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NEW SUBCLASS OF PSEUDO-TYPE MEROMORPHIC BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER

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Abstract. In the present article, we define a new subclass of pseudo-type meromorphic bi-univalent functions class Σ' of complex order $\gamma \in \mathbb{C}\backslash\{0\}$ and investigate the initial coefficient estimates $|b_0|, |b_1|$ and $|b_2|$. Furthermore we mention several new or known consequences of our result.

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1. Introduction and Definitions

Let $\mathcal{A}$ be the class of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are univalent in the open unit disc $\Delta = \{z : |z| < 1\}$. Also, let $\mathcal{S}$ be the class of all functions in $\mathcal{A}$ which are univalent and normalized by the conditions $f(0) = 0 = f'(0) - 1$ in Δ.

An analytic function φ is subordinate to an analytic function ψ, written by $\varphi(z) \prec \psi(z)$, provided there is an analytic function ω defined on Δ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $\varphi(z) = \psi(\omega(z))$. Ma and Minda [7] unified various subclasses of starlike and convex functions for which either of the quantity

$$\frac{z\, f'(z)}{f(z)} \quad \text{or} \quad 1 + \frac{z\, f''(z)}{f'(z)}$$

is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk $\Delta, \phi(0) = 1, \phi'(0) > 0$ and ϕ maps Δ onto a region starlike with respect to 1

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and symmetric with respect to the real axis. In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk Δ, satisfying $\phi(0) = 1, \phi'(0) > 0$ and $\phi(\Delta)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{1.2}$$

By setting $\phi(z)$ as given below:

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \frac{4\alpha^2 + 2\alpha}{3} z^3 + \cdots \quad (0 < \alpha \leq 1), \tag{1.3}$$

we have $B_1 = 2\alpha, \quad B_2 = 2\alpha^2$ and $B_3 = \frac{4\alpha^2+2\alpha}{3}$.

On the other hand, if we take

$$\phi(z) = \frac{1 + (1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \leq \beta < 1), \tag{1.4}$$

then $B_1 = B_2 = B_3 = 2(1-\beta)$.

Let Σ' denote the class of all meromorphic univalent functions g of the form

$$g(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \tag{1.5}$$

defined on the domain $\Delta^* = \{z : 1 < |z| < \infty\}$. Since $g \in \Sigma'$ is univalent, it has an inverse $g^{-1} = h$ that satisfy

$$g^{-1}(g(z)) = z, \ (z \in \Delta^*)$$

and

$$g(g^{-1}(w)) = w, (M < |w| < \infty, M > 0)$$

where

$$g^{-1}(w) = h(w) = w + \sum_{n=0}^{\infty} \frac{C_n}{w^n}, \quad (M < |w| < \infty). \tag{1.6}$$

Analogous to the bi-univalent analytic functions, a function $g \in \Sigma'$ is said to be meromorphic bi-univalent if $g^{-1} \in \Sigma'$. We denote the class of all meromorphic bi-univalent functions by $\mathcal{M}_{\Sigma'}$. Estimates on the coefficients of meromorphic univalent functions were widely investigated in the literature, for example, Schiffer[10] obtained the estimate $|b_2| \leq \frac{2}{3}$ for meromorphic univalent functions $g \in \Sigma'$ with $b_0 = 0$ and Duren [3] gave an elementary proof of the inequality $|b_n| \leq \frac{2}{(n+1)}$ on the coefficient of meromorphic univalent functions $g \in \Sigma'$ with $b_k = 0$ for $1 \leq k < \frac{n}{2}$. For the coefficient of the inverse of meromorphic univalent functions $h \in \mathcal{M}_{\Sigma'}$, Springer [12] proved that $|C_3| \leq 1$; $|C_3 + \frac{1}{2}C_1^2| \leq \frac{1}{2}$ and conjectured that $|C_{2n-1}| \leq \frac{(2n-1)!}{n!(n-1)!}$, $(n = 1, 2, ...)$.

In 1977, Kubota [6] has proved that the Springer's conjecture is true for $n = 3, 4, 5$ and subsequently Schober [11] obtained a sharp bound for the coefficients

$C_{2n-1}, 1 \leq n \leq 7$ of the inverse of meromorphic univalent functions in Δ^*. Recently, Kapoor and Mishra [5] (see [14]) found the coefficient estimates for a class consisting of inverses of meromorphic starlike univalent functions of order α in Δ^*.

