Novi Sad J. Math. Vol. 48, No. 2, 2018, 21-39
https://doi.org/10.30755/NSJOM.05660

ON SPLIT EQUALITY MINIMIZATION AND FIXED POINT PROBLEMS

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Abstract. In this paper, iterative algorithm for approximating a solution of a split equality minimization problem and split equality fixed point problem for demi-contractive mappings is introduced. Using our iterative algorithm, we state and prove a strong convergence theorem for approximating an element in the intersection of the solution set of a split equality minimization problem (SEMP) and the solution set of split equality fixed point problem (SEFP) for demicontractive maps. Our result do not require any compactness assumption and does not require the prior knowledge of the operator norm. Our result complements and extends some recent results in literature.

AMS Mathematics Subject Classification (2010): 47H06; 47H09; 47J05; 47J25

Key words and phrases: Split equality minimization problem; demi-contractive mapping; Proximal map; Hilbert spaces

1. Introduction

In this paper, we let H be a Hilbert space endowed with inner product $\langle .,.\rangle$ and induced norm $\|.\|$ and $\mathbb{R}$ be the set of real numbers.

Definition 1.1. Let H be a real Hilbert space and Q a nonempty, closed and convex subset of H. A mapping $T : Q \to Q$ is said to be L-Lipschitzian (see [24], and some of the references therein) if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall\, x, y \in Q. \tag{1.1}$$

If $L = 1$, we say that T is nonexpansive, i.e

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in Q. \tag{1.2}$$

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A mapping $T : Q \to Q$ is said to be *k-strictly pseudo-contractive* if there exists a constant $k \in (0,1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I-T)x - (I-T)y||^2 \ \forall\ x, y \in Q.$$

A mapping $T : Q \to Q$ is said to be *demi-contractive* if $F(T) \neq \emptyset$ and there exists a constant $k \in (0,1)$ such that

$$(1.3)\ \ \|Tx - y\|^2 \leq \|x - y\|^2 + k||x - Tx||^2, \qquad \forall\ \ x \in Q,\ y \in F(T).$$

In a real Hilbert space H, it is known that (1.3) is equivalent to

$$(1.4) \qquad \langle Tx - y, x - y\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|x - Tx\|^2.$$

A point $x \in Q$ is called a *fixed point* of T if $Tx = x$. It is well known that if T is demicontractive and $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.

Definition 1.2. Let $h : H \to \mathbb{R}$ be a proper, convex and lower semi-continuous function, then the proximal map $Prox_h$ associated with h is the function $Prox_h : H \to H$ defined by

$$Prox_h(x) = \underset{y\in H}{argmin}\Big(h(y) + \frac{1}{2}\|x - y\|^2\Big), \quad (x \in H),$$

and the proximal operator of the scaled function τh, where $\tau > 0$, is given as

$$Prox_{\tau h}(x) = \underset{y\in H}{argmin}\Big(h(y) + \frac{1}{2\tau}\|x - y\|^2\Big), \quad (x \in H).$$

Let $f : H \to \mathbb{R}$ and $g : H \to \mathbb{R}$ be two convex and lower semi-continuous functions such that f is differentiable with L-Lipschitz continuous gradient and g is "simple" meaning that its "proximal mapping"

$$x \to \underset{y\in H}{argmin}\Big(g(y) + \frac{\|x - y\|^2}{2\tau}\Big)$$

can easily be computed. In this paper, we shall consider the minimization problem of the form

$$(1.5) \qquad \min_{x\in H} F(x) := f(x) + g(x),$$

and assume the solution of (1.5) exists, we denote this solution set by Γ. For more on minimization problem see [18].

The proximal mapping defined in Definition 1.2 is uniquely defined and generalizes the projection on a closed convex set to convex functions. The proximal-gradient method [10] has been employed in solving (1.5), by generating a sequence $\{x_n\}$ via the following algorithm: For an initial point $x_1 \in H$,

$$(1.6) \qquad x_{n+1} = (prox_{\lambda_n g} o(I - \lambda_n \nabla f))x_n,$$

where ∇f is the gradient of f and $\{\lambda_n\}$ is a sequence of positive real numbers. If $\Gamma \neq \emptyset$ and the following conditions are satisfied

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad \forall x, y \in H \tag{1.7}$$

and

$$0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{2}{L}, \tag{1.8}$$

then the sequence $\{x_n\}$, (see, [10, 30]) converges weakly to a point in Γ. This proximal gradient algorithm can also be interpreted as a fixed point iteration. A point x^* is a solution to the problem (1.5) that is x^* is a minimizer of $f(x) + g(x)$, if and only if $0 \in \nabla f(x^*) + \partial g(x^*)$. For any $\gamma > 0$ this optimality condition holds if and only if the following equivalent statements hold:

$$\begin{aligned} 0 &\in \gamma\nabla f(x^*) + \gamma\partial g(x^*); \\ 0 &\in \gamma\nabla f(x^*) - x^* + x^* + \gamma\partial g(x^*); \\ (I + \partial g)(x^*) &\in (I - \gamma\nabla f)(x^*); \\ x^* &= (I + \gamma\partial g)^{-1}(I - \gamma\nabla f)(x^*); \\ x^* &= Prox_{\gamma g}(x^* - \gamma\nabla f(x^*)). \end{aligned} \tag{1.9}$$

The last two expressions in (1.9) hold with equality and not just containment because the proximal operator is single valued. The final statement says that x^* minimizes $f + g$ if and only if it is a fixed point of the forward-backward operator $(I + \gamma\partial g)^{-1}(I - \gamma\nabla f)$.

The proximal gradient method repeatedly applies this operator to obtain a fixed point and thus a solution to original problem. The condition $\gamma \in (0, \frac{1}{L}]$, where L is the Lipschitz constant of ∇f guarantees that the forward-backward operator is averaged and thus the iteration converges to a fixed point (if it exists).

The Split Feasibility Problem (SFP) introduced in 1994 by Censor and Elfving [5] is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{1.10}$$

where C and Q are nonempty closed convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and A is an $m \times n$ real matrix. The SFP has wide applications in many fields, such as phase retrieval, medical image reconstruction, signal processing, and radiation therapy treatment planning (for example see [3, 4, 5, 6, 11, 32] and the references therein).

