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UNIQUENESS OF ENTIRE FUNCTIONS SHARING A SMALL FUNCTION WITH ITS DERIVATIVES

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Abstract. In the paper we study the uniqueness of entire functions sharing a small function with their derivatives. The results of the paper improve the corresponding results of Jank, Mues and Volkman (Complex Variables Theory Appl. 6, 1 (1986), 51–71), Zhong (Kodai Math. J. 18, 2 (1995), 250–259) and Lahiri-Ghosh (Analysis (Munich) 31, 1 (2011), 47–59).

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1. Introduction

In the paper, by meromorphic functions we shall always mean meromorphic functions in the complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [2]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function h, we denote by $T(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$, as $r \to \infty$ and $r \notin E$.

Let f and g be two nonconstant meromorphic functions and let a be a small function of f. We denote by $E(a;f)$ the set of a-pionts of f, where each point is counted according its multiplicity. We denote by $\overline{E}(a;f)$ the reduced form of $E(a;f)$. We say that f, g share a CM, provided that $E(a;f) = E(a;g)$, and we say that f and g share a IM, provided that $\overline{E}(a;f) = \overline{E}(a;g)$.

2. Definitions and Results

We require the following definitions.

Definition 2.1. A meromorphic function $a = a(z)$ is called a small function of f if $T(r,a) = S(r,f)$.

Definition 2.2. For two subsets A and B of $\mathbb{C}$, we denote by $A\Delta B$ the set $(A - B) \cup (B - A)$, which is called the symmetric difference of the sets A and B.

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In 1977, L. A. Rubel and C. C. Yang [8] first investigated the uniqueness of entire functions, which share certain values with their derivatives. They proved the following theorem.

Theorem 2.3. *[8] Let f be a nonconstant entire function. If $E(a; f) = E(a; f^{(1)})$ and $E(b; f) = E(b; f^{(1)})$, for distinct finite complex numbers a and b, then $f \equiv f^{(1)}$.*

In 1979, E. Mues and N. Steinmetz [7] took up the case of IM shared values in the place of CM shared values and proved the following theorem.

Theorem 2.4. *[7] Let f be a nonconstant entire function. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(b; f) = \overline{E}(b; f^{(1)})$, for distinct finite complex numbers a and b, then $f \equiv f^{(1)}$.*

Afterwards in 1986 G. Jank, E. Mues and L. Volkman [3] considered the case of a single shared value by the first two derivatives of an entire function. They proved the following result:

Theorem 2.5. *[3] Let f be a nonconstant entire function and $a(\neq 0)$ be a finite number. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(2)})$, then $f \equiv f^{(1)}$.*

In [11] it was observed by the following example that in Theorem C the second derivative can not be straightway replaced by a higher order derivative.

Example 2.6. Let $(k \geq 3)$ be a positive integer and $w(\neq 1)$ be a root of the algebraic equation $w^{k-1} = 1$. We put $f = e^{wz} + w - 1$, then $E(w; f) = E(w; f^{(1)}) = E(w; f^{(k)})$ but $f \not\equiv f^{(1)}$.

In this context Zhong [11] extended Theorem 2.5 to higher order derivatives and proved the following result.

Theorem 2.7. *[11] Let f be a nonconstant entire function and $a(\neq 0)$ be a finite complex number. If $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})$ for $n(\geq 1)$, then $f \equiv f^{(n)}$.*

For $A \subset \mathbb{C} \cup \{\infty\}$, we denote by $N_A(r, a; f)(\overline{N}_A(r, a; f))$ the counting function (reduced counting function) of those $a-points$ of f which belong to A.

In 2011, I. Lahiri and G. K. Ghosh [4] improved Theorem 2.7 in the following manner.

Theorem 2.8. *[4] Let f be a nonconstant entire function and a be a nonzero finite number. Suppose that $A = \overline{E}(a; f) \setminus \overline{E}(a; f^{(1)})$ and $B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})\}$ for $n(\geq 1)$. If each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity and $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$, then $f = \lambda e^z$ or $f = \lambda e^z + a$, where $\lambda(\neq 0)$ is a constant.*

In the paper we extend Theorem 2.5 and Theorem 2.7 by considering shared small function instead of value sharing also by considering a weaker kind of sharing.

We now state the main result of the paper.

Theorem 2.9. *Let f be a nonconstant entire function and $a = a(z)(\not\equiv 0, \infty)$ be a small function of f such that $a^{(1)} \not\equiv a$. Suppose further that*

(i) $N_{A\cup B}(r, a; f) + N_A(r, a; f^{(1)}) = S(r, f)$, *where* $A = \overline{E}(a; f)\Delta\overline{E}(a; f^{(1)})$ *and* $B = \overline{E}(a; f) \setminus \overline{E}(a; f^{(2)})$,

(ii) $E_{1)}(a; f) \subset \overline{E}(a; f^{(1)})$, *and*

(iii) each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity.

Then $f \equiv f^{(1)}$.

Theorem 2.10. *Let f be a nonconstant entire function and $a = a(z)(\not\equiv 0, \infty)$ be a small function of f such that $a^{(1)} \not\equiv a$. Suppose further that*

(i) $N_{A\cup B}(r, a; f) + N_A(r, a; f^{(1)}) = S(r, f)$, *where* $A = \overline{E}(a; f)\Delta\overline{E}(a; f^{(1)})$, $B = \overline{E}(a; f) \setminus \{\overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})\}$ *and $n \geq 1$ is an integer,*

(ii) $E_{1)}(a; f) \subset \overline{E}(a; f^{(1)})$, *and*

(iii) each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity.

