x,x)\gamma(y,y)\psi(p(fx, fy)) \leq \alpha\left(p(x,y)\right) - \beta\left(p(x,y)\right)$$

for all comparable $x, y \in X$. Suppose that either
(i) f is continuous, or
(ii) if there exists a nondecreasing sequence $\{x_n\}$ such that

$$x_n \xrightarrow{p} x \text{ as } n \to \infty, \gamma(x_n, fx_n) \geq 1 \text{ and } \gamma(x_n, x_n) \geq 1$$

for all n, then

$$\gamma(x,x) \geq 1, \gamma(x, fx) \geq 1 \text{ and } x_n \preceq x,$$

for all $n \in \mathbb{N}$.
If there exists $x_0 \in X$ such that

$$\gamma(x_0, x_0) \geq 1, \gamma(x_0, fx_0) \geq 1 \text{ and } x_0 \preceq fx_0,$$

then f has a fixed point in X, say u such that $p\left(u,u\right) = 0$.

Corollary 2.5. *Let $(X, p, \preceq)$ be a partially ordered partial metric space such that (X, p) is $0-$complete. Assume that $f : X \to X$ and $\gamma : X \times X \to [0, \infty)$ are two mappings such that f is a nondecreasing and γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that*

$$t > 0 \text{ and } (s = t \text{ or } s = 0) \text{ implies } \psi(t) - \alpha(s) + \beta(s) > 0$$

for all $t, s \geq 0$, and

$$\gamma(x, fx)\gamma(y, fy) \Longrightarrow \psi(p(fx, fy)) \leq \alpha(p(x, y)) - \beta(p(x, y)), \tag{2.5}$$

for all comparable $x, y \in X$. Suppose that either
(i) f is continuous, or
(ii) if there exists a nondecreasing sequence $\{x_n\}$ such that

$$x_n \xrightarrow{p} x \text{ as } n \to \infty, \gamma(x_n, fx_n) \geq 1 \text{ and } \gamma(x_n, x_n) \geq 1,$$

for all n, then

$$\gamma(x, x) \geq 1, \gamma(x, fx) \geq 1 \text{ and } x_n \preceq x,$$

for all $n \in \mathbb{N}$.
If there exists $x_0 \in X$ such that

$$\gamma(x_0, x_0) \geq 1, \gamma(x_0, fx_0) \geq 1 \text{ and } x_0 \preceq fx_0,$$

then f has a fixed point u in X and $p(u, u) = 0$.

Corollary 2.6. *Let $(X, p, \preceq)$ be a partially ordered partial metric space such that (X, p) is $0-$complete. Assume that $f : X \to X$ and $\gamma : X \times X \to [0, \infty)$ are two mappings such that f is a nondecreasing and γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that*

$$t > 0 \text{ and } (s = t \text{ or } s = 0) \text{ implies } \psi(t) - \frac{\alpha(s)}{1 + \beta(s)} > 0$$

for all $t, s \geq 0$, and

$$\gamma(x, fx)\gamma(y, fy) \Longrightarrow (\psi(p(fx, fy)) + \ell)^{\gamma(x,x)\gamma(y,y)} \leq \frac{\alpha(p(x, y))}{1 + \beta(p(x, y))}, \tag{2.6}$$

for all comparable $x, y \in X$. Suppose that either
(i) f is continuous, or
(ii) if there exists a nondecreasing sequence $\{x_n\}$ such that

$$x_n \xrightarrow{p} x \text{ as } n \to \infty, \gamma(x_n, fx_n) \geq 1 \text{ and } \gamma(x_n, x_n) \geq 1,$$

for all n, then

$$\gamma(x, x) \geq 1, \gamma(x, fx) \geq 1 \text{ and } x_n \preceq x,$$

for all $n \in \mathbb{N}$.
If there exists $x_0 \in X$ such that

$$\gamma(x_0, x_0) \geq 1, \gamma(x_0, fx_0) \geq 1 \text{ and } x_0 \preceq fx_0,$$

then f has a fixed point u in X and $p(u, u) = 0$.

According to the Theorem 2.1, we provide the following corollary:

Corollary 2.7. *Let $(X, p, \preceq)$ be a partially ordered partial metric space such that (X, p) is $0-$complete. Assume that $f : X \to X$ and $\gamma : X \times X \to [0, \infty)$ are two mappings such that f is a nondecreasing and γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that*

$$t > 0 \text{ and } (s = t \quad \text{or} \quad s = 0) \text{ implies } \psi(t) - F(\alpha(s), \beta(s)) > 0,$$

for all $t, s \geq 0$, and

$$\begin{aligned} &\gamma(x, fx)\gamma(y, fy)h(\gamma(x, x), \gamma(y, y), \psi(p(fx, fy))) \\ &\quad \leq \mathcal{F}(\mu(x, x)\mu(y, y), F(\alpha\,(p(x, y))\,, \beta\,(p(x, y))) \end{aligned} \tag{2.7}$$

for all comparable $x, y \in X$, where pair $(\mathcal{F}, h)$ is an upclass of type-II and $F \in \mathcal{C}$. Suppose that either