The SFP has also been studied by numerous authors in both finite and infinite dimensional Hilbert spaces (for examples see [2, 6, 7, 12, 14, 15, 16, 21, 22, 23, 20, 26, 25, 27, 28, 31, 33]). It has been shown (see [29]) that if the SFP (1.10) has a solution, then $x^* \in C$ solves SFP (1.10) if and only if it solves the fixed point equation

$$x^* = P_C(I - \gamma A^*(I - P_Q)A)x^*, \tag{1.11}$$

where P_C and P_Q are the metric projections onto C and Q respectively, γ is any positive real number, A is a bounded linear operator and A^* is the adjoint of A.

Byrne [2] applied the forward-backward method, a type of projected gradient method, to introduce the so-called CQ-iterative procedure for approximating a solution of (1.10), which is defined by

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \ n \in \mathbb{N}, \tag{1.12}$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A.

In 2009, Censor and Segal [7] introduced an important form of the SFP called Split Common Fixed Point Problem (SCFPP), which is to find a point

$$x^* \in F(T) \ \text{ such that } \ Ax^* \in F(S), \tag{1.13}$$

where T and S are some nonlinear operators on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, A is a real $m \times n$ matrix. Based on the properties of the operators T and S, called directed operators, they presented the following algorithm for solving the SCFPP:

$$x_{n+1} = T(x_n + \gamma A^T(S - I)Ax_n), \quad \forall n \geq 1, \ x_1 \in \mathbb{R}^n, \tag{1.14}$$

where $\gamma \in (0, \frac{2}{||A||^2})$. They also obtained a convergence result for this algorithm.

Motivated by the work of Censor and Segal [7], Moudafi [15] presented the following iterative scheme which does not involve the metric projections P_C and P_Q:

(1.15)

$$x_{n+1} = (1 - \alpha_n)\left(x_n + \gamma A^*(S - I)Ax_n\right) + \alpha_n T\left(x_n + \gamma A^*(S - I)Ax_n\right), \ n \in \mathbb{N},$$

for approximating a solution of the SCFPP (1.13) and obtained a weak convergence results when T and S are demi-contractive.

Recently, Moudafi and Al-Shemas [17] introduced the following Split Equality Fixed Point Problem (SEFPP) which generalizes the SFP (1.10):

(1.16) Find $x \in C := F(T), \ \ y \in Q := F(S)$ such that $Ax = By$,

where $A : H_1 \to H_3, \ B : H_2 \to H_3$ are two bounded linear operators, $F(T)$ and $F(S)$ denotes the sets of fixed points of operators T and S defined on H_1 and H_2 respectively. Note that if $H_2 = H_3$ and $B = I$ (where I is the identity map on H_2) in (1.16), then problem (1.16) reduces to problem (1.10). Further, Moudafi and Al-Shemas presented the following algorithm for solving the SEFPP

$$\begin{cases} x_{n+1} = T(x_n - \gamma_n A^*(Ax_n - By_n)); \\ y_{n+1} = S(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1; \end{cases} \tag{1.17}$$

where $T : H_1 \to H_1, \ S : H_2 \to H_2$ are two firmly quasi-nonexpansive mappings, $A : H_1 \to H_3, \ B : H_2 \to H_3$ are two bounded linear operators, A^* and

B^* are the adjoints of A and B respectively, $\{\gamma_n\} \subset \left(\epsilon, \frac{2}{\lambda_{A^*A}+\lambda_{B^*B}} - \epsilon\right)$, λ_{A^*A} and λ_{B^*B} denote the spectral radius of A^*A and B^*B, respectively. Moudafi established the weak convergence result for problem (1.16) using algorithm (1.17).

Yaun-Fang et al. [13] presented the following algorithm for solving problem (1.16):

$$(1.18) \qquad \begin{cases} \forall x_1 \in H_1,\ y_1 \in H_2; \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(x_n - \gamma_n A^*(Ax_n - By_n)); \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n S(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1; \end{cases}$$

where $T : H_1 \to H_1, S : H_2 \to H_2$ are two firmly quasi-nonexpansive mappings, $A : H_1 \to H_3, B : H_2 \to H_3$ are two bounded linear operators, A^* and B^* are the adjoints of A and B respectively, $\{\gamma_n\} \subset \left(\epsilon, \frac{2}{\lambda_{A^*A}+\lambda_{B^*B}} - \epsilon\right)$ (for ϵ small enough), λ_{A^*A} and λ_{B^*B} denote the spectral radius of A^*A and B^*B respectively and $\alpha_n \in [\alpha, 1]$ (for some $\alpha > 0$) and established a strong and weak convergence results. Based on the work of Moudafi and Al-Shemas [17], Chidume et al. [9] proposed the following algorithm for solving the SEFPP for demi-contractive mappings:

$$(1.19) \qquad \begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2; \\ x_{n+1} = (1-\alpha)\left(x_n - \gamma A^*(Ax_n - By_n)\right) \\ +\alpha T\left(x_n - \gamma A^*(Ax_n - By_n)\right); \\ y_{n+1} = (1-\alpha)\left(y_n + \gamma B^*(Ax_n - By_n)\right) \\ +\alpha S\left(y_n + \gamma B^*(Ax_n - By_n)\right), \quad \forall n \geq 1; \end{cases}$$

where $T : H_1 \to H_1, \quad S : H_2 \to H_2$ are two demi-contractive mappings. Chidume et al. [9] proved weak and strong convergence theorems of the iterative scheme (1.19) to a solution of the SEFPP in a real Hilbert spaces.

We now consider the following Split Equality Minimization and Fixed Point Problem (SEMFPP).

Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear maps. Let $f_1 : H_1 \to \mathbb{R}$ and $f_2 : H_2 \to \mathbb{R}$ be differentiable maps with L_1 and L_2-Lipschitz continuous gradients respectively. Let $g_i : H_i \to \mathbb{R} \ \ (i = 1, 2)$ be "simple" maps. The (SEMFPP) is to find $x^* \in F(T)$ and $y^* \in F(S)$ such that

$$(1.20) \qquad \begin{cases} f_1(x^*) + g_1(x^*) & = \min_{x \in H_1} [f_1(x) + g_1(x)], \\ f_2(y^*) + g_2(y^*) & = \min_{x \in H_2} [f_2(x) + g_2(x)], \end{cases}$$

and $Ax^* = By^*$. where $T : H_1 \to H_1$ and $S : H_2 \to H_2$ are two nonlinear mappings. Assume that this problem has a solution, let's denote the solution set of (1.20) by Υ. Furthermore, we propose an iterative scheme and using the iterative scheme, we state and prove a strong convergence result for the approximation of a solution of problem (1.20).

2. Preliminaries

We state some important results that will be needed in the proof of the main result of this paper.

