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# ALMOST RICCI SOLITON AND GRADIENT ALMOST RICCI SOLITON ON 3-DIMENSIONAL NORMAL ALMOST CONTACT METRIC MANIFOLDS [1]

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**Abstract.** The object of the present paper is to study almost Ricci solitons and gradient almost Ricci solitons in 3-dimensional non-cosymplectic normal almost contact metric manifolds.

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## 1. Introduction

The study of almost Ricci soliton was introduced by Pigola et. al. [16], where essentially they modified the definition of a Ricci soliton by adding the condition on the parameter $\lambda$ to be a variable function. More precisely, we say that a Riemannian manifold $(M^n, g)$ is an almost Ricci soliton if there exist a complete vector field $V$ and a smooth soliton function $\lambda : M^n \longrightarrow \mathbb{R}$ satisfying

$$Ric + \frac{1}{2}\pounds_V g = \lambda g, \tag{1.1}$$

where $Ric$ and $\pounds$ stand, respectively, for the Ricci tensor and Lie derivative. We shall refer to this equation as the fundamental equation of the almost Ricci soliton $(M^n, g, V, \lambda)$. It will be called expanding, steady or shrinking, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$. Otherwise it will be called indefinite. When the vector field $V$ is gradient of a smooth function $f : M^n \longrightarrow \mathbb{R}$, the manifold will be called a gradient almost Ricci soliton. In this case the preceding equation becomes

$$Ric + \nabla^2 f = \lambda g, \tag{1.2}$$

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where $\nabla^2 f$ stands for the Hessian of $f$. Sometimes the classical theory of tensorial calculus is more convenient to make computations. Then, we can write the fundamental equation in this language as follows:

$$R_{ij} + \nabla_i \nabla_j f = \lambda g_{ij}. \tag{1.3}$$

Moreover, when either the vector field $X$ is trivial, or the potential $f$ is constant, the almost Ricci soliton will be called trivial, otherwise it will be a non-trivial almost Ricci soliton. We notice that when $n \geq 3$ and $X$ is a Killing vector field, an almost Ricci soliton will be a Ricci soliton, since in this case we have an Einstein manifold, from which we can apply Schur's lemma to deduce that $\lambda$ is constant. Taking into account that the soliton function $\lambda$ is not necessarily constant, certainly comparison with soliton theory will be modified. In particular, the rigidity result contained in Theorem 1.3 of [16] indicates that almost Ricci solitons should reveal a reasonably broad generalization of the fruitful concept of a classical soliton. In fact, we refer the reader to [16] to see some of these changes.

Among the results whose purpose is to better understand the geometry of almost Ricci soliton, we mention here that Barros and Ribeiro Jr. proved in [3] that a compact gradient almost Ricci soliton with non-trivial conformal vector field is isometric to a Euclidean sphere. In the same paper they proved an integral formula for compact case, which was used to prove several rigidity results, for more details see [3].

The existence of a Ricci almost soliton has been confirmed by Pigola et. al. [16] on a certain class of warped product manifolds. Some characterization of Ricci almost soliton on a compact Riemannian manifold can be found in ([1],[3],[2]). It is interesting to note that if the potential vector field $V$ of the Ricci almost soliton $(M, g, V, \lambda)$ is Killing then the soliton becomes trivial, provided the dimension of $M > 2$. Moreover, if $V$ is conformal then $M^n$ is isometric to Euclidean sphere $S^n$. Thus the Ricci almost soliton can be considered as a generalization of Einstein metric as well as Ricci soliton.

In [9], authors studied Ricci solitons and gradient Ricci solitons on 3-dimensional normal almost contact metric manifolds. In [12] authors studied compact Ricci solitons. Beside these, A. Ghosh [13] studied $K$-contact and Sasakian manifolds whose metric is a gradient almost Ricci soliton. Conditions of $K$-contact and Sasakian manifolds are stronger than normal almost contact metric manifolds in the sense that the 1-form $\eta$ of normal almost contact metric manifolds are not contact form. So, in this paper we like to study almost Ricci solitons and gradient almost Ricci solitons on 3-dimensional normal almost contact metric manifolds which is weaker than $K$-contact and Sasakian.

