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LOCALLY ϕ-QUASICONFORMALLY SYMMETRIC SASAKIAN FINSLER STRUCTURES ON TANGENT BUNDLES

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Abstract. In this study, the notion of locally ϕ-quasiconformally symmetric Sasakian Finsler structures on the distributions of tangent bundles is introduced and its various geometric properties are studied with an example in dimension 3.

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1. Introduction

Miron [5], used the vector bundle approach in Finsler geometry. Sinha and Yadav [7], defined almost contact structures on vector bundles and studied their integrability condition. In [8], Yaliniz and Caliskan analysed almost contact and Sasakian Finsler structures on vector bundles and extended their characteristics with curvature properties and some structure theorems. Massamba and Mbatakou [4], approved pulled-back bundles to construct Sasakian Finsler structures. In this paper, tangent bundle approach is chosen to clarify locally ϕ-quasiconformal symmetry property of Sasakian Finsler structures. On the other hand, quasiconformal curvature tensor appears in the literature with Yano and Sawaki [9]. Also, ϕ-quasiconformal flatness and ϕ-quasiconformal symmetry features of several manifolds, like [2, 3], are studied quite frequently. Here, we are interested in locally ϕ-quasiconformally symmetric Sasakian Finsler structures on tangent bundles.

In this section, a brief account of Sasakian Finsler structures on tangent bundles is given:

Let M be an $m = (2n + 1)$-dimensional smooth manifold. In this manner, T_xM is denoted as the tangent space at $x \in M$ where $x = (x^1, \ldots, x^m)$ are the local coordinates of M and $y = y^i \frac{\partial}{\partial x^i} \in T_xM$. Then $u = (x, y) \in TM$ where TM is the tangent bundle.

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Definition 1.1. The function $F : TM \to [0, \infty[$, the Hessian G and the manifold $F^m = (M, F)$ are called " Finsler norm", " Finsler metric" and " Finsler manifold", respectively, if the following relations hold [1]:

1. F is smooth on the slit tangent bundle TM,
2. $F(x, \lambda y) = |\lambda| F(x, y)$, for $\lambda \in \mathbb{R}$ and $u = (x, y) \in TM$,
3. $g_{ij}(x, y) = \frac{1}{2}[\frac{\partial^2 F^2}{\partial y^i \partial y^j}]$ is positive definite on TM.

Assume that (x^i, y^i) and $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ denote the local coordinates of TM and natural bases of $T_u TM$, respectively. If $\pi : TM \to M$ is the projection map, the differential map $\pi_* : T_u TM \to T_{\pi(u)} M$ satisfies $X_u \in \pi_*(X_u)$. So, $ker(\pi) = VTM$.

The non-linear connection $HTM = (N_i^j(x, y))$ is the complementary distribution of VTM for TTM i.e. $TTM = HTM \oplus VTM$, where $N_i^j = \frac{\partial N^j}{\partial y^i}$ are obtained via the spray coefficients $N^j = \frac{1}{4} g^{jk} (\frac{\partial^2 F^2}{\partial y^k \partial x^h} y^h - \frac{\partial F^2}{\partial x^k})$ [8].

For every $u \in TM$ and $X \in T_u TM$, by using non-linear connections, $X = (X^i \frac{\partial}{\partial x^i} - N_i^j(x, y) X^i \frac{\partial}{\partial y^j}) + ((N_i^j(x, y) X^i + X^j) \frac{\partial}{\partial y^j}) = X^H + X^V$ unique decomposition is obtained as the horizontal part and the vertical part of vector field X where $X^H \in T_u^H TM$ and $X^V \in T_u^V TM$ and $T_u^H TM$ and $T_u^V TM$ are spanned by $\{\frac{\delta}{\delta x^i}\}$ and $\{\frac{\partial}{\partial y^j}\}$ respectively. In addition, their dual bases are $\{dx^i\}$ and $\{\delta y^j (= dy^j + N_i^j dx^i)\}$, respectively.

Similarly, for $\eta \in (T_u TM)^*$, $\eta = \tilde{\eta}_i dx^i + \eta_j \delta y^j = \eta^H + \eta^V$ is obtained where $\eta^H \in (T_u^H TM)^*$ and $\eta^V \in (T_u^V TM)^*$.