Lemma 2.1. *Let H be a Hilbert space, then*

$$2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x-y||^2 = ||x+y||^2 - ||x||^2 - ||y||^2, \quad \forall\ x, y \in H.$$

Lemma 2.2. *[8] Let H be a Hilbert space, then $\forall\ x, y \in H$ and $\alpha \in (0,1)$, we have*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

Lemma 2.3. *(Demi-closed principle), Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping, then $I - T$ is demi-closed on C i.e, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.4. *[28] Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\Sigma_{n=0}^{\infty}\gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ or $\Sigma_{n=0}^{\infty}|\delta_n\gamma_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Main Result

Theorem 3.1. *Let H_1, H_2, and H_3 be real Hilbert spaces. Let $T : H_1 \to H_1$, $S : H_2 \to H_2$ be demicontractive mappings with constants k_1 and k_2 respectively, such that $I - S$ and $I - T$ are demi-closed at 0. Let f_i and g_i $(i = 1, 2)$ be two convex and lower semicontinuous functions such that $f_1 : H_1 \to \mathbb{R}$ and $f_2 : H_2 \to \mathbb{R}$ are differentiable with L_1 and L_2- Lipschitz continuous gradient, $g_1 : H_1 \to \mathbb{R}$ and $g_2 : H_2 \to \mathbb{R}$ be simple maps and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Upsilon \neq \emptyset$ and let the step-size $\gamma_n \in \left(\epsilon, \frac{2||At_n - Br_n||^2}{||A^*(At_n - Br_n)||^2 + ||B^*(At_n - Br_n)||^2} - \epsilon\right), n \in \Omega$. Otherwise, $\gamma_n = \gamma$ (γ being any nonnegative value), where the set of indexes $\Omega = \{n : At_n - Br_n \neq 0\}$.*

Let $u, x_0 \in Q_1$ and $v, y_0 \in Q_2$ be arbitrary and the sequences $\{(x_n,\ y_n)\}$ be generated by

$$(3.1) \qquad \begin{cases} t_n = (1-\alpha_n)x_n + \alpha_n u; \\ r_n = (1-\alpha_n)y_n + \alpha_n v; \\ u_n = Prox_{\delta_n g_1}(I - \delta_n\nabla f_1)(t_n - \gamma_n A^*(At_n - Br_n)); \\ v_n = Prox_{\delta_n g_2}(I - \delta_n\nabla f_2)(r_n + \gamma_n B^*(At_n - Br_n)); \\ x_{n+1} = (1-\zeta_n)u_n + \zeta_n Tu_n; \\ y_{n+1} = (1-\psi_n)v_n + \psi_n Sv_n; \end{cases}$$

where $\{\delta_n\}$ *is a sequence of positive real numbers and* $\{\alpha_n\}$, $\{\zeta_n\}$ *and* $\{\psi_n\}$ *are sequences in* $(0,1)$, *with conditions*

i $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

ii $\zeta_n \in (a, 1-k_1) \subseteq (0,1)$ for some $a > 0$,

iii $\psi_n \in (b, 1-k_2) \subseteq (0,1)$ for some $b > 0$.

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y})$ *in* Υ.

Proof. Clearly γ_n is well defined since for any $(x,y) \in \Upsilon$, we have

$$\langle A^*(At_n - Br_n), t_n - x\rangle = \langle At_n - Br_n, At_n - Ax\rangle \tag{3.2}$$

and

$$\langle B^*(At_n - Br_n), y - r_n\rangle = \langle At_n - Br_n, By - Br_n\rangle. \tag{3.3}$$

Adding (3.2) and (3.3) and taking into account the fact that $Ax = By$, we obtain $\forall n \in \Omega$,

$$\begin{aligned} ||At_n - Br_n||^2 &= \langle A^*(At_n - Br_n), t_n - x\rangle + \langle B^*(At_n - Br_n), y - r_n\rangle \\ &\leq ||A^*(At_n - Br_n)||||t_n - x|| + ||B^*(At_n - Br_n)||||y - r_n||. \end{aligned}$$

Therefore, for $n \in \Omega$, that is, $||At_n - Br_n|| > 0$, we have $||A^*(At_n - Br_n)|| \neq 0$ or $||B^*(At_n - Br_n)|| \neq 0$. Thus γ_n is well defined.

Let $(p,q) \in \Upsilon$, we have from (3.1) that

$$\begin{aligned} \|u_n - p\|^2 &= \|Prox_{\delta_n g_1}(I - \delta_n \nabla f_1)(t_n - \gamma_n A^*(At_n - Br_n)) - p\|^2 \\ &\leq \|t_n - \gamma_n A^*(At_n - Br_n) - p\|^2 \\ &= \|t_n - p\|^2 - 2\gamma_n\langle t_n - p, A^*(At_n - Br_n)\rangle + \gamma_n^2\|A^*(At_n - Br_n)\|^2. \end{aligned} \tag{3.4}$$

From Lemma 2.1 and noting that A^* is adjoint of A, we have

$$\begin{aligned} -2\langle t_n - p, A^*(At_n - Br_n)\rangle &= -2\langle At_n - Ap, At_n - Br_n\rangle \\ &= -\|At_n - Ap\|^2 - \|At_n - Br_n\|^2 + \|Br_n - Ap\|^2. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$\begin{aligned} \|u_n - p\|^2 \leq \|t_n - p\|^2 &- \gamma_n\|At_n - Ap\|^2 - \gamma_n\|At_n - Br_n\|^2 \\ &+ \gamma_n\|Br_n - Ap\|^2 + {\gamma_n}^2\|A^*(At_n - Br_n)\|^2. \end{aligned} \tag{3.6}$$

Similarly, from (3.1), we have

$$\begin{aligned} \|v_n - q\|^2 \leq \|r_n - q\|^2 &- \gamma_n\|Br_n - Bq\|^2 - \gamma_n\|At_n - Br_n\|^2 \\ &+ \gamma_n\|At_n - Bq\|^2 + \gamma_n^2\|B^*(At_n - Br_n)\|^2. \end{aligned} \tag{3.7}$$

Adding inequality (3.6) and (3.7), and using the fact that $Ap = Bq$, we obtain

$$\begin{aligned}\|u_n - p\|^2 + \|v_n - q\|^2 &\le \|t_n - p\|^2 + \|r_n - q\|^2 - \gamma_n[2\|At_n - Br_n\|^2 \\ &\quad - \gamma_n(\|A^*(At_n - Br_n)\|^2 + \|B^*(At_n - Br_n)\|^2)] \\ &\le \|t_n - p\|^2 + \|r_n - q\|^2. \end{aligned} \tag{3.8}$$

