ALMOST RICCI SOLITON AND GRADIENT
ALMOST RICCI SOLITON ON 3-DIMENSIONAL
NORMAL ALMOST CONTACT METRIC
MANIFOLDS

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Abstract. The object of the present paper is to study almost Ricci
solitons and gradient almost Ricci solitons in 3-dimensional non-cosym-
plectic normal almost contact metric manifolds.

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1. Introduction

The study of almost Ricci soliton was introduced by Pigola et. al. [16],
where essentially they modified the definition of a Ricci soliton by adding the
condition on the parameter λ to be a variable function. More precisely, we say
that a Riemannian manifold (M^n, g) is an almost Ricci soliton if there exist a
complete vector field V and a smooth soliton function λ : M^n → ℝ satisfying

\[ \text{Ric} + \frac{1}{2} \mathcal{L} V g = \lambda g, \]  

where Ric and \( \mathcal{L} \) stand, respectively, for the Ricci tensor and Lie derivative.
We shall refer to this equation as the fundamental equation of the almost
Ricci soliton (M^n, g, V, λ). It will be called expanding, steady or shrinking,
respectively, if \( \lambda < 0 \), \( \lambda = 0 \) or \( \lambda > 0 \). Otherwise it will be called indefinite.
When the vector field V is gradient of a smooth function \( f : M^n \rightarrow ℝ \),
the manifold will be called a gradient almost Ricci soliton. In this case the
preceding equation becomes

\[ \text{Ric} + \nabla^2 f = \lambda g, \]  

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where $\nabla^2 f$ stands for the Hessian of $f$. Sometimes the classical theory of tensorial calculus is more convenient to make computations. Then, we can write the fundamental equation in this language as follows:

\[(1.3) \quad R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}.\]

Moreover, when either the vector field $X$ is trivial, or the potential $f$ is constant, the almost Ricci soliton will be called trivial, otherwise it will be a non-trivial almost Ricci soliton. We notice that when $n \geq 3$ and $X$ is a Killing vector field, an almost Ricci soliton will be a Ricci soliton, since in this case we have an Einstein manifold, from which we can apply Schur’s lemma to deduce that $\lambda$ is constant. Taking into account that the soliton function $\lambda$ is not necessarily constant, certainly comparison with soliton theory will be modified. In particular, the rigidity result contained in Theorem 1.3 of [16] indicates that almost Ricci solitons should reveal a reasonably broad generalization of the fruitful concept of a classical soliton. In fact, we refer the reader to [16] to see some of these changes.

Among the results whose purpose is to better understand the geometry of almost Ricci soliton, we mention here that Barros and Ribeiro Jr. proved in [3] that a compact gradient almost Ricci soliton with non-trivial conformal vector field is isometric to a Euclidean sphere. In the same paper they proved an integral formula for compact case, which was used to prove several rigidity results, for more details see [3].

The existence of a Ricci almost soliton has been confirmed by Pigola et. al. [16] on a certain class of warped product manifolds. Some characterization of Ricci almost soliton on a compact Riemannian manifold can be found in [11, 9, 2]. It is interesting to note that if the potential vector field $V$ of the Ricci almost soliton $(M, g, V, \lambda)$ is Killing then the soliton becomes trivial, provided the dimension of $M > 2$. Moreover, if $V$ is conformal then $M^n$ is isometric to Euclidean sphere $S^n$. Thus the Ricci almost soliton can be considered as a generalization of Einstein metric as well as Ricci soliton.

In [9], authors studied Ricci solitons and gradient Ricci solitons on 3-dimensional normal almost contact metric manifolds. In [12] authors studied compact Ricci solitons. Beside these, A. Ghosh [13] studied $K$-contact and Sasakian manifolds whose metric is a gradient almost Ricci soliton. Conditions of $K$-contact and Sasakian manifolds are stronger than normal almost contact metric manifolds in the sense that the 1-form $\eta$ of normal almost contact metric manifolds are not contact form. So, in this paper we like to study almost Ricci solitons and gradient almost Ricci solitons on 3-dimensional normal almost contact metric manifolds which is weaker than $K$-contact and Sasakian.