The Sasaki-Finsler metric G on TM is defined as follows:

$G = G^H + G^V$ in the type of $\binom{0 \quad 0}{2 \quad 0}$ and $\binom{0 \quad 0}{0 \quad 2}$ on TM^H and TM^V, respectively. Thus, Sasakian Finsler metric structures $(\phi^H, \xi^H, \eta^H, G^H)$ and $(\phi^V, \xi^V, \eta^V, G^V)$ can be constructed on either TM^H or TM^V, respectively where; ϕ denotes the tensor field of type $\binom{1 \quad 1}{1 \quad 1}$, ξ is the structure vector field of type $\binom{0 \quad 0}{1 \quad 1}$, η is the 1-form of type $\binom{1 \quad 1}{0 \quad 0}$, ∇ is the Finsler connection with respect to G on TM, L is the Lie differential operator, R is the Riemann curvature tensor field of type $\binom{1 \quad 1}{3 \quad 3}$, S is the Ricci tensor field of type $\binom{0 \quad 0}{2 \quad 2}$, for $X^H, Y^H, \xi^H \in T_u^H TM$ and $X^V, Y^V, \xi^V \in T_u^V TM$, respectively. The following relations hold for m-dimensional Sasakian Finsler metric manifolds $(TM^H, \phi^H, \xi^H, \eta^H, G^H)$ and $(TM^V, \phi^V, \xi^V, \eta^V, G^V)$ [8]:

$$\phi.\phi = -I + \eta^H \otimes \xi^H + \eta^V \otimes \xi^V \tag{1.1}$$

$$\phi \xi^H = 0, \phi \xi^V = 0 \tag{1.2}$$

$$\eta^H(\xi^H) = 1, \eta^V(\xi^V) = 1 \tag{1.3}$$

$$\eta^H(\phi X^H) = 0, \eta^V(\phi X^V) = 0, \eta^H(\phi X^V) = 0 \tag{1.4}$$

$$\begin{aligned} \Omega(X^H, Y^H) &= 2(\nabla_X^H \eta)(Y^H) = -2(\nabla_Y^H \eta)(X^H) \\ \Omega(X^H, Y^H) &= 2(\nabla_X^H \eta)(Y^H) = -2(\nabla_Y^H \eta)(X^H) \end{aligned} \tag{1.5}$$

$$\begin{aligned} G(X^H, Y^H) &= G(\phi^H X^H, \phi^H Y^H) + \eta^H(X^H)\eta^H(Y^H) \\ G(X^V, Y^V) &= G(\phi^V X^V, \phi^V Y^V) + \eta^V(X^V)\eta^V(Y^V) \end{aligned} \tag{1.6}$$

$$G(X^H, \xi^H) = \eta^H(X^H), G(X^V, \xi^V) = \eta^V(X^V) \tag{1.7}$$

$$\begin{aligned} G(\phi^H X^H, Y^H) &= -G(X^H, \phi^H Y^H) \\ G(\phi^V X^V, Y^V) &= -G(X^V, \phi^V Y^V) \end{aligned} \tag{1.8}$$

$$\begin{aligned} \Omega(X^H, Y^H) &= G(X^H, \phi Y^H) = d\eta^H(X^H, Y^H) = \Omega(\phi X^V, \phi Y^V) \\ \Omega(X^V, Y^V) &= G(X^V, \phi Y^V) = d\eta^V(X^V, Y^V) = \Omega(\phi X^V, \phi Y^V) \end{aligned} \tag{1.9}$$

$$\Omega(X^H, \xi^H) = \Omega(X^V, \xi^V) = 0 \tag{1.10}$$

$$G(X^H, \phi^H Y^H) = d\eta^H(X^H, Y^H), G(X^V, \phi^V Y^V) = d\eta^H(X^H, Y^H) \tag{1.11}$$

$$\nabla_X^H \xi^H = -\frac{1}{2}\phi^H X^H, \nabla_X^V \xi^V = -\frac{1}{2}\phi^V X^V \tag{1.12}$$

$$\begin{aligned} (\nabla_X^H \phi^H)Y^H &= \frac{1}{2}[G(X^H, Y^H)\xi^H - \eta^H(Y^H)X^H] \\ (\nabla_X^V \phi^V)Y^V &= \frac{1}{2}[G(X^V, Y^V)\xi^V - \eta^V(Y^V)X^V] \end{aligned} \tag{1.13}$$

$$\begin{aligned} R(X^H, Y^H)\xi^H &= \frac{1}{4}[\eta^H(Y^H)X^H - \eta^H(X^H)Y^H] \\ R(X^V, Y^V)\xi^V &= \frac{1}{4}[\eta^V(Y^V)X^V - \eta^V(X^V)Y^V] \end{aligned} \tag{1.14}$$

$$\begin{aligned} R(X^H, \xi^H)Y^H &= \frac{1}{4}[\eta^H(Y^H)X^H - G(X^H, Y^H))\xi^H] \\ R(X^V, \xi^V)Y^V &= \frac{1}{4}[\eta^V(Y^V)X^V - G(X^V, Y^V)\xi^V] \end{aligned} \tag{1.15}$$

$$S(X^H,\xi^H)=\frac{n}{2}\eta^H(X^H),S(X^V,\xi^V)=\frac{n}{2}\eta^V(X^V) \tag{1.16}$$

