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ON KENMOTSU MANIFOLDS ADMITTING A SPECIAL TYPE OF SEMI-SYMMETRIC NON-METRIC $\phi-$ CONNECTION

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Abstract. The object of the present paper is to study a special type of semi-symmetric non-metric ϕ-connection on a Kenmotsu manifold. It is shown that if the curvature tensor of Kenmotsu manifolds admitting a special type of semi-symmetric non-metric ϕ-connection $\bar{\nabla}$ vanishes, then the Kenmotsu manifold is locally isometric to the hyperbolic space $H^n(-1)$. Beside these, we consider Weyl conformal curvature tensor of a Kenmotsu manifold with respect to the semi-symmetric non-metric ϕ-connection. Among other results, we prove that the Weyl conformal curvature tensor with respect to the Levi-Civita connection and the semi-symmetric non-metric ϕ-connection are equivalent. Moreover, we deal with ϕ-Weyl semi-symmetric Kenmotsu manifolds with respect to the semi-symmetric non-metric ϕ-connection. Finally, an illustrative example is given to verify our result.

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1. Introduction

The product of an almost contact manifold M and the real line $\mathbb{R}$ carries a natural almost complex structure. However, if one takes M to be an almost contact metric manifold and suppose that the product metric G on $M \times \mathbb{R}$ is Kaehlerian, then the structure on M is cosymplectic [12] and not Sasakian. On the other hand, Oubina [16] pointed out that if the conformally related metric $e^{2t}G$, t being the coordinates on $\mathbb{R}$, is Kaehlerian, then M is Sasakian and conversely.

In [23], Tanno classified almost contact metric manifolds whose automorphism group possesses the maximum dimension. For such a manifold M, the sectional curvature of plane section containing ξ is a constant, say c. If $c > 0$, M is a homogeneous Sasakian manifold of constant sectional curvature. If $c = 0$,

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M is the product of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature. If $c < 0$, M is a warped product space $\mathbb{R} \times_f \mathbb{C}^n$. In 1972, Kenmotsu [13] abstracted the differential geometric properties of the third case. We call it a Kenmotsu manifold. Any point of a Kenmotsu manifold has a neighborhood isometric to the warped product $(-\epsilon, \epsilon) \times_f V$, where $(-\epsilon, \epsilon)$ is an open interval from $\mathbb{R}$, $f(t) = c \exp t$, $c > 0$ and V is a Kähler manifold [13].

More recently, in [9], almost contact metric manifolds such that η is closed and $d\Phi = 2\eta \wedge \Phi$ were studied and they were called *almost Kenmotsu* manifoolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold.

In 1924, Friedmann and Schouten [10] introduced the idea of a semi-symmetric connection on a differentiable manifold. A linear connection $\widetilde{\nabla}$ on a differentiable manifold M is said to be a semi-symmetric connection if the torsion tensor T of the connection $\widetilde{\nabla}$ satisfies $T(X, Y) = u(Y)X - u(X)Y$, where u is a 1-form and ρ is a vector field defined by $u(X) = g(X, \rho)$, for all vector fields $X, Y \in \chi(M)$. Here $\chi(M)$ denotes the set of all differentiable vector fields on M.

In 1932, Hayden [11] introduced the idea of semi-symmetric metric connections on a Riemannian manifold (M, g). A semi-symmetric connection $\widetilde{\nabla}$ is said to be a semi-symmetric metric connection if $\widetilde{\nabla} g = 0$.

A relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of (M, g) was given by Yano [24]: $\widetilde{\nabla}_X Y = \nabla_X Y + u(Y)X - g(X, Y)\rho$, where $u(X) = g(X, \rho)$.
In 1976, Yano [25] introduced the notion of semi-symmetric metric ϕ-connection in a Sasakian manifold. Semi-symmetric connection $\hat{\nabla}$ satisfying $\hat{\nabla} g \neq 0$, was initiated by Prvanović [20] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [22]. Semi-symmetric connection $\hat{\nabla}$ satisfying $\hat{\nabla} g \neq 0$ is said to be a semi-symmetric non-metric connection. Semi-symmetric non-metric connection have been studied by several authors such as ([6], [21], [27]) and many others.