The present paper is organized as follows:

After preliminaries in Section 2, in Section 3 we study almost Ricci solitons in 3-dimensional non-cosymplectic normal almost contact metric manifolds and prove that if in such manifolds the metric $g$ admits an almost Ricci soliton and $V$ is pointwise collinear with $\xi$, then either the manifold is Sasakian or
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V is constant multiple of ξ and the manifold is an η-Einstein manifold, provided α, β = constant. In the converse case we prove that if a 3-dimensional non-cosymplectic normal a.c.m., with α, β = constant, is η-Einstein of the form $S = \gamma g + \delta \eta \otimes \eta$ then an almost Ricci soliton $(M, g, \xi, \lambda)$ reduces to a Ricci soliton $(M, g, \xi, \gamma + \delta)$. Beside these in this section we also prove that if a 3-dimensional non-cosymplectic normal almost contact metric manifold admits an almost Ricci soliton $(g, \xi, \lambda)$, then the manifold is of constant scalar curvature. This section concludes with some interesting corollaries and a remark. Finally, Section 4 deals with the study of gradient almost Ricci soliton $(f, \xi, \lambda)$ in a 3-dimensional non-cosymplectic normal almost contact metric manifold and we prove that in such a manifold $\lambda = (\alpha^2 - \beta^2)f$ holds or the manifold becomes Sasakian.

2. Preliminaries

Let $M$ be an almost contact manifold and $(\phi, \xi, \eta)$ its almost contact structure. This means, $M$ is an odd-dimensional differentiable manifold and $\phi$, $\xi$, $\eta$ are tensor fields on $M$ of types $(1,1)$, $(1,0)$ and $(0,1)$ respectively, such that

\[(2.1) \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0.\]

Let $\mathbb{R}$ be the real line and $t$ a coordinate on $\mathbb{R}$. Define an almost complex structure $J$ on $M \times \mathbb{R}$ by

\[(2.2) \quad J(X, \lambda \frac{d}{dt}) = (\phi X - \lambda \xi, \eta(X) \frac{d}{dt}),\]

where the pair $(X, \lambda \frac{d}{dt})$ denotes a tangent vector on $M \times \mathbb{R}$, $X$ and $\lambda \frac{d}{dt}$ being tangent to $M$ and $\mathbb{R}$ respectively.

$M$ and $(\phi, \xi, \eta)$ are said to be normal if the structure $J$ is integrable ([5],[6]). The necessary and sufficient condition for $(\phi, \xi, \eta)$ to be normal is

\[(2.3) \quad [\phi, \phi] + 2d\eta \otimes \xi = 0,\]

where the pair $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$ defined by

\[(2.4) \quad [\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y],\]

for any $X, Y \in \chi(M)$; $\chi(M)$ being the Lie algebra of vector fields on $M$.

We say that the contact form $\eta$ has rank $r = 2s$ if $(d\eta)^s \neq 0$ and $\eta \wedge (d\eta)^s = 0$ and has rank $r = 2s + 1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$. We also say $r$ is the rank of the structure $(\phi, \xi, \eta)$.

A Riemannian metric $g$ on $M$ satisfying the condition

\[(2.5) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\]
for any $X, Y \in \chi(M)$, is said to be compatible with the structure $(\phi, \xi, \eta)$. If $g$ is such a metric, then the quadruple $(\phi, \xi, \eta, g)$ is called an almost contact metric structure on $M$ and $M$ is an almost contact metric manifold. On such a manifold we also have

\begin{equation}
\eta(X) = g(X, \xi),
\end{equation}

for any $X \in \chi(M)$ and we can always define the 2-form $\Phi$ by

\begin{equation}
\Phi(Y, Z) = g(Y, \phi Z),
\end{equation}

where $Y, Z \in \chi(M)$.

A normal almost contact metric structure $(\phi, \xi, \eta, g)$ satisfying additionally the condition $d\eta = \phi$ is called Sasakian. Of course, any such structure on $M$ has rank 3. Also a three dimensional smooth manifold is said to be a contact metric manifold if $\eta \wedge d\eta = 0$ and a normal contact metric manifold is said to be Sasakian [5]. Moreover a normal almost contact metric structure satisfying the condition $d\Phi = 0$ is said to be quasi-Sasakian [4].

In [14], Olszak studied the curvature properties of normal almost contact manifolds of dimension three with several examples. A non-trivial example of three dimensional normal almost contact metric manifold has been given in [7]. Normal almost contact metric manifolds of dimension three have been studied by several authors such as [9],[8],[11],[10] and many others.