$$S(\xi^H,\xi^H)=\frac{n}{2},S(\xi^V,\xi^V)=\frac{n}{2} \tag{1.17}$$

$$S(X^H,Y^H)=G(QX^H,Y^H),S(X^H,Y^H)=G(QX^V,Y^V) \tag{1.18}$$

$$Q(X^H)=\sum_{i=1}^{2n+1}R(E_i^H,X^H)E_i^H,Q(X^V)=\sum_{i=1}^{2n+1}R(E_i^V,X^V)E_i^V \tag{1.19}$$

$$r=\sum_{i=1}^{2n+1}(S(E_i^H,E_i^H)+S(E_i^v,E_i^v)) \tag{1.20}$$

Above-stated formulas can be used to construct Sasakian Finsler structures on both TM^H and TM^V. But in this paper, in second and third sections, locally ϕ-quasiconformal symmetry of TM^H and 3-dimensional TM^H is discussed briefly.

2. Locally ϕ-quasiconformally symmetric Sasakian Finsler structures on TM^H

Definition 2.1. Let TM^Hbe a Sasakian Finsler manifold, then it is locally ϕ-symmetric if and only if

$$\phi^2((\nabla_w^H R)(X^H,Y^H)Z^H)=0 \tag{2.1}$$

for all $X^H,Y^H,Z^H,W^H\in T_u^HTM$.

Definition 2.2. Let TM^Hbe a Sasakian Finsler manifold, then it is locally ϕ-symmetric if and only if

$$\phi^2((\nabla_W^H C^*)(X^H,Y^H)Z^H)=0 \tag{2.2}$$

for all vector fields $X^H,Y^H,Z^H,W^H\in T_u^HTM$ and where the quasiconformal curvature tensor C^* is defined by

$$\begin{aligned}C^*(X^H,Y^H)Z^H&=aR(X^H,Y^H)Z^H+b[S(Y^H,Z^H)X^H\\&-S(X^H,Z^H)Y^H+G(Y^H,Z^H)QX^H-G(X^H,Z^H)QY^H]\\&-\frac{r}{2n+1}(\frac{a}{2n}+2b)(G(Y^H,Z^H)X^H-G(X^H,Z^H)Y^H).\end{aligned} \tag{2.3}$$

for all $X^H,Y^H,Z^H,W^H\in T_u^HTM$ and the constants a,b.

If $a=1$ and $b=\frac{1}{2n-1}$, (2.3) can be expressed as follows:

$$C^*(X^H,Y^H)Z^H = R(X^H,Y^H)Z^H + \tfrac{1}{2n-1}[S(Y^H,Z^H)X^H - S(X^H,Z^H)Y^H + G(Y^H,Z^H)QX^H - G(X^H,Z^H)QY^H] - \tfrac{r}{(2n)(2n-1)}(G(Y^H,Z^H)X^H - G(X^H,Z^H)Y^H) = C(X^H,Y^H)Z^H$$

where C is Weyl conformal curvature tensor.

Calculating the covariant differentiation of (2.3), the following equality is obtained:

$$(\nabla^H_W C^*)(X^H,Y^H)Z^H = a(\nabla^H_W R)(X^H,Y^H)Z^H + b[(\nabla^H_W S)(Y^H,Z^H)X^H - (\nabla^H_W S)(X^H,Z^H)Y^H + G(Y^H,Z^H)(\nabla^H_W Q)X^H - G(X^H,Z^H)(\nabla^H_W Q)Y^H] - \frac{dr(W^H)}{2n+1}(\frac{a}{2n} + 2b)(G(Y^H,Z^H)X^H - G(X^H,Z^H)Y^H). \tag{2.4}$$

If $S(Y^H,W^H) = \lambda G(W^H,Y^H)$ is satisfied, where λ is a constant and $X^H,Y^H \in T^H_u TM$, the manifold TM^H is called an Einstein manifold, where $QX^H = \lambda X^H$.

By using (1.1); (2.2) takes the following form:

$$-(\nabla^H_W C)(X^H,Y^H)Z^H + \eta^H((\nabla^H_W C)(X^H,Y^H)Z^H)\xi^H = 0.$$

By virtue of (2.4), we obtain

$$0 = -a(\nabla^H_W R)(X^H,Y^H)Z^H - b(\nabla^H_W S)(Y^H,Z^H)X^H + b(\nabla^H_W S)(X^H,Z^H)Y^H - bG(Y^H,Z^H)(\nabla^H_W Q)X^H + bG(X^H,Z^H)(\nabla^H_W Q)Y^H + \tfrac{dr(W^H)}{2n+1}(\tfrac{a}{2n} + 2b)G(Y^H,Z^H)X^H - \tfrac{dr(W^H)}{2n+1}(\tfrac{a}{2n} + 2b)G(X^H,Z^H)Y^H + a\eta^H((\nabla^H_W R)(X^H,Y^H)Z^H)\xi^H + b(\nabla^H_W S)(Y^H,Z^H)\eta^H(X^H)\xi^H - b(\nabla^H_W S)(X^H,Z^H)\eta^H(Y^H)\xi^H + bG(Y^H,Z^H)\eta^H((\nabla^H_W Q)X^H)\eta^H(U^H) - bG(X^H,Z^H)\eta^H((\nabla^H_W Q)Y^H)\xi^H - \tfrac{dr(W^H)}{2n+1}(\tfrac{a}{2n} + 2b)G(Y^H,Z^H)\eta^H(X^H)\xi^H + \tfrac{dr(W^H)}{2n+1}(\tfrac{a}{2n} + 2b)G(X^H,Z^H)\eta^H(Y^H)\xi^H.$$

For $U^H \in T^H_u TM$, the last equality is expressed by

$$0 = -aG((\nabla^H_W R)(X^H,Y^H)Z^H,U^H) - b(\nabla^H_W S)(Y^H,Z^H)G(X^H,U^H) + b(\nabla^H_W S)(X^H,Z^H)G(Y^H,U^H) - bG(Y^H,Z^H)G((\nabla^H_W Q)X^H,U^H) + bG(X^H,Z^H)G((\nabla^H_W Q)Y^H,U^H) + \tfrac{dr(W^H)}{2n+1}(\tfrac{a}{2n} + 2b)G(Y^H,Z^H)G(X^H,U^H) - \tfrac{dr(W^H)}{2n+1}(\tfrac{a}{2n} + 2b)G(X^H,Z^H)G(Y^H,U^H) + a\eta^H((\nabla^H_W R)(X^H,Y^H)Z^H)\eta^H(U^H) + b(\nabla^H_W S)(Y^H,Z^H)\eta^H(X^H)\eta^H(U^H) - b(\nabla^H_W S)(X^H,Z^H)\eta^H(Y^H)\eta^H(U^H) + bG(Y^H,Z^H)\eta^H((\nabla^H_W Q)X^H)\eta^H(U^H) - bG(X^H,Z^H)\eta^H((\nabla^H_W Q)Y^H)\eta^H(U^H) - \tfrac{dr(W^H)}{2n+1}(\tfrac{a}{2n} + 2b)G(Y^H,Z^H)\eta^H(X^H)\eta^H(U^H) + \tfrac{dr(W^H)}{2n+1}(\tfrac{a}{2n} + 2b)G(X^H,Z^H)\eta^H(Y^H)\eta^H(U^H)$$

Putting $X^H = U^H = E^H_i$, where $\{E^H_i\}$, $i = 1, 2, \ldots, 2n+1$ is an orthonormal basis of $T^H_u TM$, and taking summation over i, we have

$$0 = (-a - b(2n+1))(\nabla^H_W S)(Y^H, Z^H) + b(\nabla^H_W S)(E^H_i, Z^H)G(Y^H, E^H_i)$$
$$-G(Y^H, Z^H)[bG((\nabla^H_W Q)E^H_i, E^H_i) + dr(W^H)(\frac{a}{2n} + 2b)] + bG((\nabla^H_W Q)Y^H, Z^H)$$
$$-\frac{dr(W^H)}{2n+1}(\frac{a}{2n} + 2b)G(E^H_i, Z^H)G(Y^H, E^H_i)$$
$$+a\eta^H((\nabla^H_W R)(E^H_i, Y^H)Z^H)\eta^H(E^H_i) \tag{2.5}$$

In (2.5) $a\eta^H((\nabla^H_W R)(E^H_i, Y^H)Z^H)\eta^H(E^H_i)$ is expressed by

$$\eta^H((\nabla^H_W R)(E^H_i, Y^H)Z^H) = G(\nabla^H_W(R(E^H_i, Y^H)\xi^H), \xi^H)$$
$$-G((\nabla^H_W E^H_i, Y^H)\xi^H, \xi^H)$$
$$-G(R(E^H_i, \nabla^H_W Y^H)\xi^H, \xi^H) - G(R(E^H_i, Y^H)\nabla^H_W \xi^H, \xi^H) \tag{2.6}$$

Owing to the fact that E^H_i is an orthonormal basis, it is easily seen that $\nabla^H_W E^H_i = 0$.