Recently, Babalola [1] defined a new subclass $\lambda-$pseudo starlike function of order β $(0 \leq \beta < 1)$ satisfying the analytic condition

$$\Re\left(\frac{z(f'(z))^{\lambda}}{f(z)}\right) > \beta, \qquad (z \in \mathbb{U}, \lambda \geq 1 \in \mathbb{R}) \tag{1.7}$$

and denoted by $\mathcal{L}_{\lambda}(\beta)$. Babalola [1] remarked that though for $\lambda > 1$, these classes of $\lambda-$pseudo starlike functions clone the analytic representation of starlike functions. Also, when $\lambda = 1$, we have the class of starlike functions of order β ($1-$pseudo starlike functions of order β) and for $\lambda = 2$, we have the class of functions, which is a product combination of geometric expressions for bounded turning and starlike functions.

Motivated by the earlier work of [2, 4, 5, 8, 13, 15], in the present investigation,we define a new subclass of pseudo type meromorphic bi-univalent functions class Σ' of complex order $\gamma \in \mathbb{C}\backslash\{0\}$ and the estimates for the coefficients $|b_0|, |b_1|$ and $|b_2|$ are investigated. Several new consequences of the new results are also pointed out.

Definition 1.1. For $0 < \lambda \leq 1$ and $\mu \geq 1$, a function $g(z) \in \Sigma'$ given by (1.5) is said to be in the class $\mathcal{P}^{\gamma}_{\Sigma'}(\lambda, \mu, \phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu} + \lambda\left(\frac{z(g'(z))^{\mu}}{g(z)}\right) - 1\right] \prec \phi(z) \tag{1.8}$$

and

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu} + \lambda\left(\frac{w(h'(w))^{\mu}}{h(w)}\right) - 1\right] \prec \phi(w) \tag{1.9}$$

where $z, w \in \Delta^*$, $\gamma \in \mathbb{C}\backslash\{0\}$ and the function h is given by (1.6).

By suitably specializing the parameter λ, we state new subclass of meromorphic pseudo bi-univalent functions of complex order $\mathcal{P}^{\gamma}_{\Sigma'}(\lambda, \mu, \phi)$ as illustrated in the following Examples.

Example 1.2. For $\lambda = 1$, a function $g \in \Sigma'$ given by (1.5) is said to be in the class $\mathcal{P}^{\gamma}_{\Sigma'}(1, \mu, \phi) \equiv \mathcal{P}^{\gamma}_{\Sigma'}(\mu, \phi)$ if it satisfies the following conditions:

$$1 + \frac{1}{\gamma}\left(\frac{z(g'(z))^{\mu}}{g(z)} - 1\right) \prec \phi(z) \quad \text{and} \quad 1 + \frac{1}{\gamma}\left(\frac{w(h'(w))^{\mu}}{h(w)} - 1\right) \prec \phi(w)$$

where $z, w \in \Delta^*, \mu \geq 1$, $\gamma \in \mathbb{C}\backslash\{0\}$ and the function h is given by (1.6).

Remark 1.3. We note that $\mathcal{P}^{\gamma}_{\Sigma'}(1, 1, \phi) \equiv \mathcal{S}^{\gamma}_{\Sigma'}(\phi)$

Example 1.4. For $\lambda = 1$ and $\gamma = 1$, a function $g \in \Sigma'$ given by (1.5) is said to be in the class $\mathcal{P}^{1}_{\Sigma'}(1, \mu, \phi) \equiv \mathcal{P}_{\Sigma'}(\mu, \phi)$ if it satisfies the following conditions :

$$\frac{z(g'(z))^{\mu}}{g(z)} \prec \phi(z) \quad \text{and} \quad \frac{w(h'(w))^{\mu}}{h(w)} \prec \phi(w)$$

where $z, w \in \Delta^*, \mu \geq 1$ and the function h is given by (1.6).

Example 1.5. For $\lambda = 0$ a function $g \in \Sigma'$ given by (1.5) is said to be in the class $\mathcal{P}^{\gamma}_{\Sigma'}(1, \mu, \phi) \equiv \mathcal{R}^{\gamma}_{\Sigma'}(\mu, \phi)$ if it satisfies the following conditions :

$$1 + \frac{1}{\gamma}\left[\left(\frac{g(z)}{z}\right)^{\mu} - 1\right] \prec \phi(z) \quad \text{and} \quad 1 + \frac{1}{\gamma}\left[\left(\frac{h(w)}{w}\right)^{\mu} - 1\right] \prec \phi(w)$$

where $z, w \in \Delta^*, \mu \geq 1$ and the function h is given by (1.6).

