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ON ALMOST PSEUDO m-PROJECTIVELY SYMMETRIC MANIFOLDS

Chawngthu Lalmalsawma[1] **and Jay Prakash Singh**[23]

Abstract. The object of the present paper is to study almost pseudo m-projectively symmetric manifolds. Some geometric properties of almost pseudo m-projectively symmetric manifolds have been studied under certain curvature conditions. Finally the existence of almost pseudo m-projectively symmetric manifolds is shown by examples.

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1. Introduction

In 1926, Cartan [4] studied certain class of Riemannian spaces and he introduced the notion of a symmetric space. Accoding to Cartan, an n-dimensional Riemannian manifold M is said to be locally symmetric if it has constant curvature, i.e if the curvature tensor satisfies $R_{hijk,l} = 0$, where ‘,’ denotes the covariant differentiation with respect to the metric tensor and R_{hijk} are the components of the curvature tensor. Later, symmetric manifolds have been studied by many authors such as: recurrent manifolds introduced by Walker [23], conformally symmetric manifolds by Chaki and Gupta [6], conformally recurrent manifolds by Adati and Miyazawa [1], pseudo symmetric manifolds introduced by Chaki [5], almost pseudo symmetric and almost pseudo conformally symmetric manifolds by De and Gazi [11, 12] etc.

The notion of weakly symmetric manifolds was introduced by Tamassy and Binh [22] in 1989. A non-flat Riemannian manifold (M^n, g), $(n > 2)$ is called weakly symmetric if the curvature tensor $\tilde{R}$ of type (1,3) satisfies the condition:

$$\nabla_X \tilde{R}(Y,Z)W = A(X)\tilde{R}(Y,Z)W + B(Y)\tilde{R}(X,Z)W + D(Z)\tilde{R}(Y,X)W \\ + E(W)\tilde{R}(Y,Z)X + g(\tilde{R}(Y,Z)W,X)P,$$

where ∇ denotes the Levi-Civita connection on (M^n, g) and A, B, D, E and P are 1-forms and a vector field respectively which are non-zero simultaneously.

[1]Department of Mathematics and Computer Science, Mizoram University, e-mail: sweezychawngthu@gmail.com
[2]Department of Mathematics and Computer Science, Mizoram University, e-mail: jpsmaths@gmail.com
[3]Corresponding author

Such a manifold is denoted by $(WS)_n$. Weakly symmetric manifolds have been studied by several authors ([3], [10], [2], [17], [18] and many others).

In 1987, Chaki [5] studied a type of non-flat Riemannian manifold whose curvature tensor satisfies

$$R_{hijk,l} = 2\lambda_l R_{hijk} + \lambda_h R_{lijk} + \lambda_i R_{hljk} + \lambda_j R_{hilk} + \lambda_k R_{hijl}, \tag{1.1}$$

where λ_l is a non-zero covariant vector. A manifold whose curvature tensor satisfies the above equation is called a pseudo symmetric manifold [5]. It is to be noted that (1.1) was already obtained by Sen and Chaki [20] when they studied certain kind of a conformally flat Riemannian space. A pseudo symmetric manifold is denoted by $(PS)_n$.

In index-free notation equation (1.1) is given by

$$\begin{aligned}(\nabla_X \tilde{R})(Y,Z)W &= 2A(X)\tilde{R}(Y,Z)W + A(Y)\tilde{R}(X,Z)W + A(Z)\tilde{R}(Y,X)W \\ &+ A(W)\tilde{R}(Y,Z)X + g(\tilde{R}(Y,Z)W,X)P,\end{aligned} \tag{1.2}$$

where $\tilde{R}$ is a Riemannian curvature tensor of type (1,3), ∇ denotes the Levi-Civita connection, A is a non-zero 1-form and P is a vector field defined by

$$g(X,P) = A(X),$$

for all X.

In 2008 De and Gazi [11] introduced a type of Riemannian manifold which is a generalization of pseudo symmetric manifolds. Such manifold is called an almost pseudo symmetric manifold and is denoted by $(APS)_n$. A Riemannian manifold(M^n, g), $(n > 2)$ is said to be almost pseudo symmetric [11] if its curvature tensor R of type $(0,4)$ satisfies the condition:

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= [A(X)+B(X)]R(Y,Z,U,V) \\ &+A(Y)R(X,Z,U,V) + A(Z)R(Y,X,U,V) \\ &+A(U)R(Y,Z,X,V) + A(V)R(Y,Z,U,X),\end{aligned} \tag{1.3}$$

where A, B are non-zero 1-forms defined by

$$g(X,P) = A(X), g(X,Q) = B(X),$$

for all vector fields X. In the papers ([12], [13]) it was mentioned that $(PS)_n$ is a particular case of an $(APS)_n$, but$(WS)_n$ is not a particular case of an $(APS)_n$.