In 1992, Agashe and Chafle [1] studied a semi-symmetric non-metric connection $\hat{\nabla}$, whose torsion tensor T satisfies $T(X, Y) = u(Y)X - u(X)Y$ and $(\hat{\nabla}_X g)(Y, Z) = -u(Y)g(X, Z) - u(Z)g(X, Y)$. In [15] Barua and Mukhopadhyay studied a type of semi-symmetric connection $\hat{\nabla}$ which satisfies $(\hat{\nabla}_X g)(Y, Z) = 2u(X)g(Y, Z) - u(Y)g(X, Z) - u(Z)g(X, Y)$. Since $\hat{\nabla} g \neq 0$, this is another type of semi-symmetric non-metric connection. However, the authors preferred the name semi-symmetric semimetric connection.

In 1994, Liang [14] studied another type of semi-symmetric non-metric connection $\hat{\nabla}$ for which we have $(\hat{\nabla}_X g)(Y, Z) = 2u(X)g(Y, Z)$, where u is a nonzero 1-form and he called this a semi-symmetric recurrent metric connection. In this paper we introduce a new type of semi-symmetric non-metric ϕ-connection in a Kenmotsu manifold. The paper is organized as follows:

After introduction, in Section 2, we give a brief account of Kenmotsu manifolds. In Section 3, we define a special type of semi-symmetric non-metric ϕ-connection on Kenmotsu manifolds. In section 4 we establish the relation between the curvature tensors with respect to the special type of the semi-symmetric non-metric ϕ-connection and the Levi-Civita connection and prove that if the curvature tensor with respect to the semi-symmetric non-metric ϕ-connection $\bar{\nabla}$ vanishes, then the Kenmotsu manifold is locally isometric to the hyperbolic space $H^n(-1)$. In Section 5 we consider Weyl conformal curvature tensor of a Kenmotsu manifold with respect to the semi-symmetric non-metric ϕ-connection. Among others we prove that the Weyl conformal curvature tensor with respect to the Levi-Civita connection and the semi-symmetric non-metric ϕ-connection are equivalent. Moreover, Section 6 deals with a ϕ-Weyl semi-symmetric Kenmotsu manifold with respect to the semi-symmetric non-metric ϕ-connection. Finally, an illustrative example is given to verify our result.

2. Kenmotsu Manifolds

Let M be an $(2n+1)$-dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and the Riemannian metric g on M satisfying ([4], [5])

$$\eta(\xi) = 1,\ \phi(\xi) = 0,\ \eta(\phi(X)) = 0,\ g(X, \xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = -X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for all vector fields X ,Y on $\chi(M)$. A manifold with the almost contact metric structure (ϕ, ξ, η, g) is an almost Kenmotsu manifold if the following conditions are satisfied

$$d\eta = 0; \quad d\Omega = 2\eta \wedge \Omega,$$

where Ω is the 2-form defined by $\Omega(X, Y) = g(X, \phi Y)$. Any normal almost Kenmotsu manifold is a Kenmotsu manifold. An almost contact metric structure (ϕ, ξ, η, g) is a Kenmotsu manifold [13] if and only if

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X. \tag{2.4}$$

Hereafter we denote the Kenmotsu manifold of dimension $(2n+1)$ by M. From the above relations, it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.5}$$

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.6}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.8}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.9}$$

$$S(X,\xi) = -2n\eta(X), \tag{2.10}$$

where R and S denote the curvature tensor and the Ricci tensor of M, respectively, with respect to the Levi-Civita connection.

Kenmotsu manifolds were studied by many authors such as Pitis [19], De and Pathak [8], Binh et al. [3], Ozgur ([18], [17]) and many others.

Let M be a Kenmotsu manifold. M is said to be an η-Einstein manifold if there exist real valued functions α, β such that $S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$. For $\beta = 0$, the manifold M is an Einstein manifold.

Now we state the following:

Lemma 2.1. *[13] Let M be an η-Einstein Kenmotsu manifold of the form $S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$. If α = constant (or β = constant), then M is an Einstein one.*

3. Semi-symmetric non-metric ϕ-connection on Kenmotsu manifolds

This section deals with a special type of semi-symmetric non-metric ϕ-connection on a Kenmotsu manifold. Let (M^{2n+1}, g) be a Kenmotsu Manifold with the Levi-Civita connection ∇ and we define a linear connection $\bar{\nabla}$ on M by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(Y)X - 2\eta(X)Y + g(X,Y)\xi. \tag{3.1}$$

Using (3.1), the torsion tensor T of M with respect to the connection $\bar{\nabla}$ is given by

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] = \eta(Y)X - \eta(X)Y. \tag{3.2}$$

The linear connection $\bar{\nabla}$ satisfying (3.2) is a semi-symmetric connection.