2. Coefficient estimates

In this section, we obtain the coefficient estimates $|b_0|$, $|b_1|$ and $|b_2|$ for $\mathcal{P}^{\gamma}_{\Sigma'}(\lambda, \mu, \phi)$, a new subclass of meromorphic pseudo bi-univalent functions class Σ' of complex order $\gamma \in \mathbb{C}\backslash\{0\}$. In order to prove our result, we recall the following lemma.

Lemma 2.1. *[9] If $\Phi \in \mathcal{P}$, the class of all functions with $\Re(\Phi(z)) > 0, (z \in \Delta)$ then*

$$|c_k| \leq 2, \text{ for each } k,$$

where

$$\Phi(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad \textit{for } z \in \Delta.$$

Define the functions p and q in $\mathcal{P}$ given by

$$p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \cdots$$

and

$$q(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + \frac{q_1}{z} + \frac{q_2}{z^2} + \cdots.$$

It follows that

$$u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2}\left[\frac{p_1}{z} + \left(p_2 - \frac{p_1^2}{2}\right)\frac{1}{z^2} + \cdots\right]$$

and

$$v(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2}\left[\frac{q_1}{z} + \left(q_2 - \frac{q_1^2}{2}\right)\frac{1}{z^2} + \cdots\right].$$

Note that for the functions $p(z), q(z) \in \mathcal{P}$, we have

$$|p_i| \leq 2 \text{ and } |q_i| \leq 2 \text{ for each } i.$$

Theorem 2.2. *Let g be given by (1.5) in the class* $\mathcal{P}^{\gamma}_{\Sigma'}(\lambda, \mu, \phi)$. *Then*

$$|b_0| \leq \frac{|\gamma||B_1|}{|\mu - \mu\lambda - \lambda|}, \tag{2.1}$$

$$|b_1| \leq \frac{|\gamma|}{2|\mu - \lambda - 2\mu\lambda|}\Bigg(4|(B_1 - B_2)^2| + 4|B_1^2| + 8|B_1(B_1 - B_2)| + \frac{|\mu(\mu-1)(1-\lambda) + 2\lambda|^2|\gamma^2 B_1^4|}{|\mu - \mu\lambda - \lambda|^4}\Bigg)^{\frac{1}{2}} \tag{2.2}$$

and

$$|b_2| \leq \frac{|\gamma|}{2|\mu - \lambda - 3\mu\lambda|}\Bigg(2|B_1| + 4|B_2 - B_1| + 2|B_1 - 2B_2 + B_3| + \frac{|\mu(\mu-1)(\mu-2)(1-\lambda) - 6\lambda||\gamma|^2|B_1|^3}{3|\lambda|^3}\Bigg) \tag{2.3}$$

where $\gamma \in \mathbb{C}\backslash\{0\}, 0 < \lambda \leq 1, \mu \geq 1$ *and* $z, w \in \Delta^*$.

Proof. It follows from (1.8) and (1.9) that

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu} + \lambda\left(\frac{z(g'(z))^{\mu}}{g(z)}\right) - 1\right] = \phi(u(z)) \tag{2.4}$$

and

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu} + \lambda\left(\frac{w(h'(w))^{\mu}}{h(w)}\right) - 1\right] = \phi(v(w)). \tag{2.5}$$

In light of (1.5), (1.6), (1.8) and (1.9), we have

$$\begin{aligned} &1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu} + \lambda\left(\frac{z(g'(z))^{\mu}}{g(z)}\right) - 1\right] \\ &= 1 + B_1 p_1 \frac{1}{2z} + \left[\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2\right]\frac{1}{z^2} \\ &\quad + \left[\frac{B_1}{2}\left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) + \frac{B_2}{2}\left(p_1 p_2 - \frac{p_1^3}{2}\right) + B_3\frac{p_1^3}{8}\right]\frac{1}{z^3}\cdots \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu} + \lambda\left(\frac{w(h'(w))^{\mu}}{h(w)}\right) - 1\right] \\ &= 1 + B_1 q_1 \frac{1}{2w} + \left[\frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2\right]\frac{1}{w^2} \\ &\quad + \left[\frac{B_1}{2}\left(q_3 - q_1 q_2 + \frac{q_1^3}{4}\right) + \frac{B_2}{2}\left(q_1 q_2 - \frac{q_1^3}{2}\right) + B_3\frac{q_1^3}{8}\right]\frac{1}{w^3}\cdots \end{aligned} \tag{2.7}$$