In 1971, Pokhariyal and Mishra [16] introduced a new curvature tensor of type (1,3) in an n-dimensional Riemannian manifold (M^n, g), $n > 2$ denoted by $\tilde{M}$ and defined by

$$\begin{aligned}\tilde{M}(Y,Z)U = \tilde{R}(Y,Z)U - \frac{1}{2(n-1)}[&S(Z,U)Y - S(Y,U)Z \\ &+g(Z,U)LY - g(Y,U)LZ],\end{aligned} \tag{1.4}$$

where $\tilde{R}$ and L denote the Riemannian curvature tensor of type $(1,3)$ and the Ricci operator defined by $g(LX,Y) = S(X,Y)$, respectively. Such a tensor $\tilde{M}$ is known as an m-projective curvature tensor. The m-projective curvature tensor have been studied by J.P. Singh [21], S.K. Chaubey and R.H. Ojha [8], S.K. Chaubey [7], and many others.

From (1.4) we can define a $(0,4)$ type m-projective curvature tensor M as follows:

$$\begin{aligned} M(Y,Z,U,V) &= R(Y,Z,U,V) - \frac{1}{2(n-1)}[S(Z,U)g(Y,V) \\ &-S(Y,U)g(Z,V) + S(Y,V)g(Z,U) - S(Z,V)g(Y,U)], \end{aligned} \tag{1.5}$$

where R denotes the Riemannian curvature tensor of type (0,4) defined by

$$R(Y,Z,U,V) = g(\tilde{R}(Y,Z)U,V),$$

and

$$M(Y,Z,U,V) = g(\tilde{M}(Y,Z)U,V),$$

where $\tilde{R}$ is the Riemannian curvature tensor of type (1,3) and S denotes the Ricci tensor of type (0,2) respectively.

The m-projective curvature tensor satisfies the properties of the Riemannian curvature tensor. The object of the present paper is to study a type of non-flat Riemannian manifold (M^n, g), $(n > 2)$ whose m-projective curvature tensor M of type (0,4) satisfies the condition:

$$\begin{aligned} (\nabla_X M)(Y,Z,U,V) &= [A(X) + B(X)]M(Y,Z,U,V) \\ &+A(Y)M(X,Z,U,V) + A(Z)M(Y,X,U,V) \\ &+A(U)M(Y,Z,X,V) + A(V)M(Y,Z,U,X). \end{aligned} \tag{1.6}$$

Such a manifold shall be called an almost pseudo m-projectively symmetric manifold and an n-dimensional manifold of this kind shall be denoted by $(APMPS)_n$. In a recent paper De and Mallick [9] studied almost pseudo concircularly symmetric manifolds and Prajjwal Pal [15] studied almost pseudo conharmonically symmetric manifolds. Motivated by the above studies, in the present paper we have studied a type of non-flat Riemannian manifold.

This paper is organized as follows: After preliminaries in Section 2, we obtain a necessary and sufficient condition for constant scalar curvature of a $(APMPS)_n$, $(n > 2)$. In Section 4 we study $(APMPS)_n$, $(n > 2)$ satisfying Codazzi type of Ricci tensor. The next section is devoted to the study of Einstein $(APMPS)_n$. In Section 6, we study Ricci symmetric $(APMPS)_n$, $(n > 2)$ and we proved that the scalar curvature of a Ricci symmetric $(APMPS)_n$ is constant. Finally, non-trivial examples of $(APMPS)_n$ have been constructed.

