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ON OSCULATING, NORMAL AND RECTIFYING BI-NULL CURVES IN $\mathbb{R}_2^5$

Kazım İlarslan[1], **Makoto Sakaki**[2] **and Ali Uçum**[3][4]

Abstract. In this paper we give the necessary and sufficient conditions for bi-null curves in $\mathbb{R}_2^5$ to be osculating, normal or rectifying curves in terms of their curvature functions.

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1. Introduction

In the Euclidean 3-space $\mathbb{R}^3$ there exist three classes of characterisirc curves, so called rectifying, normal and osculating curves. Rectifying curves are introduced by B. Y. Chen in [2] as a space curve whose position vector always lies in its rectifying plane. Here, the rectifying plane is spanned by the tangent vector $T(s)$ and the binormal vector $B(s)$. Thus, the position vector α of a rectifying curve in $\mathbb{R}^3$ satisfies the equation $\alpha(s) = \lambda(s) T(s) + \mu(s) B(s)$ for some differentiable functions $\lambda(s)$ and $\mu(s)$. In Minkowski 3-space $\mathbb{R}_1^3$, rectifying curves have similar geometric properties as in $\mathbb{R}^3$. Some characterizations of spacelike, timelike and null rectifying curves, lying fully in $\mathbb{R}_1^3$, are given in [7]. Moreover, the definition of a rectifying curve is generalized to 4-dimensional Euclidean and Minkowski spaces in [10, 9, 6].

In analogy with the Euclidean case, a normal curve in $\mathbb{R}_1^3$ is defined in [8] as a curve whose position vector always lies in its normal plane. Some characterizations of spacelike, timelike and null normal curves, lying fully in $\mathbb{R}_1^3$, are given in [8, 4, 3]. In addition, normal curves in Minkowski space-time $\mathbb{R}_1^4$ are defined in [5] as a curve whose position vector always lies in its normal space $T^{\perp}$, which represents the orthogonal complement of the tangent vector field of the curve.

Similarly, an osculating curve in $\mathbb{R}_1^4$ is defined in [11] as a curve whose position vector always lies in its osculating space, which represents the orthogonal complement of the first binormal or second binormal vectors field of the curve. Timelike, spacelike and null osculating curves are studied in [12] and [11].

[1]Kırıkkale University, Faculty of Sciences and Arts, Department of Mathematics, Kırıkkale-Turkey, e-mail: kilarslan@yahoo.com
[2]Graduate School of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan, e-mail: sakaki@hirosaki-u.ac.jp
[3]Kırıkkale University, Faculty of Sciences and Arts, Department of Mathematics, Kırıkkale-Turkey, e-mail: aliucum05@gmail.com
[4]Corresponding author

On the other hand, the second named author (in [13]) gave the notion of bi-null curves in semi-Euclidean spaces $\mathbb{R}_2^n$ of index 2, together with the unique Frenet frame and the curvatures. He also discussed some properties of bi-null curves in terms of the curvatures. Recently in [14], we study bi-null curves with constant curvature functions in $\mathbb{R}_2^5$.

In this paper, we characterize osculating bi-null curves in $\mathbb{R}_2^5$. We consider three types of osculating bi-null curves in $\mathbb{R}_2^5$ which we call as "the first kind of osculating curves", "the second kind of osculating curves" and "the third kind of osculating curves". Then we give the necessary and sufficient conditions for bi-null curves in $\mathbb{R}_2^5$ to be osculating curves in terms of their curvature functions for all types of osculating curves. In addition, we study normal bi-null curves and rectifying bi-null curves in $\mathbb{R}_2^5$.

2. Preliminaries

In this section, following [13] and [14], we recall the Frenet equations for bi-null curves in $\mathbb{R}_2^5$. Let $\mathbb{R}_2^5$ be the 5-dimensional semi-Euclidean space of index 2 with standard coordinate system $\{x_1, x_2, x_3, x_4, x_5\}$ and metric

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 - dx_5^2.$$

We denote by $\langle\ ,\ \rangle$ the inner product on $\mathbb{R}_2^5$. Recall that a vector $v \in \mathbb{R}_2^5 \backslash \{0\}$ can be spacelike if $\langle v, v\rangle > 0$, timelike if $\langle v, v\rangle < 0$ and null (lightlike) if $\langle v, v\rangle = 0$. In particular, the vector $v = 0$ is said to be spacelike. The norm of a vector v is given by $||v|| = \sqrt{|\langle v, v\rangle|}$. Two vectors v and w are said to be orthogonal, if $\langle v, w\rangle = 0$.