By virtue of (1.14), it is possible to obtain below relation:

$$0 = G(R(E^H_i, \nabla^H_W Y^H)\xi^H, \xi^H) =$$
$$\tfrac{1}{4}[G(\nabla^H_W Y^H, \xi^H)G(E^H_i, \xi^H) - G(E^H_i, \xi^H)G(\nabla^H_W Y^H, \xi^H)]$$

By using these equalities, teh second and third terms of the right part of (2.6) vanish. Thus (2.6) takes this form:

$$G((\nabla^H_W R)(E^H_i, Y^H)\xi^H, \xi^H) = G((\nabla^H_W R)(E^H_i, Y^H)\xi^H, \xi^H)$$
$$-G(R(E^H_i, Y^H)\nabla^H_W \xi^H, \xi^H). \tag{2.7}$$

Due to $G((\nabla^H_W R)(E^H_i, Y^H)\xi^H, \xi^H) + G(R(E^H_i, Y^H)\xi^H, \nabla^H_W \xi^H) = 0$, (2.7) can be expressed as follows:

$$0 = G((\nabla^H_W R)(E^H_i, Y^H)\xi^H, \xi^H) =$$
$$-G((R)(E^H_i, Y^H)\xi^H, \nabla^H_W \xi^H) + G(R(E^H_i, Y^H)\xi^H, \nabla^H_W \xi^H)$$

In consequence of these calculations and by putting $Z^H = \xi^H$ in (2.5) we have the following:

$$(-a - b(2n+1))(\nabla^H_W S)(Y^H, \xi^H) + b(\nabla^H_W S)(\xi^H, \xi^H)\eta^H(Y^H)$$
$$-\eta^H(Y^H)[bG((\nabla^H_W Q)\xi^H, \xi^H)$$
$$+dr(W^H)(\frac{a}{2n} + 2b)] + bG((\nabla^H_W Q)Y^H, \xi^H) = 0 \tag{2.8}$$

We calculate $(\nabla^H_W S)(\xi^H, \xi^H) = 0$ and $G((\nabla^H_W Q)\xi^H, \xi^H) = 0$ and additionally $G((\nabla^H_W Q)Y^H, \xi^H) = 0$.

So, (2.8) is expressed by

$$(\nabla^H_W S)(Y^H, \xi^H) = dr(W^H)(-\frac{a + 4bn}{(2n+1)(a + (2n-1)b)})\eta^H(Y^H) \tag{2.9}$$

where $a + 4bn \neq 0$. Because if $a + 4bn = 0$ from (2.3), we get $C^* = aC$. By putting $Y^H = \xi^H$ in (2.9), we find the following:

$$(\nabla^H_W S)(\xi^H,\xi^H) = dr(W^H)(-\tfrac{a+4bn}{(2n+1)(a+(2n-1)b)})$$

$$0 = dr(W^H).$$

This implies r is constant. So we find $(\nabla^H_W S)(Y^H,\xi^H) = 0$.

By the virtue of (1.5) and (1.9), from (2.9) we have

$$S(Y^H,\phi W^H) = \tfrac{n}{2}G(W^H,\phi Y^H)$$

By putting ϕW^H instead of W^H, we find $S(Y^H,W^H) = \frac{n}{2}G(W^H,Y^H)$. If we get $\frac{n}{2} = \lambda$ this means that a ϕ-quasiconformally symmetric manifold TM^H is an Einstein manifold. Then it is possible to have the following theorem:

Theorem 2.3. *If a Sasakian Finsler manifold TM^H is locally ϕ-quasiconformally symmetric, then it is an Einstein manifold.*

If we get $S(X^H,Y^H) = \lambda G(X^H,Y^H)$ in (2.3), the below relation is found.

$$\begin{aligned} C^*(X^H,Y^H)Z^H &= (a+4bn \\ &-\frac{4r}{2n+1}(\frac{a}{2n}+2b))R(X^H,Y^H)Z^H \end{aligned} \tag{2.10}$$

From (2.2), it is possible to say that TM^H is locally ϕ-quasiconformally symmetric because C^* satisfies $\phi^2(\nabla^H_W C^*(X^H,Y^H)Z^H) = 0$ for all vector fields $X^H,Y^H,Z^H \in T^H_u TM$. Also $\phi^2(\nabla^H_w R)(X^H,Y^H)Z^H = 0$ implies that TM^H is locally ϕ-symmetric. So, it enables to state the following corollary:

Corollary 2.4. *Let TM^H be locally ϕ-quasiconformally symmetric. Then it is locally ϕ-symmetric.*

3. Locally ϕ-quasiconformally symmetric Sasakian Finsler structures on 3-dimensional TM^H

In a 3-dimensional TM^H, due to $C = 0$ [6], we have

$$\begin{aligned} R(X^H,Y^H)Z^H &= [S(Y^H,Z^H)X^H - S(X^H,Z^H)Y^H + G(Y^H,Z^H)QX^H \\ &-G(X^H,Z^H)QY^H] - \frac{r}{2}(G(Y^H,Z^H)X^H - G(X^H,Z^H)Y^H) \end{aligned} \tag{3.1}$$