2. Preliminaries

Let S and r denote the Ricci tensor of type $(0,2)$ and the scalar curavture respectively and L denotes the symmetric tensor of type $(1,1)$ corresponding to the Ricci tensor S, that is,

$$g(LX, Y) = S(X, Y).$$

In this section, some formulas useful while studying $(APMPS)_n$ are derived. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold, where $1 \leq i \leq n$. From (1.4) we can easily verify that the tensor M satisfies the following property

$$\begin{aligned} \tilde{M}(Y,Z)U &= -\tilde{M}(Z,Y)U, \\ \tilde{M}(Y,Z)U + \tilde{M}(Z,U)Y + \tilde{M}(U,Y)Z &= 0. \end{aligned} \tag{2.1}$$

From (1.5)and(2.1) it follows that

$$\begin{aligned} (i) \quad & M(Y,Z,U,V) = -M(Z,Y,U,V), \\ (ii) \quad & M(Y,Z,U,V) = -M(Y,Z,V,U), \\ (iii) \quad & M(Y,Z,U,V) = M(U,V,Y,Z), \\ (iv) \quad & M(Y,Z,U,V) + M(Z,U,Y,V) + M(U,Y,Z,V) = 0. \end{aligned} \tag{2.2}$$

Also from the equation (1.5) we have

$$\sum_{i=1}^{n} M(Y,Z,e_i,e_i) = 0 = \sum_{i=1}^{n} M(e_i,e_i,U,V) \tag{2.3}$$

and

$$\begin{aligned} \sum_{i=1}^{n} M(e_i,Z,U,e_i) &= \sum_{i=1}^{n} M(Z,e_i,e_i,U) \\ &= \frac{n}{2(n-1)}[S(Z,U) - \frac{r}{n}g(Z,U)], \end{aligned} \tag{2.4}$$

where $r = \sum_{i=1}^{n} \epsilon_i S(e_i,e_i)$ is the scalar curvature.

3. $(APMPS)_n$, $(n > 2)$ with constant scalar curvature

From (1.5) we have,

$$\begin{aligned} (\nabla_X M)(Y,Z,U,V) &= (\nabla_X R)(Y,Z,U,V) \\ -\frac{1}{2(n-1)}&[(\nabla_X S)(Z,U)g(Y,V) - (\nabla_X S)(Y,U)g(Z,V) \\ &+(\nabla_X S)(Y,V)g(Z,U) - (\nabla_X S)(Z,V)g(Y,U)]. \end{aligned} \tag{3.1}$$

From (1.6) and (3.1) we obtain

$$
\begin{aligned}
(\nabla_X R)(Y,Z,U,V) &= [A(X)+B(X)]M(Y,Z,U,V) \\
&+A(Y)M(X,Z,U,V)+A(Z)M(Y,X,U,V)+A(U)M(Y,Z,X,V) \\
&+A(V)M(Y,Z,U,X)+\frac{1}{2(n-1)}[(\nabla_X S)(Z,U)g(Y,V) \\
&-(\nabla_X S)(Y,U)g(Z,V)+(\nabla_X S)(Y,V)g(Z,U)-(\nabla_X S)(Z,V)g(Y,U)]. \qquad (3.2)
\end{aligned}
$$

Contracting (3.2) over Y and V we get

$$
\begin{aligned}
(\nabla_X S)(Z,U) &= \frac{n}{2(n-1)}[A(X)+B(X)][S(Z,U)-\frac{r}{n}g(Z,U)] \\
&+A\big(\tilde{M}(X,Z)U\big)+\frac{n}{2(n-1)}A(Z)[S(X,U)-\frac{r}{n}g(X,U)] \\
&+\frac{n}{2(n-1)}A(U)[S(Z,X)-\frac{r}{n}g(Z,X)]+A\big(\tilde{M}(X,U)Z\big) \\
&+\frac{1}{2(n-1)}[(n-2)(\nabla_X S)(Z,U)+dr(X)g(Z,U)]. \qquad (3.3)
\end{aligned}
$$

Again contracting (3.3) over Z and U we get

$$
\frac{2n}{(n-1)}[A(LX)-\frac{r}{n}A(X)]=0.
$$

Since $n>2$, the above expression implies that

$$
A(LX)-\frac{r}{n}A(X)=0. \qquad (3.4)
$$

We have Bianchi's second identity for (0,4) Riemannian curvature tensor R as follows:

$$
(\nabla_X R)(Y,Z,U,V)+(\nabla_Y R)(Z,X,U,V)+(\nabla_Z R)(X,Y,U,V)=0. \qquad (3.5)
$$