We say that a curve $\gamma(t)$ in $\mathbb{R}_2^5$ is a bi-null curve if span$\{\gamma'(t), \gamma''(t)\}$ is an isotropic 2-plane for all t ([1]). That is,

$$\langle \gamma'(t), \gamma'(t)\rangle = \langle \gamma'(t), \gamma''(t)\rangle = \langle \gamma''(t), \gamma''(t)\rangle = 0,$$

and $\{\gamma'(t), \gamma''(t)\}$ are linearly independent.

We say that a bi-null curve $\gamma(t)$ in $\mathbb{R}_2^5$ is parametrized by the bi-null arc if $\left\langle \gamma^{(3)}(t), \gamma^{(3)}(t)\right\rangle = 1$. As noted in [13], a bi-null curve $\gamma(t)$ in $\mathbb{R}_2^5$ satisfies $\left\langle \gamma^{(3)}(t), \gamma^{(3)}(t)\right\rangle \geq 0$. If a bi-null curve $\gamma(t)$ in $\mathbb{R}_2^5$ satisfies $\left\langle \gamma^{(3)}(t), \gamma^{(3)}(t)\right\rangle > 0$, then we can see that

$$u(t) = \int_{t_0}^{t} \left\langle \gamma^{(3)}(t), \gamma^{(3)}(t)\right\rangle^{1/6} dt$$

becomes the bi-null arc parameter.

For a bi-null curve $\gamma(t)$ with bi-null arc parameter t in $\mathbb{R}_2^5$, there exists a unique Frenet frame $\{L_1, L_2, N_1, N_2, W\}$ such that

$$\begin{aligned} \gamma' &= L_1, \quad L_1' = L_2, \quad L_2' = W, \\ N_1' &= k_1 L_2, \\ N_2' &= -k_1 L_1 - N_1 + k_0 W, \\ W' &= -k_0 L_2 - N_2, \end{aligned} \tag{2.1}$$

where N_1, N_2 are null, $\langle L_1, N_1\rangle = \langle L_2, N_2\rangle = 1$, span$\{L_1, N_1\}$, span$\{L_2, N_2\}$ and span$\{W\}$ are mutually orthogonal, and W is a spacelike unit vector.

The frame $\{L_1, L_2, N_1, N_2, W\}$ is a pseudo-orthonormal frame. We say that the functions k_0 and k_1 are the curvatures of γ.

The following theorem can be shown similarly to the classical fundamental existence theorem for curves in $\mathbb{R}^3$.

Theorem 2.1. *Let $k_0(t)$ and $k_1(t)$ be differentiable functions on $(t_0-\varepsilon, t_0+\varepsilon)$. Let p_0 be a point in $\mathbb{R}^5_2$, and $\left\{L_1^0, L_2^0, N_1^0, N_2^0, W^0\right\}$ be a pseudo-orthonormal basis of $\mathbb{R}^5_2$. Then there exists a unique bi-null curve $\gamma(t)$ in $\mathbb{R}^5_2$ with $\gamma(t_0) = p_0$, bi-null arc parameter t and curvatures k_0, k_1, whose Frenet frame $\{L_1, L_2, N_1, N_2, W\}$ satisfies*

$$L_1(t_0) = L_1^0, \quad L_2(t_0) = L_2^0, \quad N_1(t_0) = N_1^0, \quad N_2(t_0) = N_2^0, \quad W(t_0) = W^0.$$

3. Osculating bi-null curves

In this section, we consider the conditions for a bi-null curve in $\mathbb{R}^5_2$ to be a osculating curve.

Let $\gamma(t)$ be a bi-null curve in $\mathbb{R}^5_2$ with bi-null arc t. Then the bi-null curve γ is called the first, second or third kind of osculating bi-null curve if its position vector (with respect to some chosen origin) always lies in the orthogonal space $N_1^\perp$ =span$\{L_2, N_1, N_2, W\}$, $N_2^\perp$ =span$\{L_1, N_1, N_2, W\}$ or $W^\perp$ =span$\{L_1, L_2, N_1, N_2\}$, respectively. As a result, the position vector of the first, second or third kind of osculating curve satisfies, respectively, the following equations

$$\gamma(t) = \mu_1(t) L_2(t) + \mu_2(t) N_1(t) + \mu_3(t) N_2(t) + \mu_4(t) W(t), \tag{3.1}$$

$$\gamma(t) = \mu_1(t) L_1(t) + \mu_2(t) N_1(t) + \mu_3(t) N_2(t) + \mu_4(t) W(t), \tag{3.2}$$

$$\gamma(t) = \mu_1(t) L_1(t) + \mu_2(t) L_2(t) + \mu_3(t) N_1(t) + \mu_4(t) N_2(t), \tag{3.3}$$

for some differentiable functions $\mu_i(t)$.