Putting $Z^H = \xi^H$ in(3.1), by the virtue of (1.14) and (1.16), we find

$$\begin{aligned} (\frac{1}{4}-\frac{r}{2})[\eta^H(Y^H)X^H - \eta^H(X^H)Y^H] = [\eta^H(X^H)QY^H \\ -\eta^H(Y^H)QX^H]. \end{aligned} \tag{3.2}$$

Changing $Y^H = \xi^H$ in (3.2), we get

$$QX^H = (\frac{r}{2} - \frac{1}{4})X^H + (\frac{3}{4} - \frac{r}{2})\eta^H(X^H)\xi^H. \tag{3.3}$$

By using(3.3), we have

$$S(X^H, Y^H) = (\frac{r}{2} - \frac{1}{4})G(X^H, Y^H) + (\frac{3}{4} - \frac{r}{2})\eta^H(X^H)\eta^H(Y^H). \tag{3.4}$$

Writing(3.3) and (3.4) in (3.1), we get the following:

$$\begin{aligned} R(X^H, Y^H)Z^H &= (\frac{r}{2} - \frac{1}{2})(G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H) \\ &+(\frac{3}{4} - \frac{r}{2})[\eta^H(Y^H)\eta^H(Z^H)X^H - \eta^H(X^H)\eta^H(Z^H)Y^H \\ &+G(Y^H, Z^H)\eta^H(X^H)\xi^H - G(X^H, Z^H)\eta^H(Y^H)\xi^H]. \end{aligned} \tag{3.5}$$

Using (3.3), (3.4) and (3.5) in (2.3), we obtain

$$\begin{aligned} C^*(X^H, Y^H)Z^H &= [\frac{(a+b)r}{3} - \frac{1}{2}(a+b)](G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H) \\ &+(\frac{3}{4} - \frac{r}{2})(a+b)[G(Y^H, Z^H)\eta^H(X^H)\xi^H - G(X^H, Z^H)\eta^H(Y^H)\xi^H \\ &+\eta^H(Y^H)\eta^H(Z^H)X^H - \eta^H(X^H)\eta^H(Z^H)Y^H]. \end{aligned} \tag{3.6}$$

By calculating covariant differentiation of both sizes of (3.6)

$$\begin{gathered} (\nabla^H_W C^*)(X^H, Y^H)Z^H = (\tfrac{a+b}{3})dr(W^H)(G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H) + \\ [r(\tfrac{a+b}{3}) - \tfrac{1}{2}(a+b)]\nabla^H_W(G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H) - \tfrac{dr(W^H)}{2}(a+ \\ b)[G(Y^H, Z^H)\eta^H(X^H)\xi^H - G(X^H, Z^H)\eta^H(Y^H)\xi^H + \eta^H(Y^H)\eta^H(Z^H)X^H - \\ \eta^H(X^H)\eta^H(Z^H)Y^H] + (\tfrac{3}{4} - \tfrac{r}{2})(a+b)[\nabla^H_W(G(Y^H, Z^H)\eta^H(X^H)\xi^H) - \\ \nabla^H_W(G(X^H, Z^H)\eta^H(Y^H)\xi^H) + \nabla^H_W(\eta^H(Y^H)\eta^H(Z^H)X^H) - \\ \nabla^H_W(\eta^H(X^H)\eta^H(Z^H)Y^H)]. \end{gathered}$$

Then we can write the following relation:

$$\begin{gathered} (\nabla^H_W C)(X^H, Y^H)Z^H = (\tfrac{a+b}{3})dr(W^H)(G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H) + \\ (\tfrac{3}{4} - \tfrac{r}{2})(a+b)[G(Y^H, Z^H)\nabla^H_W(\eta^H(X^H))\xi^H - G(X^H, Z^H)\nabla^H_W(\eta^H(X^H))\xi^H]. \end{gathered}$$

Because $X^H, Y^H, Z^H \in T^H_u TM$ are orthogonal to ξ^H, by using (1.1) we get $\phi^2(\nabla^H_W C(X^H, Y^H)Z^H) = -\nabla^H_W C(X^H, Y^H)Z^H + \eta^H(\nabla^H_W C(X^H, Y^H)Z^H)$ from which we have

$$\begin{aligned} \phi^2(\nabla^H_W C(X^H, Y^H)Z^H) = -(\frac{a+b}{3})dr(W^H)[G(Y^H, Z^H)X^H \\ -G(X^H, Z^H)Y^H] \end{aligned} \tag{3.7}$$