Using (3.2) and (3.5) we get

$$\begin{aligned}
&[-A(X)+B(X)]M(Y,Z,U,V)+[-A(Y)+B(Y)]M(Z,X,U,V)\\
&\quad+[-A(Z)+B(Z)]M(X,Y,U,V)\\
&\quad+A(U)\big[M(Y,Z,X,V)+M(Z,X,Y,V)+M(X,Y,Z,V)\big]\\
&\quad+A(V)\big[M(Y,Z,U,X)+M(Z,X,U,Y)+M(X,Y,U,Z)\big]\\
&\quad+\frac{1}{2(n-1)}\big[(\nabla_X S)(Z,U)g(Y,V)-(\nabla_X S)(Y,U)g(Z,V)\\
&\quad+(\nabla_X S)(Y,V)g(Z,U)-(\nabla_X S)(Z,V)g(Y,U)\\
&\quad+(\nabla_Y S)(X,U)g(Z,V)-(\nabla_Y S)(Z,U)g(X,V)\\
&\quad+(\nabla_Y S)(Z,V)g(X,U)-(\nabla_Y S)(X,V)g(Z,U)\\
&\quad+(\nabla_Z S)(Y,U)g(X,V)-(\nabla_Z S)(X,U)g(Y,V)\\
&\quad+(\nabla_Z S)(X,V)g(Y,U)-(\nabla_Z S)(Y,V)g(X,U)\big]=0. \qquad (3.6)
\end{aligned}$$

Making use of (2.2) in the equation (3.6) we get

$$\begin{aligned}
&[-A(X)+B(X)]M(Y,Z,U,V)+[-A(Y)+B(Y)]M(Z,X,U,V)\\
&\quad+[-A(Z)+B(Z)]M(X,Y,U,V)\\
&\quad+\frac{1}{2(n-1)}\big[(\nabla_X S)(Z,U)g(Y,V)-(\nabla_X S)(Y,U)g(Z,V)\\
&\quad+(\nabla_X S)(Y,V)g(Z,U)-(\nabla_X S)(Z,V)g(Y,U)\\
&\quad+(\nabla_Y S)(X,U)g(Z,V)-(\nabla_Y S)(Z,U)g(X,V)\\
&\quad+(\nabla_Y S)(Z,V)g(X,U)-(\nabla_Y S)(X,V)g(Z,U)\\
&\quad+(\nabla_Z S)(Y,U)g(X,V)-(\nabla_Z S)(X,U)g(Y,V)\\
&\quad+(\nabla_Z S)(X,V)g(Y,U)-(\nabla_Z S)(Y,V)g(X,U)\big]=0. \qquad (3.7)
\end{aligned}$$

Contracting (3.7) over Y and V we get

$$\begin{aligned}
&n[-A(X)+B(X)]\big[S(Z,U)-\frac{r}{n}g(Z,U)\big]\\
&\quad+2(n-1)[-A(\tilde{M}(Z,X)U)+B(\tilde{M}(Z,X)U)]\\
&\quad-n[-A(Z)+B(Z)]\big[S(X,U)-\frac{r}{n}g(X,U)\big]\\
&\quad+(n-3)\big\{(\nabla_X S)(Z,U)-(\nabla_Z S)(X,U)\big\}\\
&\quad+\frac{1}{2}\big\{dr(X)g(Z,U)-dr(Z)g(X,U)\big\}=0. \qquad (3.8)
\end{aligned}$$

Again contracting (3.8) over Z and U we get

$$2n\big[A(LX)-\frac{r}{n}A(X)\big]-2n\big[B(LX)-\frac{r}{n}B(X)\big]+(n-2)dr(X)=0. \qquad (3.9)$$

Combining the equations (3.4) and (3.9), we obtain

$$B(LX) - \frac{r}{n}B(X) = \frac{(n-2)}{2n}dr(X). \tag{3.10}$$

Thus we can state the following:

Theorem 3.1. *The scalar curvature r of an almost pseudo m-projectively symmetric manifold is constant if and only if*

$$B(LX) - \frac{r}{n}B(X) = 0 \tag{3.11}$$

holds for all vector fields.