Let us consider the first kind of osculating bi-null curve in $\mathbb{R}^5_2$. Then, the position vector $\gamma(t)$ satisfies (3.1). Differentiating (3.1) with respect to t, we get

$$\begin{aligned} L_1 &= -\mu_3 k_1 L_1 + (\mu_1' + \mu_2 k_1 - \mu_4 k_0) L_2 + (\mu_2' - \mu_3) N_1 + (\mu_3' - \mu_4) N_2 \\ &\quad + (\mu_4' + \mu_1 + k_0 \mu_3) W, \end{aligned}$$

which implies that

$$\begin{aligned} \mu_1' + \mu_2 k_1 - \mu_4 k_0 &= 0, \quad \mu_4' + \mu_1 + k_0 \mu_3 = 0, \\ \mu_2' - \mu_3 &= 0, \quad \mu_3' - \mu_4 = 0, \quad -\mu_3 k_1 = 1 \end{aligned} \tag{3.4}$$

where $k_1 \neq 0$. Solving the system of these equations, we have

$$\begin{aligned} \mu_1 &= \frac{k_0}{k_1} + \left(\frac{1}{k_1}\right)'', \quad \mu_3 = -\frac{1}{k_1}, \quad \mu_4 = -\left(\frac{1}{k_1}\right)', \\ \mu_2 &= -\frac{1}{k_1}\left(k_0\left(\frac{1}{k_1}\right)' + \left(\frac{k_0}{k_1}\right)' + \left(\frac{1}{k_1}\right)^{(3)}\right) \end{aligned}$$

and

$$k_1\left[\frac{1}{k_1}\left(k_0\left(\frac{1}{k_1}\right)' + \left(\frac{k_0}{k_1}\right)' + \left(\frac{1}{k_1}\right)^{(3)}\right)\right]' = 1.$$

Using above equations, we get the following theorem:

Theorem 3.1. *Let γ be a bi-null curve in $\mathbb{R}_2^5$ with bi-null arc parameter t and curvatures k_0, k_1. Then γ is congruent to a first kind of osculating curve if and only if*

$$k_1\left[\frac{1}{k_1}\left(k_0\left(\frac{1}{k_1}\right)' + \left(\frac{k_0}{k_1}\right)' + \left(\frac{1}{k_1}\right)^{(3)}\right)\right]' = 1. \tag{3.5}$$

Proof. Let γ be a first kind of osculating bi-null curve in $\mathbb{R}_2^5$ with bi-null arc parameter t. Then it is clear from above calculations that (3.5) holds.

Conversely, assume that (3.5) holds. Let us consider a vector field $X(t) \in \mathbb{R}_2^5$ as follows

$$\begin{aligned} X &= \gamma(t) - \left(\frac{k_0}{k_1} + \left(\frac{1}{k_1}\right)''\right)L_2 + \frac{1}{k_1}\left(k_0\left(\frac{1}{k_1}\right)' + \left(\frac{k_0}{k_1}\right)' + \left(\frac{1}{k_1}\right)^{(3)}\right)N_1 \\ &\quad + \frac{1}{k_1}N_2 + \left(\frac{1}{k_1}\right)' W. \end{aligned}$$

Then we obtain $X' = 0$ which implies that γ is congruent to a first kind of osculating curve. □

From Theorems 2.1 and 3.1, we get the following theorem.

Theorem 3.2. *Let $k_0(t)$ and $k_1(t)$ be differentiable functions which satisfy*

$$k_1\left[\frac{1}{k_1}\left(k_0\left(\frac{1}{k_1}\right)' + \left(\frac{k_0}{k_1}\right)' + \left(\frac{1}{k_1}\right)^{(3)}\right)\right]' = 1. \tag{3.6}$$

Then there exists a first kind of osculating bi-null curve $\gamma(t)$ in $\mathbb{R}_2^5$ with bi-null arc parameter t and curvatures k_0, k_1.