The present paper is organized as follows:
After preliminaries in Section 2, in Section 3 we study almost Ricci solitons in 3-dimensinal non-cosymplectic normal almost contact metric manifolds and prove that if in such manifolds the metric $g$ admits an almost Ricci soliton and $V$ is pointwise collinear with $\xi$, then either the manifold is Sasakian or

$V$ is constant multiple of $\xi$ and the manifold is an $\eta$-Einstein manifold, provided $\alpha, \beta$ =constant. In the converse case we prove that if a 3-dimensional non-cosymplectic normal a.c.m., with $\alpha, \beta$ =constant, is $\eta$-Einstein of the form $S = \gamma g + \delta\eta \otimes \eta$ then an almost Ricci soliton $(M, g, \xi, \lambda)$ reduces to a Ricci soliton $(M, g, \xi, \gamma + \delta)$. Beside these in this section we also prove that if a 3-dimensional non-cosymplectic normal almost contact metric manifold admits an almost Ricci soliton $(g, \xi, \lambda)$, then the manifold is of constant scalar curvature. This section concludes with some interesting corollaries and a remark. Finally, Section 4 deals with the study of gradient almost Ricci solition $(f, \xi, \lambda)$ in a 3-dimensional non-cosymplectic normal almost contact metric manifold and we prove that in such a manifold $\lambda = (\alpha^2 - \beta^2)f$ holds or the manifold becomes Sasakian.

## 2. Preliminaries

Let $M$ be an almost contact manifold and $(\phi, \xi, \eta)$ its almost contact structure. This means, $M$ is an odd-dimensional differentiable manifold and $\phi$, $\xi$, $\eta$ are tensor fields on $M$ of types $(1, 1)$, $(1, 0)$ and $(0, 1)$ respectively, such that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0. \tag{2.1}$$

Let $\mathbb{R}$ be the real line and $t$ a coordinate on $\mathbb{R}$. Define an almost complex structure $J$ on $M \times \mathbb{R}$ by

$$J(X, \lambda\frac{d}{dt}) = (\phi X - \lambda\xi, \eta(X)\frac{d}{dt}), \tag{2.2}$$

where the pair $(X, \lambda\frac{d}{dt})$ denotes a tangent vector on $M \times \mathbb{R}$, $X$ and $\lambda\frac{d}{dt}$ being tangent to $M$ and $\mathbb{R}$ respectively.

$M$ and $(\phi, \xi, \eta)$ are said to be normal if the structure $J$ is integrable ([5],[6]).

The necessary and sufficient condition for $(\phi, \xi, \eta)$ to be normal is

$$[\phi, \phi] + 2d\eta \otimes \xi = 0, \tag{2.3}$$

where the pair $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$ defined by

$$[\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \tag{2.4}$$

for any $X, Y \in \chi(M)$; $\chi(M)$ being the Lie algebra of vector fields on $M$.

We say that the contact form $\eta$ has rank $r = 2s$ if $(d\eta)^s \neq 0$ and $\eta\wedge(d\eta)^s = 0$ and has rank $r = 2s+1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$. We also say $r$ is the rank of the structure $(\phi, \xi, \eta)$.

A Riemannian metric $g$ on $M$ satisfying the condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.5}$$

for any $X, Y \in \chi(M)$, is said to be compatible with the structure $(\phi, \xi, \eta)$. If $g$ is such a metric, then the quadruple $(\phi, \xi, \eta, g)$ is called an almost contact metric structure on $M$ and $M$ is an almost contact metric manifold. On such a manifold we also have

$$\eta(X) = g(X, \xi), \tag{2.6}$$

for any $X \in \chi(M)$ and we can always define the 2-form $\Phi$ by

$$\Phi(Y, Z) = g(Y, \phi Z), \tag{2.7}$$

where $Y, Z \in \chi(M)$.

