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APPLICATIONS OF JACK'S LEMMA FOR THE HOLOMORPHIC FUNCTIONS

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Abstract. In this paper, we give some results on $\frac{zf'(z)}{f(z)}$ for certain classes of holomorphic functions on the unit disc $U = \{z : |z| < 1\}$ and on $\partial U = \{z : |z| = 1\}$. For the function $f(z) = z+c_2z^2+c_3z^3+...$ defined on the unit disc U such that $f \in \mathcal{M}$, we estimate a modulus of the angular derivative $\frac{zf'(z)}{f(z)}$ function at the boundary point b with $f'(b) = 0$. Moreover, Schwarz lemma for the class $\mathcal{M}$ is given. The sharpness of these inequalities is also proved.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z) = z + c_2z^2 + c_3z^2 + ...$ which are holomorphic on $U = \{z : |z| < 1\}$. Also, let $\mathcal{M}$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\Re\left(\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right) < \frac{3}{2}, \quad z \in U. \tag{1.1}$$

Under condition (1.1), in other words when the defining property of the class $\mathcal{M}$ is satisfied, the functions $f(z)$ in the class $\mathcal{M}$ are starlike functions. Two of the simplest results of the complex function theory for holomorphic functions are both the classical Schwarz lemma and Jack's lemma. The Schwarz lemma and Jack's lemma have a very important role in the geometric function theory. A general form for these two lemmas, which is very simple and commonly used, is given as follows:

Lemma 1 (Schwarz lemma)**.** *Let U be the unit disc in the complex plane $\mathbb{C}$. Let $f : U \to U$ be a holomorphic function with $f(0) = 0$. Under these circumstances $|f(z)| \leq |z|$ for all $z \in U$, and $|f'(0)| \leq 1$. In addition, if the equality $|f(z)| = |z|$ holds for any $z \neq 0$, or $|f'(0)| = 1$ then f is a rotation, that is, $f(z) = ze^{i\gamma}$, γ real ([7], p329, [16]).*

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Lemma 2 (Jack's Lemma). *Let $f(z)$ be a non-constant and holomorphic function on the unit disc U with $f(0) = 0$. If $|f(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0, then*

$$\frac{z_0 f'(z_0)}{f(z_0)} = k,$$

where $k \geq 1$ is a real number ([9]).

For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to [1, 6, 15, 17]. Moreover, in [8] the authors proved an analogue of the generalized Schwarz lemma for meromorphic functions. Their results improved the classical generalized Schwarz lemma.

In this work, we show an application of Jack's lemma for holomorphic functions that provide inequality (1.1). Also, we will give the Schwarz lemma for this class. Moreover, we will give the Schwarz lemma at the boundary for this class. Let $f(z) = z + c_2 z^2 + c_3 z^2 + ..$ be a holomorphic function on the unit disc U. Consider the function

$$\varphi(z) = \frac{p(z) - 1}{p(z) + 1}, \tag{1.2}$$

where $p(z) = \frac{zf'(z)}{f(z)}$. $\varphi(z)$ is holomorphic on the unit disc and $\varphi(0) = 0$. We show that $|\varphi(z)| < 1$ for $|z| < 1$. We suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |\varphi(z)| = |\varphi(z_0)| = 1.$$

From Jack's lemma, we have

$$\varphi(z_0) = e^{i\theta}, \quad \frac{z_0 \varphi'(z_0)}{\varphi(z_0)} = k.$$

Therefore, from (1.2) we obtain

$$\begin{aligned} \Re\left(\frac{1 + \frac{z_0 f''(z_0)}{f'(z_0)}}{\frac{z_0 f'(z_0)}{f(z_0)}}\right) &= \Re\left(1 + \frac{2z_0\varphi'(z_0)}{(1+\varphi(z_0))^2}\right) \\ &= \Re\left(1 + \frac{2k\varphi(z_0)}{(1+\varphi(z_0))^2}\right) \\ &= \Re\left(1 + \frac{2ke^{i\theta}}{(1+e^{i\theta})^2}\right). \end{aligned}$$