From (3.1) and the fact that T is demi-contractive, we obtain

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &= \|(1-\zeta_n)u_n + \zeta_n Tu_n - p\|^2 \\ &= \|(1-\zeta_n)(u_n - p) + \zeta_n(Tu_n - p)\|^2 \\ &= (1-\zeta_n)^2\|u_n - p\|^2 + \zeta_n^2\|Tu_n - p\|^2 + 2\zeta_n(1-\zeta_n)\langle u_n - p, Tu_n - p\rangle \\ &\le (1-\zeta_n)^2\|u_n - p\|^2 + \zeta_n^2[\|u_n - p\|^2 + k_1\|u_n - Tu_n\|^2] \\ &\quad +2\zeta_n(1-\zeta_n)\left[\|u_n - p\|^2 - \frac{1-k_1}{2}\|u_n - Tu_n\|^2\right] \\ &= (1 - 2\zeta_n + \zeta_n^2)\|u_n - p\|^2 + \zeta_n^2[\|u_n - p\|^2 + k_1\|u_n - Tu_n\|^2] \\ &\quad +2\zeta_n\|u_n - p\|^2 - 2\zeta_n^2\|u_n - p\|^2 - \zeta_n(1-\zeta_n)(1-k_1)\|u_n - Tu_n\|^2 \\ &= \|u_n - p\|^2 - \zeta_n(1 - \zeta_n - k_1)\|u_n - Tu_n\|^2 \\ &\le \|u_n - p\|^2. \end{aligned} \tag{3.9}$$

Similarly, we have that

$$\|y_{n+1} - q\|^2 \le \|v_n - q\|^2. \tag{3.10}$$

Adding (3.9) and (3.10), and using (3.8), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 + \|y_{n+1} - q\|^2 &\le \|u_n - p\|^2 + \|v_n - q\|^2 \\ &\le \|t_n - p\|^2 + \|r_n - q\|^2. \end{aligned} \tag{3.11}$$

From (3.1), (3.11) and Lemma 2.2 we obtain

$$\begin{aligned}&\|x_{n+1} - p\|^2 + \|y_{n+1} - q\|^2 \\ &\le \|(1-\alpha_n)(x_n - p) + \alpha_n(u - p)\|^2 \\ &\quad +\|(1-\alpha_n)(y_n - q) + \alpha_n(v - q)\|^2 \\ &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|u - p\|^2 + (1-\alpha_n)\|y_n - q\|^2 \\ &\quad +\alpha_n\|v - q\|^2 \\ &= (1-\alpha_n)[\|x_n - p\|^2 + \|y_n - q\|^2] + \alpha_n[\|u - p\|^2 + \|v - q\|^2] \\ &\le \max\{\|x_n - p\|^2 + \|y_n - q\|^2, \|u - p\|^2 + \|v - q\|^2\} \\ &\ \vdots \\ &\le \max\{\|x_0 - p\|^2 + \|y_0 - q\|^2, \|u - p\|^2 + \|v - q\|^2\}. \end{aligned}$$

Hence, $\{\|x_n - p\|^2 + \|y_n - q\|^2\}$ is bounded. Consequently $\{x_n\}$, $\{y_n\}$, $\{t_n\}$, $\{r_n\}$, $\{u_n\}$, $\{v_n\}$, $\{Ax_n\}$, $\{By_n\}$ are bounded. From (3.8), we obtain

$$\begin{aligned}
&\|x_{n+1} - p\|^2 + \|y_{n+1} - q\|^2 \\
&\leq \|t_n - p\|^2 + \|r_n - q\|^2 - \gamma_n[2\|At_n - Br_n\|^2 \\
&\quad -\gamma_n(\|A^*(At_n - Br_n)\|^2 + \|B^*(At_n - Br_n)\|^2)] \\
&\leq (1-\alpha_n)[\|x_n - p\|^2 + \|y_n - q\|^2] + \alpha_n[\|u - p\|^2 + \|v - q\|^2] \\
&\quad -\gamma_n[2\|At_n - Br_n\|^2 - \gamma_n(\|A^*(At_n - Br_n)\|^2 \\
&\quad +\|B^*(At_n - Br_n)\|^2)],
\end{aligned}$$

which implies

$$\begin{aligned}
&\epsilon^2\left(||A^*(At_n - Br_n)||^2 + ||B^*(At_n - Br_n)||^2\right) \\
&\leq (1-\alpha_n)\left[||x_n - p||^2 + ||y_n - q||^2\right] \\
&\quad +\alpha_n\left[||u - p||^2 + ||v - q||^2\right] \\
&\quad -\left[||x_{n+1} - p||^2 + ||y_{n+1} - q||^2\right].
\end{aligned} \tag{3.12}$$

We divided the remaining part of the proof into two cases to establish strong convergence.

Case 1: Assume that $\left\{||x_n - p||^2 + ||y_n - q||^2\right\}$ is monotone decreasing, then $\left\{||x_n - p||^2 + ||y_n - q||^2\right\}$ is convergent, thus

$$\lim_{n\to\infty}\left[\left(||x_{n+1} - p||^2 + ||y_{n+1} - q||^2\right) - \left(||x_n - p||^2 + ||y_n - q||^2\right)\right] = 0.$$

It follows from (3.12) that $\left(||A^*(At_n - Br_n)||^2 + ||B^*(At_n - Br_n)||^2\right) \to 0$, as $n \to \infty$.

Since $At_n - Br_n = 0$, if $n \notin \Omega$, we have

$$\lim_{n\to\infty} ||A^*(At_n - Br_n)||^2 = \lim_{n\to\infty} ||B^*(At_n - Br_n)||^2 = 0. \tag{3.13}$$