A normal almost contact metric structure $(\phi, \xi, \eta, g)$ satisfying additionally the condition $d\eta = \phi$ is called Sasakian. Of course, any such structure on $M$ has rank 3. Also a three dimensional smooth manifold is said to be a contact metric manifold if $\eta \wedge d\eta = 0$ and a normal contact metric manifold is said to be Sasakian [5]. Moreover a normal almost contact metric structure satisfying the condition $d\Phi = 0$ is said to be quasi-Sasakian [4].

In [14], Olszak studied the curvature properties of normal almost contact manifolds of dimension three with several examples. A non-trivial example of three dimensional normal almost contact metric manifold has been given in [7]. Normal almost contact metric manifolds of dimension three have been studied by several authors such as ([9],[8],[11],[10]) and many others.

For a normal almost contact metric structure $(\phi, \xi, \eta, g)$ on $M$, we have [14]

$$(\nabla_X \phi)(Y) = g(\phi \nabla_X \xi, Y) - \eta(Y)\phi \nabla_X \xi, \tag{2.8}$$

$$\nabla_X \xi = \alpha[X - \eta(X)\xi] - \beta\phi X, \tag{2.9}$$

where $2\alpha = div\xi$ and $2\beta = tr(\phi\nabla\xi)$, $div\xi$ is the divergent of $\xi$ defined by $div\xi = trace\{X \longrightarrow \nabla_X \xi\}$ and $tr(\phi\nabla\xi) = trace\{X \longrightarrow \phi\nabla_X \xi\}$. Using (2.9) in (2.8) we get

$$(\nabla_X \phi)(Y) = \alpha[g(\phi X, Y)\xi - \eta(Y)\phi X] + \beta[g(X, Y)\xi - \eta(Y)X]. \tag{2.10}$$

Also in this manifold the following relations hold:

$$\begin{aligned} R(X, Y)\xi &= [Y\alpha + (\alpha^2 - \beta^2)\eta(Y)]\phi^2 X \\ &\quad -[X\alpha + (\alpha^2 - \beta^2)\eta(X)]\phi^2 Y \\ &\quad +[Y\beta + 2\alpha\beta\eta(Y)]\phi X \\ &\quad -[X\beta + 2\alpha\beta\eta(X)]\phi Y, \end{aligned} \tag{2.11}$$

$$\begin{aligned} S(X, \xi) &= -X\alpha - (\phi X)\beta \\ &\quad -[\xi\alpha + 2(\alpha^2 - \beta^2)]\eta(X), \end{aligned} \tag{2.12}$$

$$\xi\beta + 2\alpha\beta = 0, \tag{2.13}$$

where $R$ denotes the curvature tensor and $S$ is the Ricci tensor.

$$(\nabla_X \eta)(Y) = \alpha g(\phi X, \phi Y) - \beta g(\phi X, Y). \tag{2.14}$$

On the other hand, the curvature tensor in a three dimensional Riemannian manifold always satisfies

$$\begin{aligned} R(X,Y)Z &= S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.15}$$

where $r$ is the scalar curvature of the manifold.

By (2.11), (2.12) and (2.15) we can derive

$$\begin{aligned} S(Y,Z) &= (\frac{r}{2} + \xi\alpha + \alpha^2 - \beta^2)g(\phi Y, \phi Z) \\ &\quad -\eta(Y)(Z\alpha + (\phi Z)\beta) - \eta(Z)(Y\alpha + (\phi Y)\beta) \\ &\quad -2(\alpha^2 - \beta^2)\eta(Y)\eta(Z). \end{aligned} \tag{2.16}$$

From (2.13) it follows that if $\alpha, \beta$ =constant, then the manifold is either $\beta$-Sasakian or $\alpha$-Kenmotsu [18] or cosymplectic [5].