For a normal almost contact metric structure $(\phi, \xi, \eta, g)$ on $M$, we have [14]

\begin{equation}
(\nabla X \phi)(Y) = g(\phi \nabla_X \xi, Y) - \eta(Y)\phi \nabla_X \xi,
\end{equation}

\begin{equation}
\nabla_X \xi = \alpha [X - \eta(X)\xi] - \beta \phi X,
\end{equation}

where $2\alpha = div \xi$ and $2\beta = tr(\phi \nabla \xi)$, $div \xi$ is the divergent of $\xi$ defined by $div \xi = \text{trace}\{X \rightarrow \nabla_X \xi\}$ and $tr(\phi \nabla \xi) = \text{trace}\{X \rightarrow \phi \nabla_X \xi\}$. Using (2.9) in (2.8) we get

\begin{equation}
(\nabla X \phi)(Y) = \alpha [g(\phi X, Y)\xi - \eta(Y)\phi X] + \beta [g(X, Y)\xi - \eta(Y)X].
\end{equation}

Also in this manifold the following relations hold:

\begin{equation}
R(X, Y)\xi = [Y \alpha + (\alpha^2 - \beta^2)\eta(Y)]\phi^2 X - [X \alpha + (\alpha^2 - \beta^2)\eta(X)]\phi^2 Y + [Y \beta + 2\alpha \beta \eta(Y)]\phi X - [X \beta + 2\alpha \beta \eta(X)]\phi Y,
\end{equation}

\begin{equation}
S(X, \xi) = -X \alpha - (\phi X)\beta - [\xi \alpha + 2(\alpha^2 - \beta^2)]\eta(X),
\end{equation}

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\end{equation}
\[ (2.13) \quad \xi \beta + 2 \alpha \beta = 0, \]

where \( R \) denotes the curvature tensor and \( S \) is the Ricci tensor.

\[ (2.14) \quad (\nabla_X \eta)(Y) = \alpha g(\phi X, \phi Y) - \beta g(\phi X, Y). \]

On the other hand, the curvature tensor in a three dimensional Riemannian manifold always satisfies

\[ (2.15) \quad R(X, Y)Z = S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y], \]

where \( r \) is the scalar curvature of the manifold.

By (2.11), (2.12) and (2.15) we can derive

\[ \begin{align*}
S(Y, Z) &= \left( \frac{r}{2} + \xi \alpha + \alpha^2 - \beta^2 \right) g(\phi Y, \phi Z) \\
&\quad - \eta(Y)Z\alpha + (\phi Z)\beta - \eta(Z)Y\alpha + (\phi Y)\beta \\
&\quad - 2(\alpha^2 - \beta^2)\eta(Y)\eta(Z).
\end{align*} \]

From (2.13) it follows that if \( \alpha, \beta = \text{constant} \), then the manifold is either \( \beta \)-Sasakian or \( \alpha \)-Kenmotsu [18] or cosymplectic [5].

Also we know that a 3-dimensional normal almost contact metric manifold is quasi-Sasakian if and only if \( \alpha = 0 \) [14, 15].

An almost contact metric manifold is said to be \( \eta \)-Einstein if its Ricci tensor \( S \) is of the form

\[ (2.17) \quad S = \lambda g + \mu \eta \otimes \eta, \]

where \( \lambda \) and \( \mu \) are smooth functions on the manifold.

### 3. Almost Ricci Soliton

In this section we consider almost Ricci solitons on 3-dimensional normal almost contact metric manifolds \( (M, \phi, \xi, \eta, g) \) with \( \alpha, \beta = \text{constants} \). In particular, let the potential vector field \( V \) be point-wise collinear with \( \xi \) i.e., \( V = b \xi \), where \( b \) is a function on \( M \). Then from (1.1) we have

\[ (3.1) \quad g(\nabla_X b \xi, Y) + g(\nabla_Y b \xi, X) + 2S(X, Y) = 2\lambda g(X, Y). \]

Using (2.6) and (2.9) in (3.1), we get

\[ (3.2) \quad 2ab[g(X, Y) - \eta(X)\eta(Y)] + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X, Y) = 2\lambda g(X, Y). \]

Putting \( Y = \xi \) in (3.2) and using (2.1), (2.6) and (2.12) yields
(3.3) \((Xb) + (\xi b)\eta(X) - 4(\alpha^2 - \beta^2)\eta(X) = 2\lambda\eta(X)\).

Putting \(X = \xi\) in (3.3) and using (2.1) we obtain

(3.4) \(\xi b = 2(\alpha^2 - \beta^2) + \lambda\).

Putting the value of \(\xi b\) in (3.3) yields

(3.5) \(db = [\lambda + 2(\alpha^2 - \beta^2)]\eta\).

Applying \(d\) on (3.5) and using \(d^2 = 0\), we get

(3.6) \([\lambda + 2(\alpha^2 - \beta^2)]d\eta + (d\lambda)\eta = 0\).

Taking wedge product of (3.6) with \(\eta\), we have

(3.7) \([\lambda + 2(\alpha^2 - \beta^2)]\eta \wedge d\eta = 0\).