4. $(APMPS)_n$, $(n > 2)$ with Codazzi type of Ricci tensor

In 1978, Gray [14] introduced two classes of Riemannian manifolds. The class A consisting of all Riemannian manifolds whose Ricci tensor S satisfies,

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0,$$

and the class B consisting of all Riemannian manifolds whose Ricci tensor is a Codazzi tensor, that is,

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = 0.$$

Suppose that the Ricci tensor of the $(APMPS)_n$ is a Codazzi type tensor, that is,

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z). \tag{4.1}$$

Now from (3.1) we get

$$\begin{aligned}(\nabla_X M)(Y,Z,U,V) &= (\nabla_X R)(Y,Z,U,V)\\ -\frac{1}{2(n-1)}&[(\nabla_X S)(Z,U)g(Y,V) - (\nabla_X S)(Y,U)g(Z,V)\\ &+(\nabla_X S)(Y,V)g(Z,U) - (\nabla_X S)(Z,V)g(Y,U)].\end{aligned} \tag{4.2}$$

By (4.2) we obtain

$$\begin{aligned}&(\nabla_X M)(Y,Z,U,V) + (\nabla_Y M)(Z,X,U,V) + (\nabla_Z M)(X,Y,U,V)\\ &= [(\nabla_X R)(Y,Z,U,V) + (\nabla_Y R)(Z,X,U,V) + (\nabla_Z R)(X,Y,U,V)]\\ &\quad -\frac{1}{2(n-1)}\big[(\nabla_X S)(Z,U)g(Y,V) - (\nabla_X S)(Y,U)g(Z,V)\\ &\quad +(\nabla_X S)(Y,V)g(Z,U) - (\nabla_X S)(Z,V)g(Y,U)\\ &\quad +(\nabla_Y S)(X,U)g(Z,V) - (\nabla_Y S)(Z,U)g(X,V)\\ &\quad +(\nabla_Y S)(Z,V)g(X,U) - (\nabla_Y S)(X,V)g(Z,U)\\ &\quad +(\nabla_Z S)(Y,U)g(X,V) - (\nabla_S)(X,U)g(Y,V)\\ &\quad +(\nabla_Z S)(X,V)g(Y,U) - (\nabla_Z S)(Y,V)g(X,U)\big].\end{aligned} \tag{4.3}$$

Using (4.1) and (3.5) in (4.3) we get

$$(\nabla_X M)(Y,Z,U,V) + (\nabla_Y M)(Z,X,,U,V) + (\nabla_Z M)(X,Y,U,V) = 0. \tag{4.4}$$

Hence we have the following theorem.

Theorem 4.1. *In an $(APMPS)_n$, $(n > 2)$ satisfying Codazzi type of Ricci tensor, the m-projective curvature tensor satisfies Bianchi's second identity.*

5. Einstein $(APMPS)_n$, $(n > 2)$

If a $(APMPS)_n$, $(n > 2)$ is an Einstein manifold, then the Ricci tensor satisfies

$$S(Y,Z) = \frac{r}{n} g(Y,Z). \tag{5.1}$$

Therefore,

$$(\nabla_X S)(Y,Z) = 0, \tag{5.2}$$

and

$$dr(X) = 0. \tag{5.3}$$

Using (5.1) and (5.2) we get from (1.5)

$$(\nabla_X M)(Y,Z,U,V) = (\nabla_X R)(Y,Z,U,V), \tag{5.4}$$

which yields

$$\begin{aligned}[A(X)+B(X)]M(Y,Z,U,V) + A(Y)M(X,Z,U,V)\\ +A(Z)M(Y,X,U,V) + A(U)M(Y,Z,X,V)\\ +A(V)M(Y,Z,U,X) = [A(X)+B(X)]R(Y,Z,U,V)\\ +A(Y)R(X,Z,U,V) + A(Z)R(Y,X,U,V)\\ +A(U)R(Y,Z,X,V) + A(V)R(Y,Z,U,X).\end{aligned} \tag{5.5}$$

In light of the equation (5.1), the equation (1.5) assumes the following form

$$\begin{aligned}M(Y,Z,U,V) = R(Y,Z,U,V) - \frac{r}{n(n-1)}[g(Z,U)g(Y,V)\\ -g(Y,U)g(Z,V)].\end{aligned} \tag{5.6}$$

Now using (5.6) in (5.5) we get

$$\begin{aligned}\frac{r}{n(n-1)}[\{A(X)+B(X)\}\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\}\\ +A(Y)\{g(Z,U)g(X,V) - g(X,U)g(Z,V)\}\\ +A(Z)\{g(X,U)g(Y,V) - g(Y,U)g(X,V)\}\\ +A(U)\{g(Z,X)g(Y,V) - g(Y,X)g(Z,V)\}\\ +A(V)\{g(Z,U)g(Y,X) - g(Y,U)g(Z,X)\} = 0,\end{aligned}$$

which implies

$$
\begin{aligned}
r[\{A(X)+B(X)\}\{g(Z,U)g(Y,V)-g(Y,U)g(Z,V)\}\\
+A(Y)\{g(Z,U)g(X,V)-g(X,U)g(Z,V)\}\\
+A(Z)\{g(X,U)g(Y,V)-g(Y,U)g(X,V)\}\\
+A(U)\{g(Z,X)g(Y,V)-g(Y,X)g(Z,V)\}\\
+A(V)\{g(Z,U)g(Y,X)-g(Y,U)g(Z,X)\}=0.
\end{aligned}
\tag{5.7}
$$