Now, equating the coefficients in (2.6) and (2.7), we get

$$\frac{(\mu-\mu\lambda-\lambda)}{\gamma}b_0=\frac{1}{2}B_1p_1, \tag{2.8}$$

$$\frac{1}{2\gamma}\Big[\big(\mu(\mu-1)(1-\lambda)+2\lambda\big)b_0^2+2(\mu-\lambda-2\lambda\mu)b_1\Big]=\frac{1}{2}B_1\left(p_2-\frac{p_1^2}{2}\right)+\frac{1}{4}B_2p_1^2, \tag{2.9}$$

(2.10)
$$\frac{1}{6\gamma}\Big[\big(\mu(\mu-1)(\mu-2)(1-\lambda)-6\lambda\big)b_0^3+6\big(\mu(\mu-1)(1-\lambda)+2\lambda+\lambda\mu\big)b_0b_1$$
$$+6(\mu-\lambda-3\lambda\mu)b_2\Big]=\left[\frac{B_1}{2}\left(p_3-p_1p_2+\frac{p_1^3}{4}\right)+\frac{B_2}{2}\left(p_1p_2-\frac{p_1^3}{2}\right)+B_3\frac{p_1^3}{8}\right],$$

$$\frac{-(\mu-\mu\lambda-\lambda)}{\gamma}b_0=\frac{1}{2}B_1q_1, \tag{2.11}$$

$$\frac{1}{2\gamma}\Big[\big(\mu(\mu-1)(1-\lambda)+2\lambda\big)b_0^2+2(\lambda-\mu+2\lambda\mu)b_1\Big]=\frac{1}{2}B_1\left(q_2-\frac{q_1^2}{2}\right)+\frac{1}{4}B_2q_1^2 \tag{2.12}$$

and

$$\frac{1}{6\gamma}\big[\big(6\lambda-\mu(\mu-1)(\mu-2)(1-\lambda)\big)b_0^3 \tag{2.13}$$
$$+6\big(\mu(\mu-1)(1-\lambda)-\mu(1-\lambda)+3\lambda+3\lambda\mu\big)b_0b_1+6(\lambda-\mu+3\lambda\mu)b_2\big]$$
$$=\left[\frac{B_1}{2}\left(q_3-q_1q_2+\frac{q_1^3}{4}\right)+\frac{B_2}{2}\left(q_1q_2-\frac{q_1^3}{2}\right)+B_3\frac{q_1^3}{8}\right]$$

From (2.8) and (2.11), we get

$$p_1=-q_1 \tag{2.14}$$

and

$$b_0^2=\frac{\gamma^2B_1^2}{8(\mu-\mu\lambda-\lambda)^2}(p_1^2+q_1^2). \tag{2.15}$$

Applying Lemma (2.1) for the coefficients p_1 and q_1, we have

$$|b_0|\le\frac{|\gamma||B_1|}{|\mu-\mu\lambda-\lambda|}.$$

Next, in order to find the bound on $|b_1|$ from (2.9), (2.12), (2.14) and (2.15), we obtain

$$2(\mu-\lambda-2\lambda\mu)^2\,\frac{b_1^2}{\gamma^2}+[\mu(\mu-1)(1-\lambda)+2\lambda]^2\frac{b_0^4}{2\gamma^2} \tag{2.16}$$
$$=(B_1-B_2)^2\,\frac{p_1^4}{8}+\frac{B_1^2}{4}(p_2^2+q_2^2)+B_1(B_2-B_1)\frac{(p_1^2p_2+q_1^2q_2)}{4}.$$

Using (2.15) and applying Lemma (2.1) once again for the coefficients p_1, p_2 and q_2, we get

$$|b_1|^2 \leq \frac{|\gamma^2|}{4|\mu-\lambda-2\lambda\mu|^2}\times \left(4|(B_1-B_2)^2|+4|B_1|^2+8|B_1(B_1-B_2)|+\frac{|\mu(\mu-1)(1-\lambda)+2\lambda|^2|\gamma^2 B_1^4|}{|\mu-\mu\lambda-\lambda|^4}\right).$$

That is,

$$|b_1| \leq \frac{|\gamma|}{2|\mu-\lambda-2\lambda\mu|}\times \sqrt{4|(B_1-B_2)^2|+4|B_1|^2+8|B_1(B_1-B_2)|+\frac{|\mu(\mu-1)(1-\lambda)+2\lambda|^2|\gamma^2 B_1^4|}{|\mu-\mu\lambda-\lambda|^4}}.$$