Since

$$\frac{e^{i\theta}}{(1+e^{i\theta})^2} = \frac{e^{i\theta}}{1 + 2e^{i\theta} + e^{2i\theta}} = \frac{1}{e^{-i\theta} + 2 + e^{i\theta}} = \frac{1}{2 + 2\cos\theta},$$

we take

$$\Re\left(\frac{1 + \frac{z_0 f''(z_0)}{f'(z_0)}}{\frac{z_0 f'(z_0)}{f(z_0)}}\right) = 1 + \frac{2k}{2\,(1+\cos\theta)} \geq \frac{3}{2}$$

which is a contradiction to (1.1). This means that there is no point $z_0 \in U$ such that $|\varphi(z_0)| = 1$ for all $z \in U$. Thus, we obtain $|\varphi(z)| < 1$ for $z \in U$. By the Schwarz lemma, we obtain

$$|c_2| \leq 2.$$

The result is sharp and the extremal function is

$$f(z) = \frac{z}{(1-z)^2}.$$

That proves

Lemma 3. *If $f(z) \in \mathcal{M}$, then we have*

$$|c_2| \leq 2. \tag{1.3}$$

The result is sharp and the extremal function is

$$f(z) = \frac{z}{(1-z)^2}.$$

This lemma yields a "$\mathcal{M}$ *version*" of the classical Schwarz lemma for hololomorphic function of one complex variable.

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point b with $|b| = 1$, and if $|f(b)| = 1$ and $f'(b)$ exists, then $|f'(b)| \geq 1$, which is known as the Schwarz lemma on the boundary. In [20], R. Osserman proposed the boundary refinement of the classical Schwarz lemma as follows:

Let $f : U \to U$ be holomorphic function with $f(0) = 0$. Assume that there is a $b \in \partial U$ so that f extends continuously to b, $|f(b)| = 1$ and $f'(b)$ exists. Then

$$|f'(b)| \geq \frac{2}{1 + |f'(0)|}. \tag{1.4}$$

Thus, by the classical Schwarz lemma, it follows that

$$|f'(b)| \geq 1. \tag{1.5}$$

Inequality (1.4) is sharp, with equality possible for each value of $|f'(0)|$. In addition, for $b = 1$ in the inequality (1.4), equality occurs for the function $f(z) = z\frac{z+\gamma}{1+\gamma z}$, $\gamma \in [0,1]$. Also, $|f'(b)| > 1$ unless $f(z) = ze^{i\theta}$, θ real. Inequality (1.5) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1, 4, 5, 6, 18, 19].

Let us give the definitions needed for our results. A *Stolz angle* Δ at $b \in \partial U$ is the interior of any triangle in U symmetric to $[0, b]$ whose closure lies in U except for the vertex b. Basic for this paper are the notions of the angular limit and the angular derivative. Let $b \in \partial U$. We say that the angular limit $f(b)$ exists if

$$f(b) = \lim_{z \to b,\ z \in \Delta} f(z)$$

for every Stolz angle Δ at b and we say that the angular derivative $f'(b)$ exists if the angular limit $f(b)$ exists and

$$f'(b) = \lim_{z \to b,\ z \in \Delta} \frac{f(z) - f(b)}{z - b}$$

for every Stolz angle Δ at b.

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see [21]).

Lemma 4 (Julia-Wolff lemma)**.** *Let f be a holomorphic function on U, $f(0) = 0$ and $f(U) \subset U$. If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial U$, $|f(b)| = 1$, then the angular derivative $f'(b)$ exists and $1 \leq |f'(b)| \leq \infty$.*

Corollary 1. *The holomorphic function f has a finite angular derivative $f'(b)$ if and only if f' has the finite angular limit $f'(b)$ at $b \in \partial U$.*

D. M. Burns and S. G. Krantz [2] and D. Chelst [3] studied the uniqueness part of the Schwarz lemma. According to M. Mateljevic's studies, some other types of results which are related to the subject can be found in ([14] and [13]). In addition, [12] was posted on ResearchGate where more general aspects of these results are discussed.

The inequality (1.5) is a particular case of a result due to Vladimir N. Dubinin in [4], who strengthened the inequality $|f'(b)| \geq 1$ by involving zeros of the function f.

X. Tang, T. Liu and J. Lu [22] established a new type of the classical boundary Schwarz lemma for holomorphic self-mappings of the unit polydisk E^n in $\mathbb{C}^n$. They extended the classical Schwarz lemma at the boundary to high dimensions.

Also, M. Jeong [10] showed some inequalities at a boundary point for a different form of holomorphic functions and found the condition for equality and in [11] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

2. Main Results

In this section, we give some results on $\frac{zf'(z)}{f(z)}$ for certain classes of holomorphic functions on the unit disc on $\partial U = \{z : |z| = 1\}$. For the function $f(z) = z + c_2 z^2 + c_3 z^3 + ...$ defined on the unit disc U such that $f(z) \in \mathcal{M}$, we estimate a modulus of the angular derivative function $\frac{zf'(z)}{f(z)}$ at the boundary point b with $f'(b) = 0$. The sharpness of these inequalities is also proved.