Also we know that a 3-dimensional normal almost contact metric manifold is quasi-Sasakian if and only if $\alpha = 0$ ([14],[15]).

An almost contact metric manifold is said to be $\eta$-Einstein if its Ricci tensor $S$ is of the form

$$S = \lambda g + \mu\eta \otimes \eta, \tag{2.17}$$

where $\lambda$ and $\mu$ are smooth functions on the manifold.

## 3. Almost Ricci Soliton

In this section we consider almost Ricci solitons on 3-dimensional normal almost contact metric manifolds $(M, \phi, \xi, \eta, g)$ with $\alpha, \beta$ =constants. In particular, let the potential vector field $V$ be point-wise collinear with $\xi$ i.e., $V = b\xi$, where $b$ is a function on $M$. Then from (1.1) we have

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X,Y) = 2\lambda g(X,Y). \tag{3.1}$$

Using (2.6) and (2.9) in (3.1), we get

$$\begin{aligned} 2\alpha b[g(X,Y) - \eta(X)\eta(Y)] &+ (Xb)\eta(Y) + (Yb)\eta(X) \\ &+2S(X,Y) = 2\lambda g(X,Y). \end{aligned} \tag{3.2}$$

Putting $Y = \xi$ in (3.2) and using (2.1), (2.6) and (2.12) yields

$$(Xb) + (\xi b)\eta(X) - 4(\alpha^2 - \beta^2)\eta(X) = 2\lambda\eta(X). \tag{3.3}$$

Putting $X = \xi$ in (3.3) and using (2.1) we obtain

$$\xi b = 2(\alpha^2 - \beta^2) + \lambda. \tag{3.4}$$

Putting the value of $\xi b$ in (3.3) yields

$$db = [\lambda + 2(\alpha^2 - \beta^2)]\eta. \tag{3.5}$$

Applying $d$ on (3.5) and using $d^2 = 0$, we get

$$[\lambda + 2(\alpha^2 - \beta^2)]d\eta + (d\lambda)\eta = 0. \tag{3.6}$$

Taking wedge product of (3.6) with $\eta$, we have

$$[\lambda + 2(\alpha^2 - \beta^2)]\eta \wedge d\eta = 0. \tag{3.7}$$

Then either $\eta \wedge d\eta = 0$ i.e., the manifold is Sasakian or

$$\lambda + 2(\alpha^2 - \beta^2) = 0. \tag{3.8}$$

Using (3.8) in (3.5) gives $db = 0$ i.e., $b =$constant. Therefore from (3.2) we have

$$S(X, Y) = (\lambda - \alpha b)g(X, Y) + \alpha b\eta(X)\eta(Y). \tag{3.9}$$

In view of (3.9) we state the following:

**Theorem 3.1.** *If in a 3-dimensional non-cosymplectic normal almost contact metric manifold the metric $g$ admits almost Ricci soliton and $V$ is pointwise collinear with $\xi$, then either the manifold is Sasakian or $V$ is constant multiple of $\xi$ and the manifold is $\eta$-Einstein, provided $\alpha, \beta =$constant.*

Conversely, let $M$ be an $\eta$-Einstein 3-dimensional non-cosymplectic normal almost contact metric manifold with $\alpha, \beta =$constant. Then

$$S(X, Y) = \gamma g(X, Y) + \delta\eta(X)\eta(Y), \tag{3.10}$$

where $\gamma$ and $\delta$ are certain smooth functions defined on $M$.