Now contracting (5.7) over Y and V we get

$$
\begin{aligned}
r[(n-1)\big[A(X)+B(X)\big]g(Z,U)+A(X)g(Z,U)\\
-A(Z)g(X,U)+(n-1)\big[A(Z)g(X,U)+A(U)g(Z,X)\big]\\
+A(X)g(Z,U)-A(U)g(Z,X)=0,
\end{aligned}
$$

which implies

$$
\begin{aligned}
r\big[\{(n+1)A(X)+(n-1)B(X)\}g(Z,U)\\
+(n-2)\{A(Z)g(X,U)+A(U)g(Z,X)\}\big]=0.
\end{aligned}
\tag{5.8}
$$

Again contracting (5.8) over Z and U we get

$$r[n(n+1)A(X)+n(n-1)B(X)+2(n-2)A(X)]=0,$$

which, in turn implies

$$r(n-1)[(n+4)A(X)+nB(X)]=0,$$

which implies

$$r[(n+4)A(X)+nB(X)]=0. \tag{5.9}$$

Again contracting (5.8) over Z and X, we get

$$r[(n+1)A(U)+(n-1)B(U)+(n-2)A(U)+n(n-2)A(U)]=0,$$

which implies

$$r[(n+1)A(U)+B(U)]=0. \tag{5.10}$$

Replacing U by X we get

$$r[(n+1)A(X)+B(X)]=0. \tag{5.11}$$

Again contracting (5.8) over X and U we get

$$r[(n+1)A(Z)+(n-1)B(Z)+n(n-2)A(Z)+(n-2)A(Z)]=0,$$

which implies

$$r[(n+1)A(Z)+B(Z)]=0. \tag{5.12}$$

Replacing Z by X we get

$$r[(n+1)A(X)+B(X)]=0. \tag{5.13}$$

Adding (5.9), (5.11) and (5.13) yields

$$r[(3n+6)A(X)+(n+2)B(X)]=0,$$

which implies

$$r(n+2)[3A(X)+B(X)]=0.$$

Therefore, either $r=0$ or $3A(X)+B(X)=0$. Thus, we can state the following theorem:

Theorem 5.1. *If an Einstein $(APMPS)_n$, $(n>2)$ is an almost pseudo symmetric manifold, then the scalar curvature of the manifold vanishes provided $3A(X)+B(X)\neq 0$.*

Again, if in an $(APMPS)_n$ $r=0$, then using (1.6) and (5.6) in (5.4), we get

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= [A(X)+B(X)]R(Y,Z,U,V)\\ &+ A(Y)R(X,Z,U,V)+A(Z)R(Y,X,U,V)\\ &+ A(U)R(Y,Z,X,V)+A(V)R(Y,Z,U,X).\end{aligned}$$

Hence we have the following theorem:

Theorem 5.2. *If in an Einstein $(APMPS)_n$, $(n>2)$ the scalar curvature vanishes, then it is an almost pseudo symmetric manifold.*

Now, Let ρ be a vector field defined by

$$g(X,\rho)=\alpha(X),$$

where $\alpha(X)=A(X)-B(X)$.
Further, we suppose that in an Einstein $(APMPS)_n$, the vector field ρ defined above is parallel. Then we have

$$\nabla_X \rho = 0 \tag{5.14}$$

for all X. By Ricci identity and (5.4) we get

$$R(X, Y, \rho, U) = 0, \tag{5.15}$$

which implies

$$S(Y, \rho) = 0. \tag{5.16}$$

Using (5.1) in (5.16) we get

$$rg(Y, \rho) = 0.$$

Thus, either $r = 0$ or $\parallel \rho \parallel^2 \neq 0$. If $r = 0$ then using (1.6) and (5.6) in (5.4), it follows that the manifold is an almost pseudo symmetric manifold. Thus we have:

Theorem 5.3. *If the vector field defined by $g(X, \rho) = A(X) - B(X)$ is a parallel vector field in an Einstein $(APMPS)_n$, $(n > 2)$, then it is an almost pseudo symmetric manifold provided $\parallel \rho \parallel \neq 0$.*

6. Examples of $(APMPS)_n$

Example1

Let us consider a Lorentzian metric g on $\mathbb{R}^4$ defined by

$$ds^2 = g_{ij}dx^i dx^j = x^1(dx^1)^2 + x^1(dx^2)^2 + x^1(dx^3)^2 - (dx^4)^2 \tag{6.1}$$

where $i, j = 1, 2, 3, 4$. Then the only non-vanishing components of the Christoffel symbols, the Riemannian curvature tensors and the Ricci tensor are

$$\Gamma^1_{22} = \Gamma^1_{33} = -\frac{1}{2x^1}, \qquad \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{2x^1},$$

$$R_{1221} = R_{1331} = -\frac{1}{2x^1}, \qquad R_{2332} = \frac{1}{4x^1},$$

and

$$S_{22} = S_{33} = -\frac{1}{4(x^1)^2}, \qquad S_{11} = -\frac{1}{(x^1)^2}, \qquad S_{44} = 0.$$

And the scalar curvature of the resulting manifold (R^4, g) is

$$r = -\frac{3}{2(x^1)^3}.$$

Now, the non vanishing components of m-projective curvature tensor and their covariant derivatives are:

$$M_{1221} = M_{1331} = -\frac{7}{24x^1}, \qquad M_{2332} = \frac{1}{3x^1},$$

$$M_{1221,1} = M_{2332,1} = \frac{7}{24x^2}, \qquad M_{2332,1} \quad = \quad -\frac{1}{3x^2},$$

where ',' denotes the covariant derivative with respect to the metric tensor.

Let us choose the associated 1-forms as follows:

$$A_i(x) \quad = \quad \begin{cases} 0, & for\ i=1 \\ x^1, & otherwise, \end{cases}$$

(6.2)

$$B_i(x) \quad = \quad \begin{cases} -\frac{1}{x^1}, & for\ i=1 \\ -x^1, & otherwise, \end{cases}$$

(6.3)

at any point $x \in \mathbb{R}^4$. To verify the relation (1.6), it is sufficient to check the following equations:

$$M_{1221,1} = \big[A_1 + B_1\big] M_{1221} + A_1 M_{1221} + A_2 M_{1121} + A_2 M_{1211} + A_1 M_{1221}, \tag{6.4}$$

and

$$M_{2332,1} = \big[A_1 + B_1\big] M_{2332} + A_2 M_{1332} + A_3 M_{2132} + A_3 M_{2312} + A_2 M_{2331}. \tag{6.5}$$

Since for the other cases (1.5) holds trivially. By (6.2) and (6.3) we get

$$\begin{aligned} R.H.S.\ of\ (6.4) &= \big[A_1 + B_1\big] M_{1221} + A_1 M_{1221} + A_1 M_{1221} \\ &= \big[3A_1 + B_1\big] M_{1221} \\ &= 3(0)\big(-\frac{7}{24x^1}\big) + \big(-\frac{1}{x^1}\big)\big(-\frac{7}{24x^1}\big) \\ &= \frac{7}{24x^2} \\ &= M_{1212,1} \\ &= L.H.S.\ of\ (6.4). \end{aligned}$$

By a similar argument it can be shown that (6.5) is also true. So $(\mathbb{R}4, g)$ is a $(APMPS)_n$.

Example2

Consider a Riemannian space V_n, $(n \geq 4)$ with the metric is given by

$$ds^2 = \phi(dx^1)^2 + K_{\alpha\beta} dx^\alpha dx^\beta + 2dx^1 dx^n, \tag{6.6}$$

where $[K_{\alpha\beta}]$ is a symmetric and non singular matrix consisting of constants and ϕ is a function of $x^1, x^2, ..., x^{n-1}$ and independent of x^n, and $1 < \alpha, \beta < n$.