Theorem 2.1. *Let $f(z) \in \mathcal{M}$. Suppose that, for some $b \in \partial U$, f has an angular limit $f(b)$ at b, $f'(b) = 0$. Then we have the inequality*

$$\left| \left(\frac{zf'(z)}{f(z)} \right)'_{z=b} \right| \geq \frac{1}{2}. \tag{2.1}$$

The inequality (2.1) is sharp with extremal function

$$f(z) = \frac{z}{(1-z)^2}.$$

Proof. Let us consider the following function

$$\varphi(z) = \frac{p(z)-1}{p(z)+1},$$

where $p(z) = \frac{zf'(z)}{f(z)}$. Then $\varphi(z)$ is holomorphic function on the unit disc U and $\varphi(0) = 0$. By Jack's lemma and since $f(z) \in \mathcal{M}$, we take $|\varphi(z)| < 1$ for $|z| < 1$. Also, we have $|\varphi(b)| = 1$ for $b \in \partial U$. It is clear that

$$\varphi'(z) = \frac{2p'(z)}{(p(z)+1)^2}.$$

Therefore, we take from (1.5), we obtain

$$1 \leq |\varphi'(b)| = \frac{2|p'(b)|}{|p(b)+1|^2} = 2|p'(b)|$$

and

$$|p'(b)| \geq \frac{1}{2}.$$

Now, we shall show that the inequality (2.1) is sharp. Equality holds true for Koebe function f given by

$$f(z) = \frac{z}{(1-z)^2} \tag{2.2}$$

which is the extremal function for the class of $\mathcal{M}$ on U.

Differentiating (2.2) logarithmically, we obtain

$$\ln f(z) = \ln \frac{z}{(1-z)^2} = \ln z - \ln(1-z)^2 = \ln z - 2\ln(1-z),$$

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \frac{2}{1-z}$$

and

$$p(z) = \frac{zf'(z)}{f(z)} = 1 + \frac{2z}{1-z}.$$

Therefore, we take

$$p'(z) = \frac{2}{(1-z)^2}$$

and

$$|p'(-1)| = \frac{1}{2}.$$

□

The inequality (2.1) can be strengthened as below by taking into account c_2 which is the second coefficient in the expansion of the function $f(z)$.

Theorem 2.2. *Let $f(z) \in \mathcal{M}$. Suppose that, for some $b \in \partial E$, f has an angular limit $f(b)$ at b, $f'(b) = 0$. Then we have the inequality*

$$\left| \left(\frac{zf'(z)}{f(z)} \right)'_{z=b} \right| \geq \frac{2}{2+|c_2|}. \tag{2.3}$$

The inequality (2.3) is sharp with extremal function

$$f(z) = \frac{z}{z^2+2az+1},$$

where $a = \frac{|c_2|}{2}$ is an arbitrary number from $[0,1]$ (see, (1.3)).

Proof. Let $\varphi(z)$ be the same as in the proof of Theorem 2.1. From (1.4) we obtain

$$\frac{2}{1+|\varphi'(0)|} \leq |\varphi'(b)| = \frac{2\,|p'(b)|}{|p(b)+1|^2} = 2\,|p'(b)|\,.$$

Since

$$\begin{aligned} \varphi(z) &= \frac{p(z)-1}{p(z)+1} = \frac{\frac{zf'(z)}{f(z)}-1}{\frac{zf'(z)}{f(z)}+1} \\ &= \frac{c_2z + \left(2c_3 - c_2^2\right) z^2 + \ldots}{2 + c_2z + (2c_3 - c_2^2)\, z^2 + \ldots}, \end{aligned}$$

and

$$|\varphi'(0)| = \frac{|c_2|}{2},$$

we take

$$\frac{2}{1+\frac{|c_2|}{2}} \leq 2\,|p'(b)|$$

and

$$\left| \left(\frac{zf'(z)}{f(z)} \right)'_{z=b} \right| \geq \frac{2}{2+|c_2|}.$$

Now, we shall show that the inequality (2.3) is sharp. Let

$$f(z) = \frac{z}{z^2+2az+1}. \tag{2.4}$$

Differentiating (2.4) logarithmically, we obtain

$$\ln f(z) = \ln z - \ln\left(z^2 + 2az + 1\right),$$

$$\frac{f'(z)}{f(z)} = \frac{1}{z} - \frac{2z + 2a}{z^2 + 2az + 1}$$

and

$$p(z) = \frac{zf'(z)}{f(z)} = 1 - \frac{2z^2 + 2za}{z^2 + 2az + 1} = \frac{1 - z^2}{z^2 + 2az + 1}.$$

Thus, since $a = \frac{|c_2|}{2}$ we get

$$p'(z) = \frac{-2z\left(z^2 + 2az + 1\right) - (2z + 2a)\left(1 - z^2\right)}{\left(z^2 + 2az + 1\right)^2}$$

and

$$|p'(1)| = \frac{2}{2 + |c_2|}.$$

□

The inequality (2.3) can be strengthened as below by taking into account c_3 which is the third coefficient in the expansion of the function $f(z)$.