In view of (3.10) we have

$$(3.11)\quad (\pounds_\xi g)(X,Y) + 2S(X,Y) - 2\lambda g(X,Y) = 2(\alpha+\gamma-\lambda)g(X,Y) - 2(\alpha-\delta)\eta(X)\eta(Y).$$

From (3.11), it follows that $M$ admits an almost Ricci soliton $(g,\xi,\lambda)$ if $\alpha+\gamma-\lambda=0$ and $\delta=\alpha=$constant. Again putting $X=Y=\xi$ in (3.10) and using (2.1) and (2.12) for $X=\xi$ yields $\gamma=-2(\alpha^2-\beta^2)-\alpha=$constant. Therefore $\lambda=\gamma+\delta=$constant. Hence we state the following:

**Theorem 3.2.** *If a 3-dimensional non-cosymplectic normal a.c.m. manifold with $\alpha,\beta=$constant is $\eta$-Einstein of the form $S=\gamma g+\delta\eta\otimes\eta$ then a Ricci almost soliton $(M,g,\xi,\lambda)$ reduces to a Ricci soliton $(M,g,\xi,\gamma+\delta)$.*

Now let $V=\xi$. Then (3.1) reduces to

$$(3.12)\quad (\pounds_\xi g)(X,Y) + 2S(X,Y) = 2\lambda g(X,Y).$$

Now, in view of (2.9) we have

$$(3.13)\quad (\pounds_\xi g)(X,Y) = 2\alpha\{g(X,Y) - \eta(X)\eta(Y).$$

In view of (2.16) for $\alpha,\beta=$constants and (3.13) we have

$$(3.14)\quad \begin{aligned} A(X,Y) &= (\pounds_\xi g)(X,Y) + 2S(X,Y) \\ &= [r+2(\alpha^2-\beta^2+\alpha)]g(X,Y) \\ &\quad -[r+2\{3(\alpha^2-\beta^2)+\alpha\}]\eta(X)\eta(Y). \end{aligned}$$

Using (3.14) in (3.12), we obtain

$$(3.15)\quad \begin{aligned} &[r+2(\alpha^2-\beta^2+\alpha)-2\lambda]g(X,Y) \\ &-[r+2\{3(\alpha^2-\beta^2)+\alpha\}]\eta(X)\eta(Y) = 0. \end{aligned}$$

Putting $X=Y=\xi$ in (3.15) and using (2.1), we obtain

$$(3.16)\quad \lambda = -2(\alpha^2-\beta^2).$$

Since we consider $\alpha,\beta=$constants, hence $\lambda=$constant. Therefore the almost Ricci soliton becomes a Ricci soliton. Hence we state the following:

**Proposition 3.1.** *If a 3-dimensional non-cosymplectic normal almost contact metric manifold with $\alpha,\beta=$constant admits an almost Ricci soliton then the manifold admits a Ricci soliton.*

In [9] the authors proved that a 3-dimensional non-cosymplectic normal almost contact metric manifold admitting a Ricci soliton $(g,\xi,\lambda)$ is of constant scalar curvature. Hence we state the following:

**Theorem 3.3.** *If a 3-dimensional non-cosymplectic normal almost contact metric manifold admits an almost Ricci soliton $(g, \xi, \lambda)$, then then the manifold is of constant scalar curvature.*

Again, we know that [2] non-trivial compact almost Ricci solitons with constant scalar curvature are gradient. Therefore we state the following:

**Corollary 3.1.** *A 3-dimensional non-cosymplectic normal almost contact metric manifold admitting the almost Ricci soliton $(M, g, \xi, \lambda)$ is necessarily admits gradient Ricci soliton.*

In [17], author proved that a compact Ricci soliton of constant curvature is Einstein. Hence in view of Theorem 3.2 and Theorem 3.3 we state the following:

**Corollary 3.2.** *If a 3-dimensional non-cosymplectic normal almost contact metric manifold with $\alpha, \beta$ =constant admits a compact almost Ricci soliton $(M, g, \xi, \lambda)$, then the manifold is Einstein.*

Again, in view of (3.16) we state the following:

**Remark 3.1.** *If a 3-dimensional non-cosymplectic normal almost contact metric manifold admits the almost Ricci soliton $(g, \xi, \lambda)$, then the almost Ricci soliton is shrinking, steady and expanding when $\alpha > \beta$, $\alpha = \beta$ and $\alpha < \beta$ respectively, provided $\alpha, \beta$ =constant.*

## 4. Gradient Almost Ricci Soliton

This section is devoted to the study 3-dimensional normal almost contact metric manifolds admitting gradient almost Ricci soliton. For a gradient almost Ricci soliton, we have

$$\nabla_Y Df = \lambda Y - QY, \tag{4.1}$$

where $D$ denotes the gradient operator of $g$.