Example 3.3. The following pairs satisfy the equation (3.6).
(i) $k_0 = t^2/2$, $k_1 = 1$ (ii) $k_0 = t/6$, $k_1 = 1/t$ (iii) $k_0 = 0$, $k_1 = 120/t^4$

From Theorem 3.1, we get the following corollary.

Corollary 3.4. *There exists no first kind of osculating bi-null curve in $\mathbb{R}^5_2$ with constant curvatures k_0, k_1.*

Let us consider the second kind of osculating bi-null curve in $\mathbb{R}^5_2$. Then, the position vector $\gamma(t)$ satisfies (3.2). Differentiating (3.2) with respect to t, we get

$$\begin{aligned} L_1 \quad = \quad & (\mu_1' - \mu_3 k_1) L_1 + (\mu_1 + \mu_2 k_1 - \mu_4 k_0) L_2 + (\mu_2' - \mu_3) N_1 + (\mu_3' - \mu_4) N_2 \\ & + (\mu_4' + \mu_3 k_0) W, \end{aligned}$$

which implies that

$$\begin{aligned} \mu_1 + \mu_2 k_1 - \mu_4 k_0 &= 0, \quad \mu_4' + \mu_3 k_0 = 0, \\ \mu_2' - \mu_3 &= 0, \quad \mu_3' - \mu_4 = 0, \quad \mu_1' - \mu_3 k_1 = 1. \end{aligned}$$

Solving the system of these equations, we have

$$\mu_1 = k_0 \mu_2'' - k_1 \mu_2, \quad \mu_3 = \mu_2', \quad \mu_4 = \mu_2'',$$

and

$$\mu_2^{(3)} + k_0 \mu_2' = 0, \quad k_0' \mu_2'' - \left(2k_1 + k_0^2\right) \mu_2' - k_1' \mu_2 = 1.$$

Using above equations, we get the following theorem:

Theorem 3.5. *Let γ be a bi-null curve in $\mathbb{R}^5_2$ with bi-null arc parameter t and curvatures k_0, k_1. Then γ is congruent to a second kind of osculating curve if and only if there exists a function μ_2 satisfying the following simultaneous equations:*

$$\mu_2^{(3)} + k_0 \mu_2' = 0, \quad k_0' \mu_2'' - \left(2k_1 + k_0^2\right) \mu_2' - k_1' \mu_2 = 1. \tag{3.7}$$

Proof. Let γ be a second kind of osculating bi-null curve in $\mathbb{R}^5_2$ with bi-null arc parameter t. Then it is clear from above calculations that (3.7) holds.

Conversely, assume that (3.7) holds. Let us consider a vector field $X(t) \in \mathbb{R}^5_2$ as follows

$$X = \gamma(t) - (k_0 \mu_2'' - k_1 \mu_2) L_1 - \mu_2 N_1 - \mu_2' N_2 - \mu_2'' W.$$

Then we obtain $X' = 0$ which implies that γ is congruent to a second kind of osculating curve. $\square$

When $\mu_2' \neq 0$, from the first equation of (3.7), we have

$$k_0 = -\frac{\mu_2^{(3)}}{\mu_2'}.$$

From the second equation of (3.7),

$$k_0' \mu_2'' - k_0^2 \mu_2' - 1 = k_1' \mu_2 + 2k_1 \mu_2'.$$

When $\mu_2 \neq 0$, multiplying by μ_2, we have

$$\left(\mu_2^2 k_1\right)' = k_0' \mu_2 \mu_2'' - k_0^2 \mu_2 \mu_2' - \mu_2,$$

and

$$k_1 = \frac{1}{\mu_2^2} \int \left(k_0' \mu_2 \mu_2'' - k_0^2 \mu_2 \mu_2' - \mu_2\right) dt.$$

From the above and Theorem 2.1, we have

Theorem 3.6. *For a differentiable function $\mu_2(t)$ with $\mu_2 \neq 0$, $\mu_2' \neq 0$, set*

$$k_0 = -\frac{\mu_2^{(3)}}{\mu_2'}, \quad k_1 = \frac{1}{\mu_2^2} \int \left(k_0' \mu_2 \mu_2'' - k_0^2 \mu_2 \mu_2' - \mu_2\right) dt. \tag{3.8}$$

Then there exists a second kind of osculating bi-null curve $\gamma(t)$ in $\mathbb{R}_2^5$ with bi-null arc parameter t and curvatures k_0, k_1.

Example 3.7. The following triples satisfy the equation (3.8).
(*i*) $\mu_2 = t$, $k_0 = 0$, $k_1 = -1/2$ (*ii*) $\mu_2 = t^2$, $k_0 = 0$, $k_1 = -1/(3t)$.