Theorem 2.3. *Let $f(z) \in \mathcal{M}$. Suppose that, for some $b \in \partial E$, f has an angular limit $f(b)$ at b, $f'(b) = 0$. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right| \geq \frac{1}{2}\left(1 + \frac{2\left(2 - |c_2|\right)^2}{4 - |c_2|^2 + |4c_3 - 3c_2^2|}\right). \tag{2.5}$$

The equality in (2.5) occurs for the function

$$f(z) = \frac{z}{1 - z^2}.$$

Proof. Let $\varphi(z)$ be the same as in the proof of Theorem 2.1. Let us consider the function

$$g(z) = \frac{\varphi(z)}{B(z)},$$

where $B(z) = z$. The function $g(z)$ is holomorphic on U. According to the maximum principle, we have $|g(z)| < 1$ for each $z \in U$. In particular, we have

$$|g(0)| = \frac{|c_2|}{2} \leq 1 \tag{2.6}$$

and

$$|h'(0)| = \frac{|4c_3 - 3c_2^2|}{4}.$$

Furthermore, it can be seen that

$$\frac{b\varphi'(b)}{\varphi(b)} = |\varphi'(b)| \geq |B'(b)| = \frac{bB'(b)}{B(b)}.$$

Consider the function

$$t(z) = \frac{g(z) - g(0)}{1 - \overline{g(0)}g(z)}.$$

This function is holomorphic on U, $|t(z)| \leq 1$ for $|z| < 1$, $t(0) = 0$, and $|t(b)| = 1$ for $b \in \partial U$. From (1.4), we obtain

$$\begin{aligned}\frac{2}{1+|t'(0)|} &\leq |t'(b)| = \frac{1-|g(0)|^2}{\left|1-\overline{g(0)}g(b)\right|^2}|g'(b)| \\ &\leq \frac{1+|g(0)|}{1-|g(0)|}\left\{|\varphi'(b)| - |B'(b)|\right\}.\end{aligned}$$

Since

$$t'(z) = \frac{1-|g(0)|^2}{\left(1-\overline{g(0)}g(z)\right)^2}g'(z)$$

and

$$|t'(0)| = \frac{|g'(0)|}{1-|g(0)|^2} = \frac{\frac{\left|4c_3-3c_2^2\right|}{4}}{1-\left(\frac{|c_2|}{2}\right)^2} = \frac{\left|4c_3-3c_2^2\right|}{4-|c_2|^2},$$

we take

$$\begin{aligned}\frac{2}{1+\frac{\left|4c_3-3c_2^2\right|}{4-|c_2|^2}} &\leq \frac{1+\frac{|c_2|}{2}}{1-\frac{|c_2|}{2}}\left\{2|p'(b)|-1\right\}, \\ \frac{2\left(4-|c_2|^2\right)}{4-|c_2|^2+|4c_3-3c_2^2|} &\leq \frac{2+|c_2|}{2-|c_2|}\left\{2|p'(b)|-1\right\} \\ \frac{2(2-|c_2|)^2}{4-|c_2|^2+|4c_3-3c_2^2|} &\leq 2|p'(b)|-1 \\ 1+\frac{2(2-|c_2|)^2}{4-|c_2|^2+|4c_3-3c_2^2|} &\leq 2|p'(b)|\end{aligned}$$

and

$$|p'(b)| \geq \frac{1}{2}\left(1+\frac{2(2-|c_2|)^2}{4-|c_2|^2+|4c_3-3c_2^2|}\right).$$

Now, we shall show that the inequality (2.3) is sharp. Let

$$f(z) = \frac{z}{1-z^2}. \tag{2.7}$$

Differentiating (2.7) logarithmically, we obtain

$$\ln f(z) = \ln z - \ln\left(1 - z^2\right),$$

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \frac{2z}{1-z^2} = \frac{1+z^2}{z\left(1-z^2\right)}$$

and

$$p(z) = \frac{zf'(z)}{f(z)} = \frac{1+z^2}{1-z^2}.$$

Therefore, we take

$$p'(z) = \frac{2z\left(1-z^2\right) - 2z\left(1+z^2\right)}{\left(1-z^2\right)^2} = \frac{4z}{\left(1-z^2\right)^2},$$

$$p'(i) = \frac{4i}{\left(1-i^2\right)^2} = i$$

and

$$|p'(i)| = 1.$$

Since $c_2 = 0$ and $c_3 = 1$, we take

$$\frac{1}{2}\left(1 + \frac{2\left(2-|c_2|\right)^2}{4 - |c_2|^2 + |4c_3 - 3c_2^2|}\right) = 1.$$

□

If $f(z) - z$ has no zeros different from $z = 0$ in Theorem 2.3, the inequality (2.5) can be further strengthened. This is given by the following Theorem.