Differentiating (4.1) covariantly in the direction of $X$ yields

$$\nabla_X \nabla_Y Df = d\lambda(X)Y + \lambda \nabla_X Y - (\nabla_X Q)Y. \tag{4.2}$$

Similarly, we get

$$\nabla_Y \nabla_X Df = d\lambda(Y)X + \lambda \nabla_Y X - (\nabla_Y Q)X. \tag{4.3}$$

and

$$\nabla_{[X,Y]} Df = \lambda[X, Y] - Q[X, Y]. \tag{4.4}$$

In view of (4.2),(4.3) and (4.4), we have

$$\begin{aligned} R(X, Y)Df &= \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df \\ &= (\nabla_Y Q)X - (\nabla_X Q)Y - (Y\lambda)X + (X\lambda Y). \end{aligned} \tag{4.5}$$

For $\alpha, \beta$ =constants, we get from (2.16)

$$QY = (\frac{r}{2} + \alpha^2 - \beta^2)Y - (\frac{r}{2} - \alpha^2 + \beta^2)\eta(Y)\xi. \tag{4.6}$$

Differentiating (4.6) covariantly in the direction of $X$ and using (2.9) and (2.14), we get

$$\begin{aligned}(\nabla_X Q)Y &= (\frac{r}{2} + \alpha^2 - \beta^2)\nabla_X Y - \frac{dr(X)}{2}\phi^2 X \\ &\quad -(\frac{r}{2} - \alpha^2 + \beta^2)[\alpha g(X,Y)\xi - 2\alpha\eta(X)\eta(Y)\xi \\ &\quad +2\alpha\eta(Y)X - \beta g(\phi X, Y)\xi - \beta\eta(Y)\phi X].\end{aligned} \tag{4.7}$$

Replacing $X$ by $\xi$ in (4.7) and using (2.9), we obtain

$$(\nabla_X Q)\xi = 2(\alpha^2 - \beta^2)[\alpha\{X - \eta(X)\xi\} - \beta\phi X]. \tag{4.8}$$

In view of (4.5) and (4.8), we have

$$g(R(X,Y)Df, \xi) = 4\beta(\alpha^2 - \beta^2)g(\phi X, Y) - (Y\lambda)\eta(X) + (X\lambda)\eta(Y). \tag{4.9}$$

Replacing $Y$ by $\xi$ in (4.9) and using (2.1), we have

$$g(R(X,\xi)Df, \xi) = -X\lambda + (\xi\lambda)\eta(X). \tag{4.10}$$

Putting $Y = \xi$ in (2.11) and using in (4.9), we obtain

$$(\alpha^2 - \beta^2)df - d\lambda = [(\alpha^2 - \beta^2)\xi f - (\xi\lambda)]\eta. \tag{4.11}$$

Operating (4.11) by $d$ and using the Poincare lemma $d^2 \equiv 0$, we get

$$d[(\alpha^2 - \beta^2)(\xi f) - (\xi\lambda)]\eta \wedge d\eta = 0. \tag{4.12}$$

Then either $\eta \wedge d\eta = 0$ i.e., the manifold is Sasakian or

$$(\alpha^2 - \beta^2)f - \lambda = \text{constant}. \tag{4.13}$$

In view of above discussion we state the following:

**Theorem 4.1.** *If a 3-dimensional normal almost contact metric manifold admits gradient almost Ricci soliton $(f, \xi, \lambda)$ then either the manifold is Sasakian or $\lambda = (\alpha^2 - \beta^2)f$.*

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