So the equation (3.1) turns into

$$\begin{aligned}(\bar{\nabla}_X g)(Y,Z) = \bar{\nabla}_X g(Y,Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z)\\ = 4\eta(X)g(Y,Z) \neq 0.\end{aligned} \tag{3.3}$$

The linear connection $\bar{\nabla}$ satisfying (3.2) and (3.3) is called a semi-symmetric non-metric connection.

By making use of (2.1), (2.4) and (3.1), it is obvious that

$$(\bar{\nabla}_X \phi)(Y) = \bar{\nabla}_X \phi Y - \phi(\bar{\nabla}_X Y) = 0. \tag{3.4}$$

The linear connection $\bar{\nabla}$ defined by (3.1) satisfying (3.2), (3.3) and (3.4) is a special type of semi-symmetric non-metric ϕ-connection on Kenmotsu manifolds.

Conversely, we show that a linear connection $\bar{\nabla}$ defined on M satisfying (3.2), (3.3) and (3.4) is given by (3.1). Let H be a tensor field of type $(1,2)$ and

$$\bar{\nabla}_X Y = \nabla_X Y + H(X,Y). \tag{3.5}$$

Then we conclude that

$$T(X,Y) = H(X,Y) - H(Y,X). \tag{3.6}$$

Further using (3.5), it follows that

$$\begin{aligned}(\bar{\nabla}_X g)(Y,Z) = \bar{\nabla}_X g(Y,Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) = &-g(H(X,Y),Z)\\ &-g(Y,H(X,Z)).\end{aligned} \tag{3.7}$$

In view of (3.3) and (3.7) yields

$$g(H(X,Y),Z) + g(Y,H(X,Z)) = -4\eta(X)g(Y,Z). \tag{3.8}$$

Also using (3.8) and (3.6), we derive that

$$\begin{aligned}g(T(X,Y),Z)+g(T(Z,X),Y)+g(T(Z,Y),X) = 2g(H(X,Y),Z)+4\eta(X)g(Y,Z)\\ -4\eta(Y)g(X,Z) - 4\eta(Z)g(X,Y).\end{aligned}$$

The above equation yields

$$\begin{aligned}g(H(X,Y),Z) = &\frac{1}{2}[g(T(X,Y),Z) + g(T(Z,X),Y) + g(T(Z,Y),X)]\\ &-2\eta(X)g(Y,Z) + 2\eta(Y)g(X,Z) + 2\eta(Z)g(X,Y).\end{aligned} \tag{3.9}$$

Let T' be a tensor field of type $(1,2)$ given by

$$g(T'(X,Y),Z) = g(T(Z,X),Y). \tag{3.10}$$

Adding (2.1), (3.2) and (3.10), we obtain

$$T'(X,Y) = \eta(X)Y - g(X,Y)\xi. \tag{3.11}$$

From (3.9) we have by using (3.10) and (3.11)

$$g(H(X,Y),Z) = \frac{1}{2}[g(T(X,Y),Z) + g(T'(X,Y),Z) + g(T'(Y,X),Z)]$$
$$-2\eta(X)g(Y,Z) + 2\eta(Y)g(X,Z) + 2\eta(Z)g(X,Y) = -\eta(Y)g(X,Z)$$
$$-2\eta(X)g(Y,Z) + \eta(Z)g(X,Y). \tag{3.12}$$

Now contracting Z in (3.12) and using (2.1), we obtain that

$$H(X,Y) = -\eta(Y)X - 2\eta(X)Y + g(X,Y)\xi. \tag{3.13}$$

Combining (3.5) and (3.13), it follows that

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(Y)X - 2\eta(X)Y + g(X,Y)\xi.$$

From the above discussions we conclude the following:

Theorem 3.1. *The linear connection $\bar{\nabla}_X Y = \nabla_X Y - \eta(Y)X - 2\eta(X)Y + g(X,Y)\xi$ is a special type of semi-symmetric non-metric ϕ-connection on a Kenmotsu manifold.*

4. Curvature tensor of a Kenmotsu manifold with respect to the semi-symmetric non-metric ϕ-connection

In this section we obtain the expressions of the curvature tensor, Ricci tensor and scalar curvature of M with respect to the semi-symmetric non-metric ϕ-connection defined by (3.1).