Let us consider the third kind of osculating bi-null curve in $\mathbb{R}_2^5$. Then, the position vector $\gamma(t)$ satisfies (3.3). Differentiating (3.3) with respect to t, we get

$$\begin{aligned} L_1 &= \left(\mu_1' - k_1 \mu_4\right) L_1 + \left(\mu_1 + \mu_2' + k_1 \mu_3\right) L_2 + \left(\mu_3' - \mu_4\right) N_1 + \mu_4' N_2 \\ &\quad + \left(\mu_2 + k_0 \mu_4\right) W, \end{aligned}$$

which implies that

$$\begin{aligned} \mu_1 + \mu_2' + k_1 \mu_3 &= 0, \quad \mu_3' - \mu_4 = 0, \\ \mu_2 + k_0 \mu_4 &= 0, \quad \mu_1' - k_1 \mu_4 = 1, \quad \mu_4' = 0. \end{aligned} \tag{3.9}$$

From (3.9), we get

$$\mu_1 = c_4 k_0' - k_1 \left(c_4 t + c_3\right), \quad \mu_2 = -c_4 k_0, \quad \mu_3 = c_4 t + c_3, \quad \mu_4 = c_4$$

and

$$c_4 k_0'' - k_1' \left(c_4 t + c_3\right) - 2 c_4 k_1 = 1$$

where $c_3, c_4 \in \mathbb{R}$.

From the above, we have

Theorem 3.8. *Let γ be a bi-null curve in $\mathbb{R}_2^5$ with bi-null arc parameter t and curvatures k_0, k_1. Then γ is congruent to a third kind of osculating curve if and only if*

$$c_4 k_0'' - k_1' \left(c_4 t + c_3\right) - 2 c_4 k_1 = 1 \tag{3.10}$$

where $c_3, c_4 \in \mathbb{R}$.

Proof. Let γ be a third kind of osculating bi-null curve in $\mathbb{R}_2^5$ with bi-null arc parameter t. Then it is clear from above calculations that (3.10) holds.

Conversely, assume that (3.10) holds. Let us consider a vector field $X(t) \in \mathbb{R}_2^5$ as follows

$$X = \gamma(t) - (c_4 k_0' - k_1 (c_4 t + c_3)) L_1 + c_4 k_0 L_2 - (c_4 t + c_3) N_1 - c_4 N_2.$$

Then we obtain $X' = 0$ which implies that γ is congruent to a third kind of osculating curve. □

When $(c_3, c_4) \neq (0, 0)$, multiplying (3.10) by $c_4 t + c_3$, we have

$$\left((c_4 t + c_3)^2 k_1\right)' = (c_4 t + c_3)(c_4 k_0'' - 1),$$

and

$$k_1 = \frac{1}{(c_4 t + c_3)^2} \int (c_4 t + c_3)(c_4 k_0'' - 1)\, dt.$$

From the above and Theorem 2.1, we get the following theorem.

Theorem 3.9. *For a differentiable function $k_0(t)$ and real numbers c_3, c_4 with $(c_3, c_4) \neq (0, 0)$, set*

$$k_1 = \frac{1}{(c_4 t + c_3)^2} \int (c_4 t + c_3)(c_4 k_0'' - 1)\, dt. \tag{3.11}$$

Then there exists a third kind of osculating bi-null curve $\gamma(t)$ in $\mathbb{R}_2^5$ with bi-null arc parameter t and curvatures k_0, k_1.

Example 3.10. The following satisfy the equation (3.11).
(*i*) $c_3 = 1$, $c_4 = 0$, $k_0 = t$, $k_1 = -t$ (*ii*) $c_3 = 0$, $c_4 = 1$, $k_0 = (t^2 + t^3)/2$, $k_1 = t$.

4. Normal or rectifying bi-null curves

In this section, we consider the conditions for a bi-null curve in $\mathbb{R}_2^5$ to be a normal curve or a rectifying curve.

Let $\gamma(t)$ be a bi-null curve in $\mathbb{R}_2^5$ with bi-null arc t. Then γ is called a normal bi-null curve if its position vector (with respect to some chosen origin) always lies in its normal space defined as the orthogonal space $L_1^{\perp}$ of the tangent vector field L_1. Since the normal space is a 4-dimensional subspace of $\mathbb{R}_2^5$, spanned by $\{L_1, L_2, N_2, W\}$, the position vector γ of a normal bi-null curve satisfies the following equation

$$\gamma(t) = \mu_1(t) L_1(t) + \mu_2(t) L_2(t) + \mu_3(t) N_2(t) + \mu_4(t) W(t) \tag{4.1}$$

where $\mu_i(t)$ are some differentiable functions for $i \in \{1, 2, 3, 4\}$.