From (3.1), we have

$$\begin{aligned}
||t_n - x_n||^2 &= ||(1-\alpha_n)x_n + \alpha_n u - x_n||^2 \\
&= \alpha_n^2||u - x_n||^2 \to 0, \text{ as } n \to \infty.
\end{aligned}$$

$$\Longrightarrow \lim_{n\to\infty} ||t_n - x_n||^2 = 0. \tag{3.14}$$

Similarly, we have

$$\lim_{n\to\infty} ||r_n - y_n||^2 = 0. \tag{3.15}$$

Also, from (3.1), Lemma 2.1 and Lemma 2.2, we have that

$$\begin{aligned}
&||u_n - p||^2 \\
&= ||Prox_{\delta_n g_1}(I - \delta_n \nabla f_1)(t_n - \gamma_n A^*(At_n - Br_n)) - p||^2 \\
&\leq \langle u_n - p, t_n - \gamma_n A^*(At_n - Br_n) - p\rangle \\
&= \frac{1}{2}[||u_n - p||^2 + ||t_n - \gamma_n A^*(At_n - Br_n) - p||^2 \\
&\quad -||u_n - p - (t_n - \gamma_n A^*(At_n - Br_n) - p)||^2] \\
&\leq \frac{1}{2}[||u_n - p||^2 + ||t_n - p||^2 + \gamma_n^2||A^*(At_n - Br_n)||^2 \\
&\quad +2\gamma_n||t_n - p||||A^*(At_n - Br_n)|| \\
&\quad -(||u_n - t_n||^2 + \gamma_n^2||A^*(At_n - Br_n)||^2 - 2\gamma_n\langle u_n - t_n, A^*(At_n - Br_n)\rangle]) \\
&= \frac{1}{2}[||u_n - p||^2 + ||t_n - p||^2 + 2\gamma_n||t_n - p||||A^*(At_n - Br_n)|| \\
&\quad -||u_n - t_n||^2 + 2\gamma_n\langle u_n - t_n, A^*(At_n - Br_n)\rangle] \\
&\leq \frac{1}{2}[||u_n - p||^2 + ||t_n - p||^2 + 2\gamma_n||t_n - p||||A^*(At_n - Br_n)|| \\
&\quad -||u_n - t_n||^2 + 2\gamma_n||u_n - t_n||||A^*(At_n - Br_n)||] \\
&\leq \frac{1}{2}[||u_n - p||^2 + (1 - \alpha_n)||x_n - p||^2 + \alpha_n||u - p||^2 \\
&\quad +2\gamma_n||t_n - p||||A^*(At_n - Br_n)|| - ||u_n - t_n||^2 \\
&\quad +2\gamma_n||u_n - t_n||||A^*(At_n - Br_n)||] \\
&\leq \frac{1}{2}[||u_n - p||^2 + ||x_n - p||^2 + \alpha_n||u - p||^2 \\
&\quad +2\gamma_n||t_n - p||||A^*(At_n - Br_n)|| \\
&\quad -||u_n - t_n||^2 + 2\gamma_n||u_n - t_n||||A^*(At_n - Br_n)||], \qquad (3.16)
\end{aligned}$$

which implies

$$\begin{aligned}
||u_n - p||^2 \leq\ & ||x_n - p||^2 + \alpha_n||u - p||^2 + 2\gamma_n||w_n - p||||A^*(At_n - Br_n)|| \\
& -||u_n - t_n||^2 + 2\gamma_n||u_n - t_n||||A^*(At_n - Br_n)||. \qquad (3.17)
\end{aligned}$$

From (3.9) and (3.17), we have

$$\begin{aligned}
||x_{n+1} - p||^2 \leq\ & ||x_n - p||^2 + \alpha_n||u - p||^2 + 2\gamma_n||t_n - x^*||||A^*(At_n - Br_n)|| \\
& -||u_n - t_n||^2 + 2\gamma_n||u_n - t_n||||A^*(At_n - Br_n)||. \qquad (3.18)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
||y_{n+1} - q||^2 \leq\ & ||y_n - q||^2 + \alpha_n||v - q||^2 + 2\gamma_n||r_n - q||||B^*(At_n - Br_n)|| \\
& -||v_n - r_n||^2 + 2\gamma_n||v_n - r_n||||B^*(At_n - Br_n)||. \qquad (3.19)
\end{aligned}$$

Adding (3.18) and (3.19), we have

$$\begin{aligned} & ||x_{n+1}-p||^2+||y_{n+1}-q||^2 \\ \leq\; & ||x_n-p||^2+||y_n-q||^2+\alpha_n[||u-p||^2+||v-q||^2] \\ & +2\gamma_n[||t_n-x^*||||A^*(At_n-Br_n)|| \\ & +||r_n-y^*||||B^*(At_n-Br_n)||] \\ & -[||u_n-t_n||^2+||v_n-r_n||^2] \\ & +2\gamma_n[||u_n-t_n||||A^*(At_n-Br_n)|| \\ & +||v_n-r_n||||B^*(At_n-Br_n)||]. \end{aligned} \tag{3.20}$$

Using (3.13) together with the fact that $\alpha_n \to 0,$ as $n \to \infty$ in (3.20), we have

$$\lim_{n\to\infty}\left[||u_n-t_n||^2+||v_n-r_n||^2\right]=0,$$

which implies

$$\lim_{n\to\infty}||u_n-t_n||^2=0 \tag{3.21}$$

and

$$\lim_{n\to\infty}||v_n-r_n||^2=0. \tag{3.22}$$

Observe that since T is demicontractive and $p \in F(T)$, so we have

$$\begin{aligned} ||Tx-p||^2 &\leq ||x-p||^2+k_1||x-Tx||^2 \\ \implies \langle Tx-p,Tx-p\rangle &\leq \langle x-p,x-p\rangle+k_1||x-Tx||^2 \\ \implies \langle Tx-p,Tx-x\rangle+\langle Tx-p,x-p\rangle &\leq \langle x-p,x-p\rangle+k_1||x-Tx||^2 \\ \implies \langle Tx-p,Tx-x\rangle &\leq \langle x-Tx,x-p\rangle+k_1||x-Tx||^2 \\ \implies \langle Tx-x,Tx-x\rangle+\langle x-p,Tx-x\rangle &\leq \langle x-Tx,x-p\rangle+k_1||x-Tx||^2 \\ ||Tx-x||^2 &\leq \langle x-p,x-Tx\rangle-\langle x-p,Tx-x\rangle \\ & \quad +k_1||x-Tx||^2 \\ \implies (1-k_1)||Tx-x||^2 &\leq 2\langle x-p,x-Tx\rangle. \end{aligned} \tag{3.23}$$

From (3.1) and (3.23), we have

$$\begin{aligned} & ||x_{n+1}-p||^2 \\ =\; & ||(1-\zeta_n)u_n+\zeta_nTu_n-p||^2 \\ =\; & ||u_n-p+\zeta_n(Tu_n-u_n)||^2 \\ =\; & ||u_n-p||^2+\zeta_n^2||Tu_n-u_n||^2-2\zeta_n\langle u_n-p,u_n-Tu_n\rangle \\ \leq\; & ||u_n-p||^2+\zeta_n^2||Tu_n-u_n||^2-(1-k_1)\zeta_n||Tu_n-u_n||^2 \\ =\; & ||u_n-p||^2+\zeta_n(\zeta_n-(1-k_1))||u_n-Tu_n||^2. \end{aligned} \tag{3.24}$$