Analogous to the definitions of the curvature tensor of M with respect to the Levi-Civita connection ∇, we define the curvature tensor $\bar{R}$ of M with respect to the semi-symmetric non-metric ϕ-connection $\bar{\nabla}$ by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z, \tag{4.1}$$

where $X, Y, Z \in \chi(M)$, the set of all differentiable vector fields on M.

Using (2.1), (2.2) and (3.1) in (4.1), we obtain

$$\bar{R}(X,Y)Z = R(X,Y)Z + (\nabla_Y \eta)(Z)X - (\nabla_X \eta)(Z)Y + \eta(Y)\eta(Z)X$$
$$-\eta(X)\eta(Z)Y. \tag{4.2}$$

By making use of (2.4) and (2.6) in (4.2), we have

$$\bar{R}(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y. \tag{4.3}$$

So the equation (4.3) turns into

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z, \tag{4.4}$$

and

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0. \tag{4.5}$$

We call (4.5) the first Bianchi identity with respect to $\bar{\nabla}$ on Kenmotsu manifolds.

Taking the inner product of (4.3) with U, it follows that

$$\tilde{\bar{R}}(X,Y,Z,U) = \tilde{R}(X,Y,Z,U) + g(Y,Z)g(X,U) - g(X,Z)g(Y,U), \tag{4.6}$$

where $U \in \chi(M)$, $\tilde{\bar{R}}(X,Y,Z,U) = g(\bar{R}(X,Y)Z,U)$ and $\tilde{R}(X,Y,Z,U) = g(R(X,Y)Z,U)$.

Equation (4.6) yields

$$\tilde{\bar{R}}(X,Y,Z,U) = -\tilde{\bar{R}}(X,Y,U,Z).$$

Let $\{e_1, ..., e_{2n+1}\}$ be a local orthonormal basis of the tangent space at a point of the manifold M. Then by putting $X = U = e_i$ in (4.6) and taking summation over i, $1 \leq i \leq 2n+1$ and also using (2.1), we get

$$\bar{S}(Y,Z) = S(Y,Z) + 2ng(Y,Z), \tag{4.7}$$

where $\bar{S}$ and S denote the Ricci tensor of M with respect to $\bar{\nabla}$ and ∇, respectively.

Equation (4.7) implies that

$$\bar{S}(Y,Z) = \bar{S}(Z,Y).$$

Let $\bar{r}$ and r denote the scalar curvature of M with respect to $\bar{\nabla}$ and ∇, respectively, i.e., $\bar{r} = \sum_{i=1}^{2n+1} \bar{S}(e_i,e_i)$ and $r = \sum_{i=1}^{2n+1} S(e_i,e_i)$.

Again let $\{e_1, ..., e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Then by putting $Y = Z = e_i$ in (4.7) and taking summation over i, $1 \leq i \leq 2n+1$ and also using (2.1), it follows that

$$\bar{r} = r + 2n(2n+1).$$

Summing up all of the above equations, we can state the following proposition:

Proposition 4.1. *For a Kenmotsu manifold M with respect to a special type of semi-symmetric non-metric ϕ-connection $\bar{\nabla}$*

(i) The curvature tensor $\bar{R}$ is given by

$$\bar{R}(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y,$$

(ii) The Ricci tensor $\bar{S}$ is given by

$$\bar{S}(Y,Z) = S(Y,Z) + 2ng(Y,Z),$$

(iii) The scalar curvature $\bar{r}$ is given by

$$\bar{r} = r + 2n(2n+1),$$

(iv)$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$*,*
(v)$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0$*,*
(vi) The Ricci tensor $\bar{S}$ is symmetric,
(vii) $\tilde{\bar{R}}(X,Y,Z,U) = -\tilde{\bar{R}}(X,Y,U,Z)$*.*

Definition 4.2. A Kenmotsu manifold with respect to the Levi-Civita connection is of constant curvature if its curvature tensor R is of the form

$$g(R(X,Y)Z,U) = k[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)],$$

where k is a constant.