Theorem 2.4. *Let $f(z) \in \mathcal{M}$, $f(z) - z$ has no zeros on U except $z = 0$ and $c_2 > 0$. Suppose that, for some $b \in \partial U$, f has an angular limit $f(b)$ at b, $f'(b) = 0$. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right| \geq \frac{1}{2}\left(1 - \frac{4c_2 \ln^2\left(\frac{c_2}{2}\right)}{4c_2 \ln\left(\frac{c_2}{2}\right) - |4c_3 - 3c_2^2|}\right). \tag{2.8}$$

Proof. Let $c_2 > 0$ and let us consider the function $g(z)$ as in Theorem 2.3. Taking account of the equality (2.6), we denote by $\ln g(z)$ the holomorphic branch of the logarithm normed by condition

$$\ln g(0) = \ln\left(\frac{c_2}{2}\right) = \ln\left|\frac{c_2}{2}\right| + i \arg\left(\frac{c_2}{2}\right) < 0, \quad c_2 > 0$$

and

$$\ln\left(\frac{c_2}{2}\right) < 0.$$

Take the following auxiliary function

$$r(z) = \frac{\ln g(z) - \ln g(0)}{\ln g(z) + \ln g(0)}.$$

It is obvious that $r(z)$ is a holomorphic function on U, $r(0)=0$, $|r(z)|<1$ for $|z|<1$, and also $|r(b)|=1$ for $b\in\partial U$. So, we can apply (1.4) to the function $r(z)$. Since

$$r'(z)=2\ln g(0)\frac{g'(z)}{g(z)\left(\ln g(z)+\ln g(0)\right)^2},$$

and

$$r'(b)=2\ln g(0)\frac{g'(b)}{g(b)\left(\ln g(b)+\ln g(0)\right)^2},$$

we obtain

$$\begin{aligned}
\frac{2}{1+|r'(0)|} &\leq |r'(b)|=\frac{2\left|\ln g(0)\right|}{\left|\ln g(b)+\ln g(0)\right|^2}\left|\frac{g'(b)}{g(b)}\right|,\\
&= \frac{-2\ln g(0)}{\ln^2 g(0)+\arg^2 g(b)}\left|\frac{\varphi'(b)}{B(b)}-\frac{\varphi(b)B'(b)}{B(b)^2}\right|\\
&= \frac{-2\ln g(0)}{\ln^2 g(0)+\arg^2 g(b)}\left|\frac{\varphi(b)}{b^2}\right|\left|\frac{b\varphi'(b)}{\varphi(b)}-\frac{bB'(b)}{B(b)}\right|\\
&= \frac{-2\ln g(0)}{\ln^2 g(0)+\arg^2 g(b)}\left\{|\varphi'(b)|-|B'(b)|\right\}\\
&\leq \frac{-2\ln g(0)}{\ln^2 g(0)}\left\{2\left|p'(b)\right|-1\right\}\\
&= \frac{-2}{\ln\left(\frac{c_2}{2}\right)}\left\{2\left|p'(b)\right|-1\right\}.
\end{aligned}$$

Since

$$r'(0)=\frac{g'(0)}{2g(0)\ln g(0)}$$

and thus,

$$|r'(0)|=\frac{\frac{\left|4c_3-3c_2^2\right|}{4}}{-2\frac{c_2}{2}\ln\left(\frac{c_2}{2}\right)}=\frac{\left|4c_3-3c_2^2\right|}{-4c_2\ln\left(\frac{c_2}{2}\right)},$$

we have

$$\frac{2}{1-\frac{\left|4c_3-3c_2^2\right|}{4c_2\ln\left(\frac{c_2}{2}\right)}}\leq\frac{-2}{\ln\left(\frac{c_2}{2}\right)}\left\{2\left|p'(b)\right|-1\right\},$$

$$1-\frac{4c_2\ln^2\left(\frac{c_2}{2}\right)}{4c_2\ln\left(\frac{c_2}{2}\right)-\left|4c_3-3c_2^2\right|}\leq 2\left|p'(b)\right|,$$

and

$$|p'(b)|\geq\frac{1}{2}\left(1-\frac{4c_2\ln^2\left(\frac{c_2}{2}\right)}{4c_2\ln\left(\frac{c_2}{2}\right)-\left|4c_3-3c_2^2\right|}\right).$$

□

If $f(z)-z$ a have zeros different from z = 0, taking into account these zeros, the inequality (2.5) can be strengthened in another way. This is given by the following Theorem.