Next, the bi-null curve γ is called a rectifying bi-null curve if its position vector (with respect to some chosen origin) always lies in its rectifying space defined as the orthogonal space $L_2^{\perp}$ of the principal normal vector field L_2. Since the rectifying space is a 4-dimensional subspace of $\mathbb{R}_2^5$, spanned by $\{L_1, L_2, N_1, W\}$, the position vector γ of a rectifying bi-null curve satisfies the following equation

$$\gamma(t) = \mu_1(t) L_1(t) + \mu_2(t) L_2(t) + \mu_3(t) N_1(t) + \mu_4(t) W(t) \tag{4.2}$$

where $\mu_i(t)$ are some differentiable functions for $i \in \{1,2,3,4\}$.

Let γ be a normal bi-null curve in $\mathbb{R}_2^5$ parametrized by bi-null arc parameter t and with curvatures k_0, k_1. Then the position vector satisfies (4.1). Differentiating (4.1) with respect to t, we find

$$\begin{aligned} L_1 &= (\mu_1' - \mu_3 k_1) L_1 + (\mu_1 + \mu_2' - \mu_4 k_0) L_2 + (\mu_2 + \mu_3 k_0 + \mu_4') W \\ &\quad - \mu_3 N_1 + (\mu_3' - \mu_4) N_2 \end{aligned}$$

which leads to the following system of equations

$$\begin{aligned} \mu_1 + \mu_2' - \mu_4 k_0 &= 0, \quad \mu_1' - \mu_3 k_1 = 1, \quad \mu_2 + \mu_3 k_0 + \mu_4' = 0, \\ \mu_3' - \mu_4 &= 0, \quad \mu_3 = 0. \end{aligned} \tag{4.3}$$

Solving (4.3), we obtain

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0 \tag{4.4}$$

which is a contradiction. Thus, we have the following theorem:

Theorem 4.1. *There exists no normal bi-null curve in* $\mathbb{R}_2^5$.

Let us assume that γ is a rectifying bi-null curve in $\mathbb{R}_2^5$ parametrized by bi-null arc parameter t and with curvatures k_0, k_1. Then the position vector satisfies (4.2). Differentiating (4.2) with respect to t, we find

$$L_1 = \mu_1' L_1 + (\mu_1 + \mu_2' + \mu_3 k_1 - \mu_4 k_0) L_2 + (\mu_2 + \mu_4') W + \mu_3' N_1 - \mu_4 N_2,$$

which leads to the following system of equations

$$\begin{aligned} \mu_1' &= 1, \quad \mu_1 + \mu_2' + \mu_3 k_1 - \mu_4 k_0 = 0, \\ \mu_2 + \mu_4' &= 0, \quad \mu_3' = 0, \quad \mu_4 = 0. \end{aligned} \tag{4.5}$$

Solving (4.5), we obtain

$$\mu_2 = \mu_4 = 0, \quad \mu_3 = c_3 \neq 0, \quad \mu_1 + c_3 k_1 = 0, \quad \mu_1 = t + c_0, \tag{4.6}$$

where c_0 and c_3 are constant. Thus, we can write the curve γ as

$$\gamma(t) = (t + c_0) L_1 + c_3 N_1. \tag{4.7}$$

Using the above computations, we obtain the following theorem:

Theorem 4.2. *Let γ be a bi-null curve parametrized by bi-null arc parameter t and with curvatures k_0, k_1 in $\mathbb{R}_2^5$. Then γ is a rectifying curve if and only if one of the following conditions holds:*

(i) the curvature k_1 of γ is given by $k_1(t) = -(t+c_0)/c_3$,
(ii) the distance function $\rho = \|\gamma\|$ satisfies $\rho^2 = |a_0 t + a_1|$, $a_0 \in \mathbb{R}/\{0\}$, $a_1 \in \mathbb{R}$,
(iii) $\langle \gamma, L_1 \rangle = c_3$.

Proof. Suppose that γ is rectifying. Then using (4.6), we find $k_1(t) = -(t+c_0)/c_3$. It is clear from (4.7) that the statements (ii) and (iii) hold.