Then either \(\eta \wedge d\eta = 0\) i.e., the manifold is Sasakian or

(3.8) \(\lambda + 2(\alpha^2 - \beta^2) = 0\).

Using (3.8) in (3.5) gives \(db = 0\) i.e., \(b = \text{constant}\). Therefore from (3.2) we have

(3.9) \(S(X,Y) = (\lambda - \alpha b)g(X,Y) + \alpha b\eta(X)\eta(Y)\).

In view of (3.9) we state the following:

**Theorem 3.1.** If in a 3-dimensional non-cosymplectic normal almost contact metric manifold the metric \(g\) admits almost Ricci soliton and \(V\) is pointwise collinear with \(\xi\), then either the manifold is Sasakian or \(V\) is constant multiple of \(\xi\) and the manifold is \(\eta\)-Einstein, provided \(\alpha, \beta = \text{constant}\).

Conversely, let \(M\) be an \(\eta\)-Einstein 3-dimensional non-cosymplectic normal almost contact metric manifold with \(\alpha, \beta = \text{constant}\). Then

(3.10) \(S(X,Y) = \gamma g(X,Y) + \delta \eta(X)\eta(Y)\),

where \(\gamma\) and \(\delta\) are certain smooth functions defined on \(M\).

In view of (3.10) we have
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\( (L_\xi g)(X,Y) + 2S(X,Y) = 2\lambda g(X,Y) \)
\[= 2(\alpha + \gamma - \lambda)g(X,Y) - 2(\alpha - \delta)\eta(X)\eta(Y). \]

From (3.11), it follows that \( M \) admits an almost Ricci soliton \((g, \xi, \lambda)\) if \( \alpha + \gamma - \lambda = 0 \) and \( \delta = \alpha = \text{constant} \). Again putting \( X = Y = \xi \) in (3.10) and using (2.1) and (2.12) for \( X = \xi \) yields \( \gamma = 2(\alpha^2 - \beta^2) - \alpha = \text{constant} \). Therefore \( \lambda = \gamma + \delta = \text{constant} \). Hence we state the following:

**Theorem 3.2.** If a 3-dimensional non-cosymplectic normal a.c.m. manifold with \( \alpha, \beta = \text{constant} \) is \( \eta \)-Einstein of the form \( S = \gamma g + \delta \eta \otimes \eta \) then a Ricci almost soliton \((M, g, \xi, \lambda)\) reduces to a Ricci soliton \((M, g, \xi, \gamma + \delta)\).

Now let \( V = \xi \). Then (3.1) reduces to

\[ (L_\xi g)(X,Y) + 2S(X,Y) = 2\lambda g(X,Y). \]

Now, in view of (2.9) we have

\[ (L_\xi g)(X,Y) = 2\alpha\{g(X,Y) - \eta(X)\eta(Y). \]

In view of (2.16) for \( \alpha, \beta = \text{constants} \) and (3.13) we have

\[ A(X,Y) = (L_\xi g)(X,Y) + 2S(X,Y) \]
\[= [r + 2(\alpha^2 - \beta^2 + \alpha)]g(X,Y) \]
\[-[r + 2\{3(\alpha^2 - \beta^2) + \alpha\}]\eta(X)\eta(Y). \]

Using (3.14) in (3.12), we obtain

\[ [r + 2(\alpha^2 - \beta^2 + \alpha) - 2\lambda]g(X,Y) \]
\[-[r + 2\{3(\alpha^2 - \beta^2) + \alpha\}]\eta(X)\eta(Y) = 0. \]

Putting \( X = Y = \xi \) in (3.15) and using (2.1), we obtain

\[ \lambda = -2(\alpha^2 - \beta^2). \]

Since we consider \( \alpha, \beta = \text{constants} \), hence \( \lambda = \text{constant} \). Therefore the almost Ricci soliton becomes a Ricci soliton. Hence we state the following:

**Proposition 3.1.** If a 3-dimensional non-cosymplectic normal almost contact metric manifold with \( \alpha, \beta = \text{constant} \) admits an almost Ricci soliton then the manifold admits a Ricci soliton.

In [9] the authors proved that a 3-dimensional non-cosymplectic normal almost contact metric manifold admitting a Ricci soliton \((g, \xi, \lambda)\) is of constant scalar curvature. Hence we state the following:
Theorem 3.3. If a 3-dimensional non-cosymplectic normal almost contact metric manifold admits an almost Ricci soliton \((g, \xi, \lambda)\), then the manifold is of constant scalar curvature.