Theorem 2.5. *Let $f(z) \in \mathcal{M}$. Suppose that, for some $b \in \partial U$, f has an angular limit $f(b)$ at b, $f'(b) = 0$. Let $z_1, z_2, ..., z_n$ be zeros of the function $f(z) - z$ on U that are different from zero. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right| \geq \frac{1}{2}\left(1+\sum_{k=1}^{n}\frac{1-|z_k|^2}{|b-z_k|} + \frac{2\left(2\prod_{k=.}^{n}|z_k|-|c_2|\right)^2}{4\left(\prod_{k=1}^{n}|z_k|\right)^2-|c_2|^2+\prod_{k=1}^{n}|z_k|\left|4c_3-3c_2^2+2c_2\sum_{k=1}^{n}\frac{1-|z_k|^2}{z_k}\right|}\right). \tag{2.9}$$

The equality in (2.9) occurs for the function

$$f(z) = e^{\int_0^z \frac{1-t^2\prod_{k=1}^{n}\frac{t-z_k}{1-\overline{z_k}t}}{\left(1+t^2\prod_{k=1}^{n}\frac{t-z_k}{1-\overline{z_k}t}\right)t}dt},$$

where $z_1, z_2, ..., z_n$ are positive real numbers.

Proof. Let $\varphi(z)$ be as in the proof of Theorem 2.1 and $z_1, z_2, ..., z_n$ be zeros of the function $f(z) - z$ on U that are different from zero.

$$B_1(z) = z\prod_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}$$

is a holomorphic function on U and $|B_1(z)| < 1$ for $z \in U$. By the maximum principle for each $z \in U$, we have $|\varphi(z)| \leq |B_1(z)|$. Consider the function

$$m(z) = \frac{\varphi(z)}{B_1(z)}$$

is holomorphic on U and $|m(z)| \leq 1$ for $z \in U$. In particular, we have

$$|m(0)| = \frac{|c_2|}{2\prod_{k=1}^{n}|z_k|}$$

and

$$|m'(0)| = \frac{\left|4c_3-3c_2^2+2c_2\sum_{k=1}^{n}\frac{1-|z_k|^2}{z_k}\right|}{4\prod_{k=1}^{n}|z_k|}.$$

Moreover, it can be seen that

$$\frac{b\varphi'(b)}{\varphi(b)} = |\varphi'(b)| \geq |B_1'(b)| = \frac{bB_1'(b)}{B_1(b)}.$$

In addition, with the simple calculations, we take

$$\left|B_1'(b)\right| = 1 + \sum_{k=1}^{n} \frac{1-|z_k|^2}{|b-z_k|}.$$

The composite function

$$\Theta(z) = \frac{m(z)-m(0)}{1-\overline{m(0)}m(z)}$$

is holomorphic on the unit disc U, $|\Theta(z)| < 1$ for $z \in U$, $\Theta(0) = 0$ and $|\Theta(b)| = 1$ for $b \in \partial U$. From (1.4), we obtain

$$\begin{aligned}
\frac{2}{1+|\Theta'(0)|} &\leq |\Theta'(b)| = \frac{1-|m(0)|^2}{\left|1-\overline{m(0)}m(b)\right|^2}\,|m'(b)| \\
&\leq \frac{1+|m(0)|}{1-|m(0)|}\left(|\varphi'(b)| - |B_1'(b)|\right) \\
&= \frac{1+\frac{|c_2|}{2\prod_{k=1}^{n}|z_k|}}{1-\frac{|c_2|}{2\prod_{k=1}^{n}|z_k|}}\left\{2\,|p'(b)| - \left(1+\sum_{k=1}^{n}\frac{1-|z_k|^2}{|b-z_k|}\right)\right\} \\
&= \frac{2\prod_{k=1}^{n}|z_k| + |c_2|}{2\prod_{k=1}^{n}|z_k| - |c_2|}\left\{2\,|p'(b)| - \left(1+\sum_{k=1}^{n}\frac{1-|z_k|^2}{|b-z_k|}\right)\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
|\Theta'(0)| &= \frac{|m'(0)|}{1-|m(0)|^2} = \frac{\frac{\left|4c_3-3c_2^2+2c_2\sum_{k=1}^{n}\frac{1-|z_k|^2}{z_k}\right|}{4\prod_{k=1}^{n}|z_k|}}{1-\left(\frac{|c_2|}{4\prod_{k=1}^{n}|z_k|}\right)^2} \\
&= \prod_{k=1}^{n}|z_k|\,\frac{\left|4c_3-3c_2^2+2c_2\sum_{k=1}^{n}\frac{1-|z_k|^2}{z_k}\right|}{2\left(\prod_{k=1}^{n}|z_k|\right)^2-|c_2|^2},
\end{aligned}$$