In the metric considered, the only non-vanishing components of Christoffel symbols, Riemannian curvature tensor and Ricci tensor are [19]

$$\Gamma^{\beta}_{11} = -\frac{1}{2}K^{\alpha\beta}\phi_{.\alpha}, \quad \Gamma^{n}_{11} = \frac{1}{2}\phi_{.1}, \quad ,\Gamma^{n}_{1\alpha} = \frac{1}{2}\phi_{.\alpha},$$

(6.7)
$$R_{1\alpha\beta 1} = \frac{1}{2}\phi_{.\alpha\beta}, \quad S_{11} = \frac{1}{2}K^{\alpha\beta}\phi_{.\alpha\beta},$$

where '.' denotes the partial differentiation with respect to the coordinates and $K^{\alpha\beta}$ are the elements of the matrix inverse to $[K_{\alpha\beta}]$. Here we consider $K_{\alpha\beta}$ as Kronecker symbol $\delta_{\alpha\beta}$ and

$$\phi = (M_{\alpha\beta} + \delta_{\alpha\beta})x^{\alpha}x^{\beta}e^{(x^1)^2},$$

where $M_{\alpha\beta}$ are constant and satisfy the relations

$$\begin{aligned} M_{\alpha\beta} &= 0, \; for \; \alpha \neq \beta, \\ &\neq 0, \; for \; \alpha = \beta, \\ \sum_{\alpha=1}^{n-1} M_{\alpha\alpha} &= 0. \end{aligned}$$

This is to be noted that the metric with this form of ϕ was considered by De and Gazi [12]. Thus we have

$$\phi_{\alpha\beta} = 2(M_{\alpha\beta} + \delta_{\alpha\beta})e^{(x^1)^2}, \qquad \delta_{\alpha\beta}\delta^{\alpha\beta} = n - 2,$$
$$\delta^{\alpha\beta}M_{\alpha\beta} = \sum_{\alpha=1}^{n-1} M_{\alpha\alpha} = 0.$$

Therefore

$$\delta^{\alpha\beta}\phi_{\alpha\beta} = 2(\delta^{\alpha\beta}M_{\alpha\beta} + \delta^{\alpha\beta}\delta_{\alpha\beta})e^{(x^1)^2} = 2(n-2)e^{(x^1)^2}.$$

Since $\phi_{\alpha\beta}$ vanishes for $\alpha \neq \beta$, the only non-zero components of the Riemannian curvature tensor and Ricci tensor by virtue of (6.7) are

$$R_{1\alpha\alpha 1} = \frac{1}{2}\phi_{.\alpha\alpha} = \big(1 + M_{\alpha\alpha}\big)e^{(x^1)^2},$$

$$S_{11} = \frac{1}{2}\phi_{.\alpha\beta}\delta^{\alpha\beta} = (n-2)e^{(x^1)^2}.$$

Also, the scalar curvature $r = 0$.

Hence the only non-zero components of the m-projective curvature tensor, and their covariant derivatives are

$$\begin{aligned} M_{1\alpha\alpha 1} &= \big(1 + M_{\alpha\alpha}\big)e^{(x^1)^2} + \frac{n-2}{2(n-1)}e^{(x^1)^2} \\ &= \big(1 + \frac{n}{2(n-1)}M_{\alpha\alpha}\big)e^{(x^1)^2}, \end{aligned}$$

$$\begin{aligned} M_{1\alpha\alpha1,1} &= 2x^1\big(1+\frac{n}{2(n-1)}M_{\alpha\alpha}\big)e^{(x^1)^2} \\ &= 2x^1 M_{1\alpha\alpha1}, \end{aligned}$$

where ‘,’ denotes the covariant derivative with respect to the metric tensor.

Let us choose the associated 1-forms as follows:

$$A_i(x) = \begin{cases} x^1, & for\ i=1 \\ 0, & otherwise, \end{cases}$$

(6.8)

$$B_i(x) = \begin{cases} -x^1, & for\ i=1 \\ 0, & otherwise, \end{cases}$$

at any point $x \in V_n$. To verify the relation (1.6), it is sufficient to check the following equation:

(6.9) $$M_{1\alpha\alpha1,1} = (3A_1+B_1)M_{\alpha11\alpha}$$

$$\begin{aligned} R.H.S.\ of\ (6.9) &= (3A_1+B_1)M_{1\alpha\alpha1} \\ &= (3x^1-x^1)M_{1\alpha\alpha1} \\ &= 2x^1 M_{1\alpha\alpha1} \\ &= M_{1\alpha\alpha1,1} \\ &= L.H.S.\ of\ (6.9). \end{aligned}$$

So V_n is a $(APMPS)_n$.

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