Due to $\phi^2(\nabla^H_W C(X^H, Y^H)Z^H) = 0$ if we take $a+b=0$ and $a=-b$ in (2.4), we have $C(X^H, Y^H)Z^H = aC(X^H, Y^H)Z^H$. Because of $C=0$, in (3.7)we find $dr(W^H) = 0$. This means that the curvature r is constant. Then it is possible to state the following theorem:

Theorem 3.1. *Let TM^H be a 3-dimensional Sasakian Finsler manifold. A necessary and sufficient condition to be locally ϕ-quasiconformally symmetric is that r is constant.*

Corollary 3.2. *Let TM^H be a 3-dimensional Sasakian Finsler manifold. A necessary and sufficient condition to be ϕ-symmetric is that r is constant.*

Corollary 3.3. *Let TM^H be a 3-dimensional Sasakian Finsler manifold. A necessary and sufficient condition to be locally ϕ-quasiconformally symmetric is to be locally ϕ-symmetric.*

Example 3.4. Suppose $T(TM) = \{TM, \pi, M\}$ is the tangent bundle with $M = R^3$, where $u \in TM$ is defined by $(x^1, x^2, x^3, y^1, y^2, y^3)$. Assume the adapted local frames of $T_u^H TM$ and $T_u^V TM$ are $(\frac{\delta}{\delta x^1}, \frac{\delta}{\delta x^2}, \frac{\delta}{\delta x^3})$ and $(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3})$, respectively. Then the orthonormal frame of $T_u TM$ is

$$E_j = E_j^i \frac{\delta}{\delta x^i} + E_j^i \frac{\partial}{\partial y^i} = E_j^1 \frac{\delta}{\delta x^1} + E_j^2 \frac{\delta}{\delta x^2} + E_j^3 \frac{\delta}{\delta x^3} + \tilde{E}_j^1 \frac{\partial}{\partial y^1} + \tilde{E}_j^2 \frac{\partial}{\partial y^2} + \tilde{E}_j^3 \frac{\partial}{\partial y^3}$$

where

$$E_1 = -\frac{\delta}{\delta x^1} - \frac{\partial}{\partial y^1} = E_1^H + E_1^V,$$

$$E_2 = -(x^2)^2 \frac{\delta}{\delta x^2} + x^1 \frac{\delta}{\delta x^3} - (y^2)^2 \frac{\partial}{\partial y^1} + y^1 \frac{\partial}{\partial y^3} = E_2^H + E_2^V,$$

$$E_3 = \frac{\delta}{\delta x^3} + \frac{\partial}{\partial y^3} = E_3^H + E_3^V = \xi.$$

Let $\eta = \tilde{\eta}_i dx^i + \eta_a \delta y^a = \eta_1 dx^1 + \eta_2 dx^2 + \eta_3 dx^3 + \tilde{\eta}_1 \delta y^1 + \tilde{\eta}_2 \delta y^2 + \tilde{\eta}_3 \delta y^3 = \eta^H + \eta^V$ be defined by $\eta = \frac{x^1}{(x^2)^2} dx^2 + dx^3 - \frac{y^1}{(y^2)^2} \delta y^2 + \delta y^3$.

Suppose that $\phi = \phi^H + \phi^V$ is a tensor field such that its coefficients are tensor fields ϕ^H and ϕ^V with the type of $(1,1)$. Their matrix forms are:

$$\phi^H = \begin{bmatrix} 0 & -\frac{1}{(x^2)^2} & 0 \\ (x^2)^2 & 0 & 0 \\ -x^1 & 0 & 0 \end{bmatrix} \text{ and } \phi^V = \begin{bmatrix} 0 & -\frac{1}{(y^2)^2} & 0 \\ (y^2)^2 & 0 & 0 \\ -y^1 & 0 & 0 \end{bmatrix}.$$

The Sasaki-Finsler metric is defined by the matrix forms:

$$G^H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1+(x^1)^2}{(x^2)^4} & \frac{x^1}{(x^2)^2} \\ 0 & \frac{x^1}{(x^2)^2} & 1 \end{bmatrix} \text{ and } G^V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1+(y^1)^2}{(y^2)^4} & \frac{y^1}{(y^2)^2} \\ 0 & \frac{y^1}{(y^2)^2} & 1 \end{bmatrix}.$$

It is possible to construct Sasakian Finsler manifolds on both horizontal and vertical distributions. In this example, it is shown that 3-dimensional TM^H admits the Sasakian Finsler structure $(\phi^H, \xi^H, \eta^H, G^H)$.

We calculate

$$\phi^H(\xi^H) = 0, \phi^H(E_1^H) = -E_2^H, \phi^H(E_2^H) = E_1^H,$$

so relation (1.2) is satisfied. Similarly, (1.3) holds. Also it is possible to see that

$$\phi^H(\phi^H(Z^H) = -a_1E_1^H - b_1E_2^H = -Z^H + \eta^H(Z^H)\xi^H,$$

for any $Z^H = a_1E_1^H + b_1E_2^H + c_1E_3^H \in T_u^H TM$. Hence, it is shown that (1.1) is true.