we take

$$\frac{2}{1+\prod_{k=1}^{n}|z_k|\frac{\left|4c_3-3c_2^2+2c_2\sum_{k=1}^{n}\left(\frac{1-\left|z_k^2\right|}{z_k}\right)\right|}{4\left(\prod_{k=1}^{n}|z_k|\right)^2-|c_2|^2}} \leq$$

$$\frac{2\prod_{k=1}^{n}|z_k|+|c_2|}{2\prod_{k=1}^{n}|z_k|-|c_2|}\left\{2\,|p'(b)| - \left(1+\sum_{k=1}^{n}\frac{1-|z_k|^2}{|b-z_k|}\right)\right\},$$

$$\frac{2\left(4\left(\prod\limits_{k=1}^{n}|z_k|\right)^2-|c_2|^2\right)}{4\left(\prod\limits_{k=1}^{n}|z_k|\right)^2-|c_2|^2+\prod\limits_{k=1}^{n}|z_k|\left|4c_3-3c_2^2+2c_2\sum\limits_{k=1}^{n}\left(\frac{1-\left|z_k^2\right|}{z_k}\right)\right|}\leq$$

$$\frac{2\prod\limits_{k=1}^{n}|z_k|+|c_2|}{2\prod\limits_{k=1}^{n}|z_k|-|c_2|}\left\{2\left|p'(b)\right|-\left(1+\sum_{k=1}^{n}\frac{1-|z_k|^2}{|b-z_k|}\right)\right\},$$

$$\frac{2\left(2\prod\limits_{k=1}^{n}|z_k|-|c_2|\right)^2}{4\left(\prod\limits_{k=1}^{n}|z_k|\right)^2-|c_2|^2+\prod\limits_{k=1}^{n}|z_k|\left|4c_3-3c_2^2+2c_2\sum\limits_{k=1}^{n}\left(\frac{1-\left|z_k^2\right|}{z_k}\right)\right|}$$

$$\leq 2\left|p'(b)\right|-\left(1+\sum_{k=1}^{n}\frac{1-|z_k|^2}{|b-z_k|}\right)$$

and

$$\left|p'(b)\right|\geq\frac{1}{2}\left(1+\sum_{k=1}^{n}\frac{1-|z_k|^2}{|b-z_k|}\right)$$

$$+\frac{1}{2}\left(\frac{2\left(2\prod\limits_{k=1}^{n}|z_k|-|c_2|\right)^2}{4\left(\prod\limits_{k=1}^{n}|z_k|\right)^2-|c_2|^2+\prod\limits_{k=1}^{n}|z_k|\left|4c_3-3c_2^2+2c_2\sum\limits_{k=1}^{n}\left(\frac{1-\left|z_k^2\right|}{z_k}\right)\right|}\right).$$

Now, we shall show that the inequality (2.9) is sharp. Let

$$f(z)=e^{\int\limits_0^z\frac{1-t^2\prod\limits_{k=1}^{n}\frac{t-z_k}{1-\overline{z_k}t}}{\left(1+t^2\prod\limits_{k=1}^{n}\frac{t-z_k}{1-\overline{z_k}t}\right)t}dt}. \tag{2.10}$$

Differentiating (2.10) logarithmically, we obtain

$$\ln f(z)=\ln e^{\int\limits_0^z\frac{1-t^2\prod\limits_{k=1}^{n}\frac{t-z_k}{1-\overline{z_k}t}}{\left(1+t^2\prod\limits_{k=1}^{n}\frac{t-z_k}{1-\overline{z_k}t}\right)t}dt}=\int\limits_0^z\frac{1-t^2\prod\limits_{k=1}^{n}\frac{t-z_k}{1-\overline{z_k}t}}{\left(1+t^2\prod\limits_{k=1}^{n}\frac{t-z_k}{1-\overline{z_k}t}\right)t}dt,$$

$$\frac{f'(z)}{f(z)}=\frac{1-z^2\prod\limits_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}}{\left(1+z^2\prod\limits_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}\right)z}$$

and

$$p(z)=\frac{zf'(z)}{f(z)}=\frac{1-z^2\prod\limits_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}}{1+z^2\prod\limits_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}}.$$