(i) f is continuous, or

(ii) if there exists a nondecreasing sequence $\{x_n\}$ such that

$$x_n \xrightarrow{p} x \text{ as } n \to \infty, \gamma(x_n, fx_n) \geq 1 \text{ and } \gamma(x_n, x_n) \geq 1,$$

for all n, then

$$\gamma(x, x) \geq 1, \gamma(x, fx) \geq 1 \text{ and } x_n \preceq x,$$

for all $n \in \mathbb{N}$.

If there exists $x_0 \in X$ such that

$$\gamma(x_0, x_0) \geq 1, \gamma(x_0, fx_0) \geq 1 \text{ and } x_0 \preceq fx_0,$$

then f has a fixed point.

Proof. Let $\gamma(x, fx)\gamma(y, fy) \geq 1$. In this case, we have by (2.7)

$$\begin{aligned} &h(\gamma(x, x), \gamma(y, y), \psi(p(fx, fy))) \\ &\quad \leq \quad \gamma(x, fx)\gamma(y, fy)h(\gamma(x, x), \gamma(y, y), \psi(p(fx, fy))) \\ &\quad \leq \quad \mathcal{F}(\mu(x, x)\mu(y, y), F(\alpha\,(p(x, y))\,, \beta\,(p(x, y))) \end{aligned}$$

that is,

$$\begin{aligned} \gamma(x, fx)\gamma(y, fy) &\geq 1 \\ &\Rightarrow \\ h(\gamma(x, x), \gamma(y, y), \psi(p(fx, fy))) &\leq \mathcal{F}(\mu(x, x)\mu(y, y), F(\alpha\,(p(x, y))\,, \beta\,(p(x, y))) \end{aligned}$$

Hence, all the conditions of theorem 2.1 hold and therefore f has a fixed point. □

Corollary 2.8. *Let $(X, p, \preceq)$ be a partially ordered partial metric space such that (X, p) is 0-complete. Assume that $f : X \to X$ and $\gamma : X \times X \to [0, \infty)$*

are two mappings such that f is a nondecreasing and γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that for all $t, s \geq 0$,

$$t > 0 \text{ and } (s = t \quad \text{or} \quad s = 0) \text{ implies } \psi(t) - \alpha(s) + \beta(s) > 0$$

and

(2.8)
$$\gamma(x, fx)\gamma(y, fy) \left(\psi(p(fx, fy)) + \ell\right)^{\gamma(x,x)\gamma(y,y)} \leq \alpha\left(p(x, y)\right) - \beta\left(p(x, y)\right) + \ell$$

for all comparable $x, y \in X$ where $l \geq 1$. Suppose that either

(i) f is continuous, or

(ii) if a nondecreasing sequence $\{x_n\}$ is such that $x_n \xrightarrow{p} x$ as $n \to \infty$, $\gamma(x_n, fx_n) \geq 1$ and $\gamma(x_n, x_n) \geq 1$ for all n, then $\gamma(x, x) \geq 1$, $\gamma(x, fx) \geq 1$ and $x_n \preceq x$ for all $n \in \mathbb{N}$.

If there exists $x_0 \in X$ such that $\gamma(x_0, x_0) \geq 1$, $\gamma(x_0, fx_0) \geq 1$ and $x_0 \preceq fx_0$, then f has a fixed point.

Corollary 2.9. *Let $(X, p, \preceq)$ be a partially ordered partial metric space such that (X, p) is $0-$complete. Assume that $f : X \to X$ and $\gamma : X \times X \to [0, \infty)$ are two mappings such that f is a nondecreasing and γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that for all $t, s \geq 0$,*

$$t > 0 \text{ and } (s = t \text{ or } s = 0) \text{ implies } \psi(t) - \alpha(s) + \beta(s) > 0$$

and

$$\gamma(x, fx)\gamma(y, fy) \left(\psi(p(fx, fy)) + 1\right)^{\gamma(x,x)\gamma(y,y)} \leq 2^{\alpha(p(x,y)) - \beta(p(x,y))}$$

for all comparable $x, y \in X$ where $\ell \geq 1$. Suppose that either

(i) f is continuous, or

(ii) if there exists a nondecreasing sequence $\{x_n\}$ such that

$$x_n \xrightarrow{p} x \text{ as } n \to \infty, \gamma(x_n, fx_n) \geq 1 \text{ and } \gamma(x_n, x_n) \geq 1,$$

for all n, then $\gamma(x, x) \geq 1$, $\gamma(x, fx) \geq 1$ and $x_n \preceq x$ for all $n \in \mathbb{N}$.