In order to find the estimate $|b_2|$, consider the sum of (2.10) and (2.13) with $p_1=-q_1$, we have

$$\frac{1}{\gamma}b_0b_1 = \frac{B_1[p_3+q_3]+(B_2-B_1)p_1[p_2-q_2]}{2[2\mu(\mu-1)(1-\lambda)-(1-\lambda)\mu+5\lambda+4\lambda\mu]}. \tag{2.17}$$

Subtracting (2.13) from (2.10) and using $p_1=-q_1$ we have

$$\begin{aligned}(2.18)\quad & 2(\mu-\lambda-3\lambda\mu)\frac{b_2}{\gamma} \\ &= -(\mu-\lambda-3\mu\lambda)\frac{b_0b_1}{\gamma}-[\mu(\mu-1)(\mu-2)(1-\lambda)-6\lambda]\frac{b_0^3}{3\gamma}+\frac{B_1}{2}(p_3-q_3) \\ &\quad +\frac{B_2-B_1}{2}(p_2+q_2)p_1+\frac{B_1-2B_2+B_3}{4}p_1^3.\end{aligned}$$

Substituting for $\frac{b_0b_1}{\gamma}$ and $\frac{b_0^3}{\gamma}$ in (2.18), simple computation yields,

$$\begin{aligned}\frac{b_2}{\gamma} = & \frac{-B_1}{2(\mu-\lambda-3\lambda\mu)}\left(\frac{\mu-3\lambda-4\lambda\mu-\mu(\mu-1)(1-\lambda)}{2\mu(\mu-1)(1-\lambda)-\mu+5\lambda+5\lambda\mu}p_3\right. \\ & \left.+\frac{2\lambda+\lambda\mu+\mu(\mu-1)(1-\lambda)}{2\mu(\mu-1)(1-\lambda)-\mu+5\lambda+5\lambda\mu}q_3\right) \\ & -\frac{(B_2-B_1)p_1}{2(\mu-\lambda-3\lambda\mu)}\left(\frac{\mu-3\lambda-4\lambda\mu-\mu(\mu-1)(1-\lambda)}{2\mu(\mu-1)(1-\lambda)-\mu+5\lambda+5\lambda\mu}p_2\right. \\ & \left.-\frac{2\lambda+\lambda\mu+\mu(\mu-1)(1-\lambda)}{2\mu(\mu-1)(1-\lambda)-\mu+5\lambda+5\lambda\mu}q_2\right) \\ & +\frac{B_1-2B_2+B_3}{8(\mu-\lambda-3\lambda\mu)}p_1^3-\frac{(\mu(\mu-1)(\mu-2)(1-\lambda)-6\lambda)\gamma^2\ B_1^3}{48(\mu-\lambda-3\lambda\mu)\lambda^3}p_1^3.\end{aligned}$$

Applying Lemma 2.1 in the above equation yields,

$$|b_2| \leq \frac{|\gamma|}{2|\mu-\lambda-3\lambda\mu|}\times \Big(2|B_1|+4|B_2-B_1|+2|B_1-2B_2+B_3| + \frac{|\mu(\mu-1)(\mu-2)(1-\lambda)-6\lambda||\gamma|^2|B_1|^3}{3|\lambda|^3}\Big) \tag{2.19}$$

□

By taking $\lambda = 1$, we state the following.

Theorem 2.3. *Let g be given by (1.5) in the class $\mathcal{P}^{\gamma}_{\Sigma'}(\mu,\phi)$. Then*

$$|b_0| \leq |\gamma|\ |B_1|, \tag{2.20}$$

$$|b_1| \leq \frac{|\gamma|}{|1+\mu|}\sqrt{|(B_1-B_2)^2|+|B_1^2|+2|B_1(B_1-B_2)|+|\gamma|^2\ |B_1^4|} \tag{2.21}$$

and

$$|b_2| \leq \frac{|\gamma|}{|1+2\mu|}\left(|B_1|+2|B_2-B_1|+|B_1-2B_2+B_3|+|\gamma|^2\ |B_1|^3\right) \tag{2.22}$$

where $\gamma\in\mathbb{C}\backslash\{0\}, \mu\geq 1$ and $z,w\in\Delta^*$.

By taking $\lambda = 1$ and $\gamma = 1$, we state the following results.