Therefore, we take

$$\left|p'(1)\right|=\frac{1}{2}\left(2+\sum_{k=1}^{n}\frac{1+z_k}{1-z_k}\right).$$

Since $|p'(0)|=|c_2|=0$ and $|c_3|=\prod\limits_{k=1}^{n}|z_k|$, (2.9) is satisfied with equality. □

References

[1] Harold P. Boas. Julius and julia: mastering the art of the schwarz lemma. *Amer. Math. Monthly*, 117(9):770–785, 2010.

[2] Daniel M. Burns and Steven G. Krantz. Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary. *J. Amer. Math. Soc.*, 7(3):661–676, 1994.

[3] Dov Chelst. A generalized Schwarz lemma at the boundary. *Proc. Amer. Math. Soc.*, 129(11):3275–3278, 2001.

[4] V. N. Dubinin. On the Schwarz inequality on the boundary for functions regular in the disk. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 286(Anal. Teor. Chisel i Teor. Funkts. 18):74–84, 228–229, 2002.

[5] V. N. Dubinin. Bounded holomorphic functions covering no concentric circles. *J. Math. Sci. (N.Y.)*, 207(6):825–831, 2015.

[6] Mark Elin, Fiana Jacobzon, Marina Levenshtein, and David Shoikhet. The Schwarz lemma: rigidity and dynamics. In *Harmonic and complex analysis and its applications*, Trends Math., pages 135–230. Birkhäuser/Springer, Cham, 2014.

[7] G. M. Goluzin. *Geometricheskaya teoriya funktsiĭ kompleksnogo peremennogo*. Second edition. Edited by V. I. Smirnov. With a supplement by N. A. Lebedev, G. V. Kuzmina and Ju. E. Alenicyn. Izdat. "Nauka", Moscow, 1966.

[8] Xiaojun Huang and Ling Chen. Generalized Schwarz lemmas for meromorphic functions. *Bull. Korean Math. Soc.*, 49(2):417–422, 2012.

[9] I. S. Jack. Functions starlike and convex of order α. *J. London Math. Soc. (2)*, 3:469–474, 1971.

[10] Moonja Jeong. The Schwarz lemma and boundary fixed points. *J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math.*, 18(3):275–284, 2011.

[11] Moonja Jeong. The Schwarz lemma and its application at a boundary point. *J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math.*, 21(3):219–227, 2014.

[12] M. Mateljević. Note on rigidity of holomorphic mappings & schwarz and jack lemma. *ResearchGate*, 2015 (in preparation).

[13] M. Mateljević. Hyperbolic geometry and schwarz lemma. *ResearchGate*, 2016.

[14] M. Mateljević. Schwarz lemma, the carathéodory and kobayashi metrics and applications in complex analysis. (2016), 1-12. In *XIX GEOMETRICAL SEMINAR, At Zlatibor.*, page 1, 2016.

[15] M. Mateljević. Ahlfors-schwarz lemma, hyperbolic geometry, the carathéodory and kobayashi metric. In *Zbornik radova sedmog Simpozijuma "Matematika i primene" (4-5. novembar 2016.), Univerzitet u Beogradu - Matematički fakultet*, page 1, 2017.

[16] M. Mateljević. Schwarz lemma and Kobayashi metrics for harmonic and holomorphic functions. *J. Math. Anal. Appl.*, 464(1):78–100, 2018.

[17] M. Mateljević. Schwarz lemma and distortion for harmonic functions via length and area. *arXiv*, 8 May 2018.

[18] Bülent Nafi Örnek. Sharpened forms of the Schwarz lemma on the boundary. *Bull. Korean Math. Soc.*, 50(6):2053–2059, 2013.

[19] Bülent Nafi Örnek. Inequalities for the angular derivatives of certain classes of holomorphic functions in the unit disc. *Bull. Korean Math. Soc.*, 53(2):325–334, 2016.

[20] Robert Osserman. A sharp Schwarz inequality on the boundary. *Proc. Amer. Math. Soc.*, 128(12):3513–3517, 2000.

[21] Ch. Pommerenke. *Boundary behaviour of conformal maps*, volume 299 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[22] XiaoMin Tang, TaiShun Liu, and Jin Lu. Schwarz lemma at the boundary of the unit polydisk in $\mathbb{C}^n$. *Sci. China Math.*, 58(8):1639–1652, 2015.

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