If $\tilde{\bar{R}} = 0$, then the equation (4.6) turns into

$$\tilde{R}(X,Y,Z,U) = g(X,Z)g(Y,U) - g(Y,Z)g(X,U). \tag{4.8}$$

Therefore, $g(R(X,Y)Z,U) = k[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]$, where $k = -1$. From which it follows that the Kenmotsu manifold with respect to the Levi-Civita connection is of constant curvature -1.

This leads to the following theorem:

Theorem 4.3. *If the curvature tensor of $\bar{\nabla}$ in a Kenmotsu manifold vanishes, then the Kenmotsu manifold is locally isometric to the hyperbolic space $H^n(-1)$.*

Definition 4.4. For each plane p in the tangent space $T_x(M)$, the sectional curvature $K(p)$ is defined by $K(p) = \frac{\tilde{R}(X,Y,X,Y)}{g(X,X)g(Y,Y)-g(X,Y)^2}$, where $\{X,Y\}$ is orthonormal basis for p. Clearly $K(p)$ is the independent of the choice of the orthonormal basis $\{X,Y\}$ [2].

Putting $Z = X$, $U = Y$ in (4.8), we get

$$\tilde{R}(X,Y,X,Y) = [g(X,X)g(Y,Y) - g(X,Y)g(X,Y)].$$

Then from the above equation we conclude that

$$K(p) = \frac{\tilde{R}(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2} = -1.$$

Summing up, we can state the following theorem :

Theorem 4.5. *If in a Kenmotsu manifold the curvature tensor of a special type of semi-symmetric non-metric ϕ-connection $\bar{\nabla}$ vanishes, then the sectional curvature of the plane determined by two vectors $X, Y \in \xi^{\perp}$ is -1.*

Lemma 4.6. *[19] The Kenmotsu manifold M has constant sectional curvature -1 if and only if M is obtained by a concircular structure transformation from $\mathbb{C}^n \times \mathbb{R}$ endowed with the canonical cosymplectic structure.*

Therefore from Theorem 4.5 and Lemma 4.6 we can state the following theorem:

Theorem 4.7. *If in a Kenmotsu manifold the curvature tensor of the special type of semi-symmetric non-metric ϕ-connection $\bar{\nabla}$ vanishes, then the Kenmotsu manifold is obtained by a concircular structure transformation from $\mathbb{C}^n \times \mathbb{R}$ endowed with the canonical cosymplectic structure.*

5. Weyl conformal curvature tensor of a Kenmotsu manifold with respect to the semi-symmetric non-metric ϕ-connection

In a Riemannian manifold Weyl conformal curvature tensor C is defined as follows:

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY] + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{5.1}$$

where R is the Riemannian curvature tensor of type $(1,3)$, the Ricci operator Q is defined by $g(QX,Y) = S(X,Y)$, S is the Ricci tensor of type $(0,2)$ and r denotes the scalar curvature.
Let $\bar{C}$ be the conformal curvature tensor of M with respect to the semi-symmetric non-metric ϕ-connection $\bar{\nabla}$. Then

$$\begin{aligned} \bar{C}(X,Y)Z &= \bar{R}(X,Y)Z - \frac{1}{2n-1}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X \\ &\quad -g(X,Z)\bar{Q}Y] + \frac{\bar{r}}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{5.2}$$

where $\bar{R}$ is the Riemannian curvature tensor of type $(1,3)$, the Ricci operator $\bar{Q}$ is defined by $g(\bar{Q}X,Y) = \bar{S}(X,Y)$, $\bar{S}$ is the Ricci tensor of type $(0,2)$ and $\bar{r}$ denotes the scalar curvature with respect to semi-symmetric non-metric ϕ-connection $\bar{\nabla}$.
An application of Proposition 4.1 in (5.2) yields

$$\bar{C}(X,Y)Z = C(X,Y)Z, \tag{5.3}$$

for all X, Y, Z. Thus the Weyl conformal curvature tensor with respect to the Levi-Civita connection and the semi-symmetric non-metric ϕ-connection are equivalent. Therefore, we conclude the following:

Theorem 5.1. *The Weyl conformal curvature tensor with respect to the Levi-Civita connection and the semi-symmetric non-metric ϕ-connection are equivalent.*

In [3], Binh et al. proved the following:

Proposition 5.2. *Let M be a Kenmotsu manifold. Then the following assertions are equivalent:*
(a) M has constant sectional curvature -1;
(b) M is conformally flat;
(c) M is conformally symmetric;
(d) M is conformally semi-symmetric (i. e. $R.C = 0$);
(e) $R(X, \xi).C = 0$ for any X.