Conversely, let us assume that either (i), (ii) or (iii) holds in the following.

If (i) holds, by using Frenet equations, we have

$$\frac{d}{dt}(\gamma - (t+c_0)L_1 - c_3 N_1) = 0.$$

So up to parallel translations of $\mathbb{R}_2^5$, it follows that γ is a rectifying curve.

If (ii) holds, differentiating the equation $\langle \gamma, \gamma \rangle = \pm(a_0 t + a_1)$ twice with respect to t, we have $\langle \gamma, L_2 \rangle = 0$, which means that γ is a rectifying curve.

If (iii) holds, differentiating it, we conclude that γ is a rectifying curve. □

Definition 4.3. The pseudo-sphere of radius $r > 0$ and center p_0 in $\mathbb{R}_2^5$ is given by

$$S_2^4(r) = \left\{ x \in \mathbb{R}_2^5 \,\middle|\, \langle x - p_0, x - p_0 \rangle = r^2 \right\}.$$

Finally we give a structure theorem for rectifying bi-null curves in $\mathbb{R}_2^5$ with nonzero spacelike position vector.

Theorem 4.4. (i) *Let $\gamma(t)$ be a bi-null curve in $\mathbb{R}_2^5$ parametrized by bi-null arc parameter t, and with nonzero spacelike position vector. If the curve γ is rectifying, then by changing parameter, it can be written as*

$$\gamma(s) = b e^s y(s), \quad b \in \mathbb{R}^+, \tag{4.8}$$

where $y(s)$ is a unit speed timelike curve in $S_2^4(1) = \{x \in \mathbb{R}_2^5 | \langle x, x \rangle = 1\}$ with

$$\langle \frac{d^2y}{ds^2}(s), \frac{d^2y}{ds^2}(s) \rangle = 1, \quad \langle \frac{d^3y}{ds^3}(s), \frac{d^3y}{ds^3}(s) \rangle = \frac{64 b^{10} e^{10s}}{a_0^6} - 1, \quad a_0 \in \mathbb{R}^+.$$

(ii) *Conversely, let $y(s)$ be a unit speed timelike curve in $S_2^4(1)$ which satisfies*

$$\langle \frac{d^2y}{ds^2}(s), \frac{d^2y}{ds^2}(s) \rangle = 1, \quad \langle \frac{d^3y}{ds^3}(s), \frac{d^3y}{ds^3}(s) \rangle = \frac{64 b^{10} e^{10s}}{a_0^6} - 1,$$
$$\frac{d^2y}{ds^2}(s) \neq y(s), \quad a_0, b \in \mathbb{R}^+.$$

Then the curve $\gamma(s) = b e^s y(s)$ admits a parameter change t such that it becomes a rectifying bi-null curve in $\mathbb{R}_2^5$ with respect to the bi-null arc parameter t.

Proof. (i) Suppose that γ is a rectifying bi-null curve in $\mathbb{R}_2^5$ parametrized by bi-null arc parameter t, and with nonzero spacelike position vector. By Theorem 4.2, the distance function $\rho = \|\gamma\|$ satisfies $\rho^2 = a_0 t + a_1$, $a_0 \in \mathbb{R}\setminus\{0\}$, $a_1 \in \mathbb{R}$. We may choose $a_0 \in \mathbb{R}^+$.

Let us define a curve $y(t)$ by $y(t) = \gamma(t)/\rho(t)$. The curve y lies in the pseudo-sphere $S_2^4(1)$. Then, we have

$$\gamma(t) = y(t)\sqrt{a_0 t + a_1}. \tag{4.9}$$

By differentiating (4.9) with respect to t, we find

$$L_1(t) = \frac{a_0}{2\sqrt{a_0 t + a_1}} y(t) + y'(t)\sqrt{a_0 t + a_1}. \tag{4.10}$$

From (4.10), we have

$$\langle y'(t), y'(t)\rangle = -\frac{a_0^2}{4(a_0 t + a_1)^2}, \quad \|y'(t)\| = \frac{a_0}{2(a_0 t + a_1)}$$

and y is a timelike curve.

Denote the arc length parameter of the curve y by

$$s = \int_{t_0}^{t} \|y'(u)\|\, du = \frac{1}{2}\ln\frac{a_0 t + a_1}{b^2}, \quad b \in \mathbb{R}^+.$$

Then $a_0 t + a_1 = b^2 e^{2s}$. Substituting this in (4.9), we get $\gamma(s) = be^s y(s)$.