If $\eta^H(\phi Z^H) = 0$, then (1.4) is satisfied. Thus, (ϕ^H, ξ^H, η^H) is an almost contact Finsler structure on TM^H.

Due to

$$\eta^H(Z^H) = c_1 = G^H(Z^H, \xi^H)$$

for any $Z^H \in T_u^H TM$, thus (1.7) holds.

Because of

$$G^H(\phi Z^H, \phi W^H) = a_1a_2 + b_1b_2 = G^H(Z^H, W^H) - \eta^H(Z^H)\eta^H(W^H),$$

it can be seen that (1.6) holds. This implies $(\phi^H, \xi^H, \eta^H, G^H)$ is an almost contact Finsler metric structure.

On the other hand,

$$[E_1^H, E_2^H] = -E_3^H, [E_1^H, E_3^H] = 0, [E_2^H, E_3^H] = 0.$$

Finsler connection $\nabla = \nabla^H + \nabla^V$ of metric $G = G^H + G^V$ can be expressed by the Koszul formula:

$2G^H(\nabla_X^H Y^H, Z^H) = X^H G^H(Y^H, Z^H) + Y^H G^H(Z^H, X^H) - Z^H G^H(X^H, Y^H) - G^H(X^H, [Y^H, Z^H]) - G^H(Y^H, [X^H, Z^H]) + G^H(Z^H, [X^H, Y^H])$. This yields

$$\nabla_{E_1^H}^H E_3^H = \tfrac{1}{2}E_2^H, \nabla_{E_1^H}^H E_2^H = -\tfrac{1}{2}E_3^H, \nabla_{E_1^H}^H E_1^H = 0,$$

$$\nabla_{E_2^H}^H E_3^H = -\tfrac{1}{2}E_1^H, \nabla_{E_2^H}^H E_2^H = 0, \nabla_{E_2^H}^H E_1^H = \tfrac{1}{2}E_3^H,$$

$$\nabla_{E_3^H}^H E_3^H = 0, \nabla_{E_3^H}^H E_2^H = -\tfrac{1}{2}E_1^H, \nabla_{E_3^H}^H E_1^H = \tfrac{1}{2}E_2^H.$$

In consequence of these calculations,

$$\nabla_Z^H \xi^H = -\tfrac{1}{2}(-a_1E_2^H + b_1E_1^H) = -\tfrac{1}{2}\phi Z^H$$

is satisfied, so (1.12) holds.

Due to

$$(\nabla_Z^H \phi)W^H = \tfrac{1}{2}\{-a_1c_2E_1^H - b_1c_2E_2^H + (a_1a_2 + b_1b_2)E_3^H\} = \tfrac{1}{2}[G^H(Z^H, W^H)\xi^H - \eta^H(W^H)Z^H],$$

it can be seen that (1.13) holds.

Because of

$$\nabla_Z^H \eta^H(W^H) = \tfrac{1}{2}(a_1b_2 - b_1a_2) = \tfrac{1}{2}G^H(Z^H, \phi W^H),$$

(1.5) and (1.8) hold. Hence, $(\phi^H, \xi^H, \eta^H, G^H)$ is a Sasakian Finsler structure on TM^H.

We can verify the following results:

$$R(E_1^H, E_2^H)E_1^H = \tfrac{3}{4}E_2^H, R(E_1^H, E_2^H)E_2^H = -\tfrac{3}{4}E_1^H,$$

$$R(E_1^H, E_2^H)E_3^H = 0, R(E_3^H, E_1^H)E_1^H = \tfrac{1}{4}E_3^H,$$

$$R(E_1^H, E_3^H)E_2^H = 0, R(E_1^H, E_3^H)E_3^H = \tfrac{1}{4}E_1^H, R(E_2^H, E_3^H)E_1^H = 0,$$

$$R(E_2^H, E_3^H)E_2^H = -\tfrac{1}{4}E_3^H, R(E_2^H, E_3^H)E_3^H = \tfrac{1}{4}E_2^H$$

and

$$S(E_1^H, E_1^H) = -\tfrac{1}{2},\ S(E_2^H, E_2^H) = -\tfrac{1}{2},\ S(E_3^H, E_3^H) = \tfrac{1}{2}$$

and also (1.17) holds and we get $r = -\frac{1}{2}$.

Consequently, the scalar curvature r is constant and by virtue of Corollary 3.2 and Corollary 3.3, TM^H is locally ϕ-quasiconformally symmetric. It is possible to verify that TM^V is locally ϕ-quasiconformally symmetric, similarly.

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