Suppose $\bar{R} = 0$, then from Proposition 4.1 we get $R(X,Y)Z = -\{g(Y,Z)X - g(X,Z)Y\}$. It follows that M is a manifold of constant curvature -1 with respect to the Levi-Civita connection. Then from Proposition 5.2 we conclude that M is conformally flat. Since $\bar{C} = C$, then M is conformally flat with respect to the semi-symmetric non-metric ϕ-connection. Conversely, if $\bar{C} = 0$, then $C = 0$. Hence by Proposition 5.2 M is a manifold of constant curvature -1 with respect to the Levi-Civita connection, i.e., $R(X,Y)Z = -\{g(Y,Z)X - g(X,Z)Y\}$. Again in view of Proposition 4.1 we have $\bar{R} = 0$. Thus we conclude that $C = 0$, $\bar{C} = 0$ and $\bar{R} = 0$ are equivalent. Thus we can state the following:

Theorem 5.3. *Let M be a Kenmotsu manifold. Then with respect to the semi-symmetric non-metric ϕ-connection the following assertions are equivalent:*
(a) M has constant sectional curvature -1;
(b) M is conformally flat ($\bar{C} = 0$);
(c) M is conformally symmetric ($\bar{\nabla}\bar{C} = 0$);
(d) M is conformally semi-symmetric (i. e. $\bar{R}.\bar{C} = 0$);
(e) $\bar{R}(X, \xi).\bar{C} = 0$ for any X.

6. ϕ-Weyl semisymmetric Kenmotsu manifold with respect to the semi-symmetric non-metric ϕ-connection

Definition 6.1. [26] A Riemannian manifold (M^{2n+1}, g), $n > 1$ is said to be ϕ-Weyl semisymmetric if $C(X,Y).\phi = 0$ holds on M.

First we consider ϕ-Weyl semisymmetric Kenmotsu manifolds. Then

$$(C(X,Y).\phi)Z = 0, \tag{6.1}$$

for all X, Y, Z. Putting $Z = \xi$ in (6.1) we have

$$\phi(C(X,Y)\xi) = 0. \tag{6.2}$$

Using (5.1) in (6.2) we get

$$-(1 + \frac{r}{2n})\{\eta(X)\phi Y - \eta(Y)\phi X\} = \eta(Y)\phi QX - \eta(X)\phi QY. \tag{6.3}$$

Putting $X = \xi$ in the above equation we get

$$S(X,Y) = -(1 + \frac{r}{2n})g(X,Y) - (2n - 1 - \frac{r}{2n})\eta(X)\eta(Y). \tag{6.4}$$

Thus in view of the above we can state the following:

Proposition 6.2. *A ϕ-Weyl semi-symmetric Kenmotsu manifold (M^{2n+1}, g), $n > 1$ is an η-Einstein manifold.*

Since $\bar{C} = C$, then $(C(X,Y).\phi)Z = 0$ and $(\bar{C}(X,Y).\phi)Z = 0$ are equivalent. Thus we can state the following:

Theorem 6.3. *A ϕ-Weyl semi-symmetric Kenmotsu manifold (M^{2n+1}, g), $n > 1$ with respect to the semi-symmetric non-metric ϕ-connection is an η-Einstein manifold.*

7. Example of a 5-dimensional Kenmotsu manifold with respect to the semi-symmetric non-metric ϕ-connection

We consider the 5-dimensional smooth manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$.
We choose the vector fields

$$e_1 = e^{-v}\frac{\partial}{\partial x},\;\; e_2 = e^{-v}\frac{\partial}{\partial y},\;\; e_3 = e^{-v}\frac{\partial}{\partial z},\;\; e_4 = e^{-v}\frac{\partial}{\partial u},\;\; e_5 = \frac{\partial}{\partial v},$$

which are linearly independent at each point of M [7].