Furthermore, differentiating $\gamma(s) = be^s y(s)$ twice with respect to s and using $dt/ds = 2b^2e^{2s}/a_0$, we obtain

$$L_2(t) = \frac{a_0^2}{4b^3} e^{-3s}\left(\frac{d^2y}{ds^2}(s) - y(s)\right). \tag{4.11}$$

By $\langle L_2(t), L_2(t)\rangle = 0$ and (4.11), we get

$$\left\langle \frac{d^2y}{ds^2}(s), \frac{d^2y}{ds^2}(s)\right\rangle = 1.$$

Differentiating (4.11) with respect to s and using $dt/ds = 2b^2e^{2s}/a_0$, we find

$$W(t) = \frac{a_0^3}{8b^5} e^{-5s}\left(3y(s) - \frac{dy}{ds}(s) - 3\frac{d^2y}{ds^2}(s) + \frac{d^3y}{ds^3}(s)\right). \tag{4.12}$$

By $\langle W(t), W(t)\rangle = 1$ and (4.12), we get

$$\left\langle \frac{d^3y}{ds^3}(s), \frac{d^3y}{ds^3}(s)\right\rangle = \frac{64b^{10}e^{10s}}{a_0^6} - 1.$$

(ii) First, by the condition $\left(d^2y/ds^2\right)(s) \neq y(s)$, we can see that

$$\left\{\frac{d\gamma}{ds}(s), \frac{d^2\gamma}{ds^2}(s)\right\}$$

are linearly independent. Under these conditions, choosing a new parameter t for $\gamma(s) = be^s y(s)$ as

$$s = \frac{1}{2} \ln \frac{a_0 t + a_1}{b^2}, \quad a_1 \in \mathbb{R},$$

we have

$$\gamma(t) = y(t)\sqrt{a_0 t + a_1}. \tag{4.13}$$

Using the conditions of $y(s)$, we can find that

$$\begin{aligned} \langle y'(t), y'(t)\rangle &= \frac{-a_0^2}{4(a_0 t + a_1)^2}, \quad \langle y''(t), y''(t)\rangle = \frac{-3a_0^4}{16(a_0 t + a_1)^4}, \\ \left\langle y^{(3)}(t), y^{(3)}(t)\right\rangle &= \frac{1}{(a_0 t + a_1)} - \frac{45}{64}\frac{a_0^6}{(a_0 t + a_1)^6}. \end{aligned} \tag{4.14}$$

Differentiating (4.13), we can get

$$\gamma'(t) = \frac{a_0}{2}(a_0 t + a_1)^{-\frac{1}{2}} y(t) + (a_0 t + a_1)^{\frac{1}{2}} y'(t),$$

$$\gamma''(t) = -\frac{a_0^2}{4}(a_0 t + a_1)^{-\frac{3}{2}} y(t) + a_0 (a_0 t + a_1)^{-\frac{1}{2}} y'(t) + (a_0 t + a_1)^{\frac{1}{2}} y''(t),$$

and

$$\begin{aligned} \gamma^{(3)}(t) &= \frac{3}{8}a_0^3 (a_0 t + a_1)^{-\frac{5}{2}} y(t) - \frac{3}{4}a_0^2 (a_0 t + a_1)^{-\frac{3}{2}} y'(t) \\ &\quad + \frac{3}{2}a_0 (a_0 t + a_1)^{-\frac{1}{2}} y''(t) + (a_0 t + a_1)^{\frac{1}{2}} y^{(3)}(t). \end{aligned}$$

From these equations and (4.14),

$$\langle \gamma'(t), \gamma'(t)\rangle = \langle \gamma''(t), \gamma''(t)\rangle = 0, \quad \left\langle \gamma^{(3)}(t), \gamma^{(3)}(t)\right\rangle = 1.$$

Hence γ is a bi-null curve and t is the bi-null arc parameter of γ. Then since $\langle \gamma(t), \gamma(t)\rangle = a_0 t + a_1$, from Theorem 4.2, we conclude that γ is a rectifying bi-null curve in $\mathbb{R}_2^5$. □

Remark 1. From Theorem 2.1 and Theorem 4.2 (i), we can see the existence of rectifying bi-null curves in $\mathbb{R}_2^5$. Furthermore, from Theorem 4.4, and by choosing a nonzero spacelike vector p_0 in the Theorem 2.1, we find the existence of rectifying bi-null curves in $\mathbb{R}_2^5$ with nonzero spacelike position vector.

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