Theorem 2.4. *Let g be given by (1.5) in the class $\mathcal{P}_{\Sigma'}(\mu,\phi)$. Then*

$$|b_0| \leq |B_1|,$$

$$|b_1| \leq \frac{1}{|1+\mu|}\ \sqrt{|(B_1-B_2)^2|+|B_1^2|+2|B_1(B_1-B_2)|+|B_1^4|}$$

and

$$|b_2| \leq \frac{1}{|1+2\mu|}\left(|B_1|+2|B_2-B_1|+|B_1-2B_2+B_3|+\ |B_1|^3\right)$$

where $\mu\geq 1,\ z,w\in\Delta^*$.

3. Corollaries and concluding Remarks

For functions g be given by (1.5) and $g\in\mathcal{P}^{\gamma}_{\Sigma'}\left(\lambda,\mu,\left(\frac{1+z}{1-z}\right)^{\alpha}\right)\equiv\mathcal{P}^{\gamma}_{\Sigma'}(\lambda,\mu,\alpha)$ by setting $B_1 = 2\alpha$, $B_2 = 2\alpha^2$ and $B_3 = \frac{4\alpha^2+2\alpha}{3}$ and similarly, for $g\in\mathcal{P}^{\gamma}_{\Sigma'}\left(\lambda,\mu,\frac{1+(1-2\beta)z}{1-z}\right)\equiv\mathcal{P}^{\gamma}_{\Sigma'}(\lambda,\mu,\beta)$ by setting $B_1 = B_2 = B_3 = 2(1-\beta)$ one can easily derive the results corresponding to Theorems 2.2, 2.3 and 2.4.

Corollary 3.1. *Let g be given by (1.5) in the class $\mathcal{P}^{\gamma}_{\Sigma'}(\lambda, \mu, \alpha)$. Then*

$$|b_0| \leq \frac{2|\gamma|\alpha}{|\mu - \mu\lambda - \lambda|}, \tag{3.1}$$

$$|b_1| \leq \frac{2|\gamma|\alpha}{|\mu - \lambda - 2\lambda\mu|}\sqrt{(\alpha-2)^2 + \frac{|\mu(\mu-1)(1-\lambda) + 2\lambda|^2|\gamma^2|}{|\mu - \mu\lambda - \lambda|^4}\alpha^2} \tag{3.2}$$

and

$$|b_2| \leq \frac{2|\gamma|\alpha}{|\mu - \lambda - 3\lambda\mu|}\left(3 - 2\alpha + \left(\frac{4 - 6\alpha + 2\alpha^2}{3}\right) + \frac{2|\gamma|^2\alpha^2|\mu(\mu-1)(\mu-2)(1-\lambda) - 6\lambda|}{3|\lambda|^3}\right) \tag{3.3}$$

where $\gamma \in \mathbb{C}\backslash\{0\}, 0 < \lambda \leq 1, \mu \geq 1$ and $z, w \in \Delta^*$.

Corollary 3.2. *Let g be given by (1.5) in the class $\mathcal{P}^{\gamma}_{\Sigma'}(\lambda, \mu, \beta)$. Then*

$$|b_0| \leq \frac{2|\gamma|(1-\beta)}{|\mu - \mu\lambda - \lambda|}, \tag{3.4}$$

$$|b_1| \leq \frac{2|\gamma|(1-\beta)}{|\mu - \lambda - 2\lambda\mu|}\sqrt{1 + \frac{|\mu(\mu-1)(1-\lambda) + 2\lambda|^2|\gamma^2|}{|\mu - \mu\lambda - \lambda|^4}(1-\beta)^2} \tag{3.5}$$

and

$$|b_2| \leq \frac{2|\gamma|(1-\beta)}{|\mu - \lambda - 3\lambda\mu|}\left(1 + \frac{2|\gamma|^2(1-\beta)^2|\mu(\mu-1)(\mu-2)(1-\lambda) - 6\lambda|}{3|\lambda|^3}\right) \tag{3.6}$$

where $\gamma \in \mathbb{C}\backslash\{0\}, 0 < \lambda \leq 1, \mu \geq 1$ and $z, w \in \Delta^*$.

Concluding Remarks: We remark that, when $\lambda = 1$ and $\mu = 1$, we can easily obtain the coefficient estimates b_0, b_1 and b_2 for $\mathcal{S}^{\gamma}_{\Sigma'}(\phi)$, which leads to the results discussed in Theorem 2.3 of [8]. Also, we can obtain the initial coefficient estimates for function g given by (1.5) in the subclass $\mathcal{S}^{\gamma}_{\Sigma'}(\phi)$ by taking $\phi(z)$ given in (1.3) and (1.4) respectively.

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