Let g be the Riemannian metric defined by

$$g(e_i, e_j) = 0,\;\; i \neq j,\;\; i, j = 1, 2, 3, 4, 5$$

and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1.$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, e_5),$$

for any $Z \in \chi(M)$.

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = e_3,\;\; \phi e_2 = e_4,\;\; \phi e_3 = -e_1,\;\; \phi e_4 = -e_2,\;\; \phi e_5 = 0.$$

Using the linearity of ϕ and g, we have

$$\eta(e_5) = 1,$$

$$\phi^2(Z) = -Z + \eta(Z)e_5$$

and

$$g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any $U, Z \in \chi(M)$. Thus, for $e_5 = \xi$, $M(\phi, \xi, \eta, g)$ defines an almost contact metric manifold. The 1-form η is closed.

We have

$$\Omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) = g(\frac{\partial}{\partial x}, \phi\frac{\partial}{\partial z}) = g(\frac{\partial}{\partial x}, -\frac{\partial}{\partial x}) = -e^{2v}.$$

Hence, we obtain $\Omega = -e^{2v} dx \wedge dz$. Thus, $d\Omega = -2e^{2v} dv \wedge dx \wedge dz = 2\eta \wedge \Omega$. Therefore, $M(\phi, \xi, \eta, g)$ is an almost Kenmotsu manifold. It can be seen that $M(\phi, \xi, \eta, g)$ is normal. So, it is a Kenmotsu manifold.

Then we have

$$[e_1, e_2] = [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = 0, [e_1, e_5] = e_1,$$

$$[e_4, e_5] = e_4, [e_2, e_4] = [e_3, e_4] = 0, [e_2, e_5] = e_2, [e_3, e_5] = e_3.$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Taking $e_5 = \xi$ and using the above formula we obtain the following:

$$\nabla_{e_1} e_1 = -e_5, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_1} e_4 = 0, \quad \nabla_{e_1} e_5 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_5, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = 0, \quad \nabla_{e_2} e_5 = e_2,$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = -e_5, \quad \nabla_{e_3} e_4 = 0, \quad \nabla_{e_3} e_5 = e_3,$$

$$\nabla_{e_4} e_1 = 0, \quad \nabla_{e_4} e_2 = 0, \quad \nabla_{e_4} e_3 = 0, \quad \nabla_{e_4} e_4 = -e_5, \quad \nabla_{e_4} e_5 = e_4,$$

$$\nabla_{e_5} e_1 = 0, \quad \nabla_{e_5} e_2 = 0, \quad \nabla_{e_5} e_3 = 0, \quad \nabla_{e_5} e_4 = 0, \quad \nabla_{e_5} e_5 = 0.$$

Further we obtain the following:

$$\bar{\nabla}_{e_i} e_j = 0, \;\; i, j = 1, 2, 3, 4, 5.$$

and hence

$$(\bar{\nabla}_{e_i} \phi) e_j = 0, \;\; i, j = 1, 2, 3, 4, 5.$$

By the above results, we can easily obtain the non-vanishing components of the curvature tensors as follows:

$$R(e_1, e_2)e_2 = R(e_1, e_3)e_3 = R(e_1, e_4)e_4 = R(e_1, e_5)e_5 = -e_1,$$

$$R(e_1, e_2)e_1 = e_2, \; R(e_1, e_3)e_1 = R(e_5, e_3)e_5 = R(e_2, e_3)e_2 = e_3,$$

$$R(e_2, e_3)e_3 = R(e_2, e_4)e_4 = R(e_2, e_5)e_5 = -e_2, R(e_3, e_4)e_4 = -e_3,$$

$$R(e_2, e_5)e_2 = R(e_1, e_5)e_1 = R(e_4, e_5)e_4 = R(e_3, e_5)e_3 = e_5,$$

$$R(e_1, e_4)e_1 = R(e_2, e_4)e_2 = R(e_3, e_4)e_3 = R(e_5, e_4)e_5 = e_4$$

and

$$\bar{R}(e_i, e_j)e_k = 0, \;\; i, j, k = 1, 2, 3, 4, 5.$$

From the components of the curvature tensor of the Kenmotsu manifold it can be easily seen that the manifold is of constant curvature -1. Therefore Theorem 5.1 is verified.

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