Similarly, we have that

$$||y_{n+1}-q||^2 \leq ||v_n-q||^2+\psi_n(\psi_n-(1-k_2))||v_n-Sv_n||^2. \tag{3.25}$$

Adding (3.24) and (3.25), we have

$$
\begin{aligned}
&||x_{n+1}-p||^2+||y_{n+1}-q||^2\\
&\quad\le ||u_n-p||^2+||v_n-q||^2+\zeta_n(\zeta_n-(1-k_1))||Tu_n-u_n||^2\\
&\qquad +\psi_n(\psi_n(1-k_2))||v_n-Sv_n||^2\\
&\quad\le ||t_n-p||+||r_n-q||^2+\zeta_n(\zeta_n-(1-k_1))||Tu_n-u_n||^2\\
&\qquad +\psi_n(\psi_n-(1-k_2))||v_n-Sv_n||^2\\
&\quad\le (1-\alpha_n)\big[||x_n-p||^2+||y_n-q||^2\big]+\alpha_n\big[||u-p||^2+||v-q||^2\big]\\
&\qquad +\zeta_n(\zeta_n-(1-k_1))||u_n-Tu_n||^2\\
&\qquad +\psi_n(\psi_n-(1-k_2))||v_n-Sv_n||^2. \qquad (3.26)
\end{aligned}
$$

Let $K_n=\zeta_n((1-k_1)-\zeta_n)||u_n-Tu_n||^2+\psi_n((1-k_2)-\psi_n)||v_n-Sv_n||^2$, then

$$
\begin{aligned}
&K_n\\
&\quad\le (1-\alpha_n)\left[||x_n-p||^2+||y_n-q||^2\right]\\
&\qquad -\left[||x_{n+1}-p||^2+||y_{n+1}-q||^2\right]\\
&\qquad +\alpha_n\left[||u-p||^2+||v-q||^2\right]\to 0,\ \text{as } n\to\infty,
\end{aligned}
$$

which implies

$$||u_n-Tu_n||^2+||v_n-Sv_n||^2\to 0,\ n\to\infty.$$

That is

$$\lim_{n\to\infty}||u_n-Tu_n||^2=0, \tag{3.27}$$

and

$$\lim_{n\to\infty}||v_n-Sv_n||^2=0. \tag{3.28}$$

From (3.27), we have

$$\lim_{n\to\infty}||x_{n+1}-u_n|| = \lim_{n\to\infty}\zeta_n||u_n-Tu_n||=0. \tag{3.29}$$

Similarly, from (3.28), we have

$$\lim_{n\to\infty}||y_{n+1}-v_n|| = \lim_{n\to\infty}\psi_n||v_n-Sv_n||=0. \tag{3.30}$$

From (3.14) and (3.21), we have $||x_n-u_n||\le||x_n-t_n||+||t_n-u_n||\to 0$, which implies that

$$\lim_{n\to\infty}||x_n-u_n||=0. \tag{3.31}$$

Similarly, from (3.15) and (3.22), we have

$$\lim_{n\to\infty}||y_n-v_n||=0. \tag{3.32}$$

Also, from (3.29) and (3.31), we have

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - u_n|| + ||u_n - x_n|| \to 0, \quad as \quad n \to \infty,$$

which implies that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.33}$$

Similarly, from (3.30) and (3.32), we have

$$\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0. \tag{3.34}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $\bar{x} \in H_1$. By (3.31) and (3.14), we have that $\{u_n\}$ and $\{t_n\}$ converges weakly to $\bar{x}$ and by the demi-closeness of $I-T$ at 0 and (3.27), we have that $\bar{x} \in F(T)$. Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ converges weakly to $\bar{y} \in H_2$. By (3.32) and (3.15), we have that $\{v_n\}$ and $\{r_n\}$ converges weakly to $\bar{y}$ and by the demi-closeness of $I - S$ at 0 and (3.28), we have that $\bar{y} \in F(S)$.

Also, since A and B are bounded linear operators, we have that $\{At_n\}$ converges weakly to $A\bar{x}$ and $\{Br_n\}$ converges weakly to $B\bar{y}$.

Next, we show that $A\bar{x} = B\bar{y}$.

$$\begin{aligned}
&||A\bar{x} - B\bar{y}||^2 \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - B\bar{y}\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - B\bar{y} + At_n - At_n + Br_n - Br_n\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - At_n\rangle + \langle A\bar{x} - B\bar{y}, At_n - Br_n\rangle + \langle A\bar{x} - B\bar{y}, Br_n - B\bar{y}\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - At_n\rangle + \langle A\bar{x}, At_n - Br_n\rangle - \langle B\bar{y}, At_n - Br_n\rangle \\
&\quad +\langle A\bar{x} - B\bar{y}, Br_n - B\bar{y}\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - At_n\rangle + \langle \bar{x}, A^*(At_n - Br_n)\rangle - \langle \bar{y}, B^*(At_n - Br_n)\rangle \\
&\quad +\langle A\bar{x} - B\bar{y}, Br_n - B\bar{y}\rangle \\
&\leq \langle A\bar{x} - B\bar{y}, A\bar{x} - At_n\rangle + ||\bar{x}||||A^*(At_n - Br_n)|| + ||\bar{y}||||B^*(At_n - Br_n)|| \\
&+ \langle A\bar{x} - B\bar{y}, Br_n - B\bar{y}\rangle \to 0, n \to \infty,
\end{aligned}$$

which implies that $||A\bar{x} - B\bar{y}|| = 0$. Hence $A\bar{x} = B\bar{y}$.

Let $p_n = t_n - \gamma_n A^*(At_n - Br_n)$.

Then $\|p_n - t_n\|^2 = {\gamma_n}^2 \|A^*(At_n - Br_n)\|^2 \to 0$ as $n \to \infty$, and

$$\|u_n - p_n\| \leq \|u_n - t_n\| + \|t_n - p_n\| \to 0 \quad as \quad n \to \infty. \tag{3.35}$$

We get from (3.35) that

$$\|prox_{\delta_n g_1}(p_n - \delta_n \nabla f_1(p_n)) - p_n\| \to 0. \tag{3.36}$$

Similarly, if we let $a_n = r_n + \gamma_n B^*(At_n - Br_n)$, we obtain

$$\|prox_{\delta_n g_2}(a_n - \delta_n \nabla f_2(a_n)) - a_n\| \to 0. \tag{3.37}$$

Hence by Lemma 2.3 (demiclosedness principle) we have $w_\omega(p_n) = w_\omega(x_n) \subset \Upsilon$. There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$, for some $\bar{x} \in \Upsilon$. By similar argument, we get $\bar{y} \in \Upsilon$. Hence $(\bar{x}, \bar{y}) \in \Upsilon$.