If there exists $x_0 \in X$ such that

$$\gamma(x_0, x_0) \geq 1, \gamma(x_0, fx_0) \geq 1 \text{ and } x_0 \preceq fx_0,$$

then f has a fixed point.

Corollary 2.10. *Let $(X, p, \preceq)$ be a partially ordered partial metric space such that (X, p) is $0-$complete. Assume that $f : X \to X$ and $\gamma : X \times X \to [0, \infty)$ are two mappings such that f is a nondecreasing and γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ such that for all $t, s \geq 0$,*

$$t > 0 \text{ and } (s = t \text{ or } s = 0) \text{ implies } \psi(t) - \alpha(s) + \beta(s) > 0$$

and

$$\gamma(x, fx)\gamma(y, fy)\gamma(x, x)\gamma(y, y)\psi(p(fx, fy)) \leq \alpha\left(p(x, y)\right) - \beta\left(p(x, y)\right)$$

for all comparable $x, y \in X$ where $\ell \geq 1$. Suppose that either

(i) f is continuous, or

(ii) if there exists a nondecreasing sequence $\{x_n\}$ such that

$$x_n \xrightarrow{p} x \text{ as } n \to \infty, \gamma(x_n, fx_n) \geq 1 \text{ and } \gamma(x_n, x_n) \geq 1$$

for all n, then

$$\gamma(x, x) \geq 1, \gamma(x, fx) \geq 1 \text{ and } x_n \preceq x,$$

for all $n \in \mathbb{N}$.

If there exists $x_0 \in X$ such that

$$\gamma(x_0, x_0) \geq 1, \gamma(x_0, fx_0) \geq 1 \text{ and } x_0 \preceq fx_0,$$

then f has a fixed point.

Example 2.11. Let $X = [0, +\infty)$ be endowed with the partial metric $p(x, y) = \max\{x, y\}$ for all $x, y \in X$ and $f : X \to X$ be defined by

$$fx = \begin{cases} \frac{1}{8}\left(1 + x^2\right) \text{ if } x \in [0, 1], \\ 5^x \text{ if } x \in (1, +\infty). \end{cases}$$

Define $\gamma : X \times X \to [0, +\infty)$, $\psi : [0, +\infty) \to [0, +\infty)$ and $\beta : [0, +\infty) \to [0, +\infty)$ by

$$\gamma(x, y) = \begin{cases} 1 \text{ if } x, y \in [0, 1] \\ 0 \text{ otherwise} \end{cases}$$

$$\psi(t) = t + \frac{1}{4}, \ \alpha(t) = t + \frac{1}{2} \text{ and } \beta(t) = 1.$$

We have to show that this example supports our Corollary 2.6

Obviously (X, p) is a $0-$complete partially ordered metric space. Now, we have to show that f is a $\gamma-$admissible mapping. Let $\gamma(x, y) \geq 1$ for $x, y \in [0, 1]$, then $fx, fy \in [0, 1]$, such that $\gamma(fx, fy) \geq 1$. Now we will show that condition 2.1 is satisfied. Let $t > 0$ and $s = t$. Then

$$\psi(t) - \frac{\alpha(t)}{1 + \beta(t)} = \frac{t}{2} > 0.$$

Also, for $t > 0$ and $s = 0$ we have $\psi(t) - \frac{\alpha(0)}{1+\beta(0)} = t > 0$. It remains to prove that condition (2.2) is satisfied. Let $\gamma(x, fx)\gamma(y, fy) \geq 1$. This is possible only if $x, y \in [0, 1]$. Let $\max\{x, y\} = x$. Therefore $\max\{fx, fy\} = fx$. Then

$$\begin{aligned} \psi(p(fx, fy)) &= \frac{1}{8}(1 + x^2) + \frac{1}{4} \\ &\leq \frac{x}{4} + \frac{1}{4} \\ &\leq \frac{x + \frac{1}{2}}{1 + 1} \\ &= \frac{\alpha(p(x, y))}{1 + \beta(p(x, y))}. \end{aligned}$$

Now, we have to show that condition (ii) in Corollary 2.6 is also satisfied. Let $\gamma(x_n, x_n) \geq 1$, then $x_n \in [0,1]$. Since $x_n \to x$, we have $x \in [0,1]$ and therefore we have $\gamma(x,x) \geq 1$, and $x_n \preceq x$. Let $x_0 = 4-\sqrt{15}$. Then $\gamma(x_0, x_0) \geq 0$, and $fx_0 \in [0,1]$, we have $\gamma(x_0, fx_0) \geq 1$. Also $x_0 \preceq f(x_0)$. So $4-\sqrt{15}$ is a fixed point.

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