Again, we know that \([2]\) non-trivial compact almost Ricci solitons with constant scalar curvature are gradient. Therefore we state the following:

Corollary 3.1. A 3-dimensional non-cosymplectic normal almost contact metric manifold admitting the almost Ricci soliton \((M, g, \xi, \lambda)\) is necessarily admits gradient Ricci soliton.

In \([17]\), author proved that a compact Ricci soliton of constant curvature is Einstein. Hence in view of Theorem 3.2 and Theorem 3.3 we state the following:

Corollary 3.2. If a 3-dimensional non-cosymplectic normal almost contact metric manifold with \(\alpha, \beta = \text{constant}\) admits a compact almost Ricci soliton \((M, g, \xi, \lambda)\), then the manifold is Einstein.

Again, in view of (3.16) we state the following:

Remark 3.1. If a 3-dimensional non-cosymplectic normal almost contact metric manifold admits the almost Ricci soliton \((g, \xi, \lambda)\), then the almost Ricci soliton is shrinking, steady and expanding when \(\alpha > \beta\), \(\alpha = \beta\) and \(\alpha < \beta\) respectively, provided \(\alpha, \beta = \text{constant}\).

4. Gradient Almost Ricci Soliton

This section is devoted to the study 3-dimensional normal almost contact metric manifolds admitting gradient almost Ricci soliton. For a gradient almost Ricci soliton, we have

\[
\nabla_Y Df = \lambda Y - QY,
\]

where \(D\) denotes the gradient operator of \(g\).

Differentiating (4.1) covariantly in the direction of \(X\) yields

\[
\nabla_X \nabla_Y Df = d\lambda(X)Y + \lambda \nabla_X Y - (\nabla_X Q)Y.
\]

Similarly, we get

\[
\nabla_Y \nabla_X Df = d\lambda(Y)X + \lambda \nabla_Y X - (\nabla_Y Q)X.
\]

and

\[
\nabla_{[X,Y]} Df = \lambda [X,Y] - Q[X,Y].
\]

In view of (4.2), (4.3) and (4.4), we have

\[
R(X, Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df
= (\nabla_Y Q)X - (\nabla_X Q)Y - (Y\lambda)X + (X\lambda Y).
\]
For $\alpha, \beta =$constants, we get from (2.16)

\[(4.6) \quad QY = \left(\frac{r}{2} + \alpha^2 - \beta^2\right)Y - \left(\frac{r}{2} - \alpha^2 + \beta^2\right)\eta(Y)\xi.\]

Differentiating (4.6) covariantly in the direction of $X$ and using (2.9) and (2.14), we get

\[(4.7) \quad (\nabla_X Q)Y = \left(\frac{r}{2} + \alpha^2 - \beta^2\right)\nabla_X Y - \frac{dr(X)}{2}\phi^2 X - \frac{r}{2}\mathcal{g}(X, Y)\xi - \beta\phi X.\]

Replacing $X$ by $\xi$ in (4.7) and using (2.9), we obtain

\[(4.8) \quad (\nabla_X Q)\xi = 2(\alpha^2 - \beta^2)[\alpha\{X - \eta(X)\xi\} - \beta\phi X].\]

In view of (4.5) and (4.8), we have

\[(4.9) \quad g(R(X, Y)Df, \xi) = 4\beta(\alpha^2 - \beta^2)g(\phi X, Y) - (Y\lambda)\eta(X) + (X\lambda)\eta(Y).\]

Replacing $Y$ by $\xi$ in (4.9) and using (2.1), we have

\[(4.10) \quad g(R(X, \xi)Df, \xi) = -X\lambda + (\xi\lambda)\eta(X).\]

Putting $Y = \xi$ in (2.11) and using in (4.9), we obtain

\[(4.11) \quad (\alpha^2 - \beta^2)df - d\lambda = [(\alpha^2 - \beta^2)\xi f - (\xi\lambda)]\eta.\]

Operating (4.11) by $d$ and using the Poincare lemma $d^2 \equiv 0$, we get

\[(4.12) \quad d[(\alpha^2 - \beta^2)(\xi f) - (\xi\lambda)]\eta \wedge d\eta = 0.\]

Then either $\eta \wedge d\eta = 0$ i.e., the manifold is Sasakian or

\[(4.13) \quad (\alpha^2 - \beta^2)f - \lambda = \text{constant}.\]

In view of above discussion we state the following:

**Theorem 4.1.** If a 3-dimensional normal almost contact metric manifold admits gradient almost Ricci soliton $(f, \xi, \lambda)$ then either the manifold is Sasakian or $\lambda = (\alpha^2 - \beta^2)f$. 
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