Next, we show that $(\{x_n\}, \{y_n\})$ converges strongly to $(\bar{x}, \bar{y})$.

From (3.11), we have

$$\begin{aligned} &||x_{n+1} - \bar{x}||^2 + ||y_{n+1} - \bar{y}||^2 \\ &\quad \le ||t_n - \bar{x}||^2 + ||r_n - \bar{y}||^2 \\ &\quad = (1-\alpha_n)^2||x_n - \bar{x}||^2 + \alpha_n^2||u - \bar{x}||^2 \\ &\qquad +2(1-\alpha_n)\alpha_n\langle x_n - \bar{x}, u - \bar{x}\rangle \\ &\qquad +(1-\alpha_n)^2||y_n - \bar{y}||^2 + \alpha_n^2||v - \bar{y}||^2 \\ &\qquad +2(1-\alpha_n)\alpha_n\langle y_n - \bar{y}, v - \bar{y}\rangle \\ &\quad \le (1-\alpha_n)^2\left[||x_n - \bar{x}||^2 + ||y_n - \bar{y}||^2\right] + \alpha_n\big[\alpha_n||u - \bar{x}||^2 \\ &\qquad +2(1-\alpha_n)\langle x_n - \bar{x}, u - \bar{x}\rangle + \alpha_n||v - \bar{y}||^2 \\ &\qquad +2(1-\alpha_n)\langle y_n - \bar{y}, v - \bar{y}\rangle\big]. \end{aligned} \tag{3.38}$$

Applying Lemma (2.4) to (3.38), we have that $(\{x_n\}, \{y_n\})$ converges strongly to $(\bar{x}, \bar{y})$.

Case 2. Assume that $\{||x_n - p||^2 + ||y_n - q||^2\}$ is not monotone decreasing. Set $\Gamma_n = ||x_n - p||^2 + ||y_n - q||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \ge n_0$ (for some large n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \Gamma_k \le \Gamma_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$, as $n \to \infty$ and

$$\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}, \forall n \ge n_0.$$

From (3.12), we have

$$\begin{aligned} &\epsilon^2(||A^*(At_{\tau(n)} - Br_{\tau(n)})||^2 + ||B^*(At_{\tau(n)} - Br_{\tau(n)})||^2 \\ &\quad \le (1-\alpha_{\tau(n)})[||x_{\tau(n)} - p||^2 \\ &\qquad +||y_{\tau(n)} - q||^2] - [||x_{\tau(n)+1} - p||^2 \\ &\qquad +||y_{\tau(n)+1} - q||^2] + \alpha_{\tau(n)}[||u - p||^2 + ||v - p||^2] \\ &\quad \le \alpha_{\tau(n)}[||u - p||^2 + ||v - q||^2]. \end{aligned} \tag{3.39}$$

Therefore,

$$\left(||A^*(At_{\tau(n)} - Br_{\tau(n)})||^2 + ||B^*(At_{\tau(n)} - Br_{\tau(n)})||^2\right) \to 0, \text{ as } n \to \infty.$$

Note that $At_{\tau(n)} - Br_{\tau(n)} = 0$, if $\tau(n) \notin \Omega$. Hence,

$$\lim_{n\to\infty} ||A^*(At_{\tau(n)} - Br_{\tau(n)})||^2 = 0, \tag{3.40}$$

and

$$\lim_{n\to\infty} ||B^*(At_{\tau(n)} - Br_{\tau(n)})||^2 = 0. \tag{3.41}$$

Using the same argument as in case 1, we have $(\{x_{\tau(n)}\}, \{y_{\tau(n)}\})$ converges weakly to $(\bar{x}, \bar{y}) \in \Gamma$.

Now for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq [||x_{\tau(n)+1} - p||^2 + ||y_{\tau(n)+1} - q||^2] - [||x_{\tau(n)} - p||^2 + ||y_{\tau(n)} - q||^2] \\ &\leq (1 - \alpha_{\tau(n)})[||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2] - [||x_{\tau(n)} - p||^2 + ||y_{\tau(n)} - q||^2] \\ &\quad +\alpha_{\tau(n)}[\alpha_{\tau(n)}[||u - \bar{x}||^2 + ||v - \bar{y}||^2] + 2(1 - \alpha_{\tau(n)})(\langle x_{\tau(n)} - \bar{x}, u - \bar{x}\rangle \\ &\quad +\langle y_{\tau(n)} - \bar{y}, v - \bar{y}\rangle)], \end{aligned}$$

which implies

$$\begin{aligned} &||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2 \\ &\quad \leq \alpha_{\tau(n)}[||u - \bar{x}||^2 + ||v - \bar{y}||^2] \\ &\quad\quad +2(1 - \alpha_{\tau(n)})(\langle x_{\tau(n)} \\ &\quad\quad -\bar{x}, u - \bar{x}\rangle + \langle y_{\tau(n)} - \bar{y}, v - \bar{y}\rangle) \to 0. \end{aligned}$$

Hence

$$\lim_{n\to\infty} (||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2) = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Moreover, for $n \geq n_0$, it is clear that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$.

Consequently for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Thus, $\lim_{n\to\infty} \Gamma_n = 0$. That is $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$. □

Corollary 3.2. *Let H_1, H_2, and H_3 be real Hilbert spaces Let $T : H_1 \to H_1$, $S : H_2 \to H_2$ be two nonexpansive mappings such that $I - S$ and $I - T$ are demi-closed at 0. Let f_i and g_i $(i = 1, 2)$ be two convex and lower semi-continuous functions such that $f_1 : H_1 \to \mathbb{R}$ and $f_2 : H_2 \to \mathbb{R}$ are differentiable with L_1- and L_2- Lipschitz continuous gradient, $g_1 : H_1 \to \mathbb{R}$ and $g_2 : H_2 \to \mathbb{R}$ be simple maps and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Upsilon \neq \emptyset$ and let the step-size $\gamma_n \in \left(\epsilon, \frac{2||At_n - Br_n||^2}{||A^*(At_n - Br_n)||^2 + ||B^*(At_n - Br_n)||^2} - \epsilon\right), n \in \Omega$ q otherwise, $\gamma_n = \gamma$ (γ being any nonegative value), where the set of indexes $\Omega = \{n : At_n - Br_n \neq 0\}$.*

Let $u, x_0 \in Q_1$ and $v, y_0 \in Q_2$ be arbitrary and the sequences $\{(x_n, , y_n)\}$ be *generated by*

$$\begin{cases} t_n = (1-\alpha_n)x_n + \alpha_n u \\ r_n = (1-\alpha_n)y_n + \alpha_n v \\ u_n = Prox_{\delta_n g_1}(I-\delta_n\nabla f_1)(t_n - \gamma_n A^*(At_n - Br_n)) \\ v_n = Prox_{\delta_n g_2}(I-\delta_n\nabla f_2)(r_n + \gamma_n B^*(At_n - Br_n)) \\ x_{n+1} = (1-\zeta_n)u_n + \zeta_n T u_n \\ y_{n+1} = (1-\psi_n)v_n + \psi_n S v_n \end{cases} \tag{3.42}$$

where $\{\delta_n\}$ *is a sequence of positive real numbers and* $\{\alpha_n\}$, $\{\zeta_n\}$ *and* $\{\psi_n\}$ *are sequences in* $(0,1)$, *with conditions*

i $\lim_{n\to\infty}\alpha_n = 0, \quad \sum_{n=1}^{\infty}\alpha_n = \infty,$

ii $\zeta_n \in (a, 1-k_1) \subseteq (0,1)$ for some $a > 0,$

iii $\psi_n \in (b, 1-k_2) \subseteq (0,1)$ for some $b > 0.$

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y})$ *in* Υ.

Corollary 3.3. *Let* H_1, H_2, *and* H_3 *be real Hilbert spaces. Let* $T : H_1 \to H_1$, $S : H_2 \to H_2$ *be* k_1*-strictly pseudocontractive and* k_2*-strictly pseudocontractive mappings respectively such that* $I-S$ *and* $I-T$ *are demi-closed at 0. Let* f_i *and* g_i $(i = 1,2)$ *be two convex and lower semicontinuous functions such that* $f_1 : H_1 \to \mathbb{R}$ *and* $f_2 : H_2 \to \mathbb{R}$ *are differentiable with* L_1*- and* L_2*- Lipschitz continuous gradient,* $g_1 : H_1 \to \mathbb{R}$ *and* $g_2 : H_2 \to \mathbb{R}$ *be simple maps and* $A : H_1 \to H_3$, $B : H_2 \to H_3$ *be bounded linear operators. Assume that the solution set* $\Upsilon \neq \emptyset$ *and let the step-size* $\gamma_n \in \left(\epsilon, \frac{2||At_n - Br_n||^2}{||A^*(At_n - Br_n)||^2 + ||B^*(At_n - Br_n)||^2} - \epsilon\right), n \in \Omega$. *Otherwise,* $\gamma_n = \gamma$(γ *being any nonegative value), where the set of indexes* $\Omega = \{n : At_n - Br_n \neq 0\}$.

Let $u, x_0 \in Q_1$ *and* $v, y_0 \in Q_2$ *be arbitrary and the sequences* $(\{x_n\}, \{y_n\})$ *be generated by*

$$\begin{cases} t_n = (1-\alpha_n)x_n + \alpha_n u \\ r_n = (1-\alpha_n)y_n + \alpha_n v \\ u_n = Prox_{\delta_n g_1}(I-\delta_n\nabla f_1)(t_n - \gamma_n A^*(At_n - Br_n)) \\ v_n = Prox_{\delta_n g_2}(I-\delta_n\nabla f_2)(r_n + \gamma_n B^*(At_n - Br_n)) \\ x_{n+1} = (1-\zeta_n)u_n + \zeta_n T u_n \\ y_{n+1} = (1-\psi_n)v_n + \psi_n S v_n \end{cases} \tag{3.43}$$

where $\{\delta_n\}$ *is a sequence of positive real numbers and* $\{\alpha_n\}$, $\{\zeta_n\}$ *and* $\{\psi_n\}$ *are sequences in* $(0,1)$, *with conditions*

i $\lim_{n\to\infty}\alpha_n = 0, \quad \sum_{n=1}^{\infty}\alpha_n = \infty,$

ii $\zeta_n \in (a, 1-k_1) \subseteq (0,1)$ for some $a > 0,$

iii $\psi_n \in (b, 1-k_2) \subseteq (0,1)$ for some $b > 0.$

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y})$ *in* Υ.

3.1. Application to the split equality monotone variational inclusion problem

Let H_1, H_2 and H_3 be real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Let $f_1 : H_1 \to H_1$, $f_2 : H_2 \to H_2$ be α_1, (respectively, α_2)-inverse strongly monotone mappings and $M_1 : H_1 \to 2^{H_1}$, $M_2 : H_2 \to 2^{H_2}$ be maximal monotone mappings. The split equality monotone variational inclusion problem (SEMVIP) is to find $x^* \in H_1$ and $y^* \in H_2$ such that

$$0 \in f_1(x^*) + M_1(x^*), \tag{3.44}$$

$$0 \in f_2(y^*) + M_2(y^*), \quad \text{and} \quad Ax^* = By^*. \tag{3.45}$$

Let $\mathrm{SOL}(f_i, M_i)$, $(i = 1, 2)$ be the solution set of SEMVIP. The operator $J_\sigma^{M_i}(I - \lambda\phi)$ $(i = 1, 2)$ is an averaged nonexpansive operator and $F(J_\sigma^{M_i}(I - \lambda f_i)) = \mathrm{SOL}(f_i, M_i)$, $i = 1, 2$, where $\sigma > 0$, $\lambda \in (0, 2\alpha)$ and $J_\sigma^{M_i}(I - \lambda\phi)$ is the resolvent of M_i with parameter σ (see for example [1, 19]).

Since every averaged nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive, in Corollary 3.2, if we let $T = J_\sigma^{M_1}(I - \lambda f_1)$ and $S = J_\sigma^{M_2}(I - \lambda f_2)$, then we obtain a strong convergence result for approximating a common solution of SEMVIP and SEMFPP.

Acknowledgement

The first and second author acknowledge with thanks the bursary and financial support from Department of Science and Technology and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DST-NRF COE-MaSS) Doctoral Bursary. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS. The authors are also grateful to the referees for carefully reading the submitted manuscript and giving useful comments and suggestions on how to improve it.

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Received by the editors February 10, 2017
First published online April 17, 2018