Then $f \equiv f^{(1)} \equiv f^{(n)}$.

Putting $A = B = \emptyset$ in Theorem 2.9 and Theorem 2.10 we respectively obtain the following corollaries.

Corollary 2.11. *Let f be a nonconstant entire function and $a = a(z)(\not\equiv 0, \infty)$ be a small function of f such that $a^{(1)} \not\equiv a$. If $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(2)})$, then $f \equiv f^{(1)}$.*

Corollary 2.12. *Let f be a nonconstant entire function and $a = a(z)(\not\equiv 0, \infty)$ be a small function of f such that $a^{(1)} \not\equiv a$. If $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f) \subset \overline{E}(a; f^{(n)}) \cap \overline{E}(a; f^{(n+1)})$, $n(\geq 1)$ is an integer, then $f \equiv f^{(1)} \equiv f^{(n)}$.*

We note that Corollary 2.11 is an improvement of Theorem 2.5 and Corollary 2.12 is an improvement of Theorem 2.7.

3. Lemmas

In this section we need the following lemmas.

Lemma 3.1. *{[1]; see also [9]} Let f be a meromorphic function and k be a positive integer. Suppose that f is a solution of the following differential equation: $a_0 w^{(k)} + a_1 w^{(k-1)} + \cdots + a_k w = 0$, where $a_0(\neq 0), a_1, a_2, \cdots, a_k$ are constants. Then $T(r, f) = O(r)$. Furthermore, if f is transcendental, then $r = O(T(r, f))$.*

Lemma 3.2. *[1] Let f be a meromorphic function and n be a positive integer. If there exist meromorphic functions* $a_0(\not\equiv 0), a_1, a_2, \cdots, a_n$ *such that*

$$a_0 f^n + a_1 f^{n-1} + \cdots + a_{n-1} f + a_n \equiv 0,$$

then

$$m(r, f) \leq nT(r, a_0) + \sum_{j=1}^{n} m(r, a_j) + (n-1)\log 2.$$

Lemma 3.3. *{[6]; see also p.28[10]} Let f be a nonconstant meromorphic function. If*

$$R(f) = \frac{a_0 f^p + a_1 f^{p-1} + \cdots + a_p}{b_0 f^q + b_1 f^{q-1} + \cdots + b_q}$$

is an irreducible rational function in f with the coefficients being small functions of f and $a_0 b_0 \not\equiv 0$*, then*

$$T(r, R(f)) = max\{p, q\} T(r, f) + S(r, f).$$

Lemma 3.4. *Let f,* $a_0, a_1, a_2, \cdots, a_p, b_0, b_1, b_2, \cdots, b_q$ *be meromorphic functions. If*

$$R(f) = \frac{a_0 f^p + a_1 f^{p-1} + \cdots + a_p}{b_0 f^q + b_1 f^{q-1} + \cdots + b_q} \qquad (a_0 b_0 \not\equiv 0),$$

then

$$T(r, R(f)) = O(T(r, f) + \sum_{i=0}^{p} T(r, a_i) + \sum_{j=0}^{q} T(r, b_j)).$$

Proof. The Lemma follows from the first fundamental theorem and the properties of the characteristic function. □

Lemma 3.5. *{p.68 [2]} Let f be a transcendental meromorphic function and* $f^n P(z) = Q(z)$*, where P(z), Q(z) are differential polynomials generated by f and the degree of Q is at most n. Then* $m(r, P) = S(r, f)$.

Lemma 3.6. *{p.69 [2]} Let f be a nonconstant meromorphic function and*

$$g(z) = f^n(z) + P_{n-1}(f),$$

where $P_{n-1}(f)$ *is a differential polynomial generated by f and of degree at most* $n-1$.

If $N(r, \infty; f) + N(r, 0; g) = S(r, f)$*, then* $g(z) = h^n(z)$*, where* $h(z) = f(z) + \frac{a(z)}{n}$ *and* $h^{n-1}(z)a(z)$ *is obtained by substituting* $h(z)$ *for* $f(z)$*,* $h^{(1)}(z)$ *for* $f^{(1)}(z)$ *etc. in the terms of degree* $n-1$ *in* $P_{n-1}(f)$.

Let us note the special case, where $P_{n-1}(f) = a_0(z)f^{n-1}+$ terms of degree $n-2$ at most. Then $h^{n-1}(z)a(z) = a_0(z)h^{n-1}(z)$ and so $a(z) = a_0(z)$. Hence $g(z) = (f(z) + \frac{a_0(z)}{n})^n$.

Lemma 3.7. *{p.47 [2]} Let f be a nonconstant meromorphic function and a_1,a_2,a_3 be three distinct meromorphic functions satisfying $T(r, a_\mu) = S(r, f)$ for $\mu = 1, 2, 3$. Then*

$$T(r, f) \leq \overline{N}(r, 0; f - a_1) + \overline{N}(r, 0; f - a_2) + \overline{N}(r, 0; f - a_3) + S(r, f).$$

Lemma 3.8. *{p.58, Remark 1 [5]} Let f be a solution of the following homogeneous differential equation*

$$a_n(z)f^{(n)}(z) + a_{n-1}(z)f^{(n-1)}(z) + \cdots + a_1(z)f^{(1)}(z) + a_0(z)f(z) = 0,$$

where the coefficients $a_0(z), \cdots, a_n(z)$ are polynomials and are not all identically equal to zero. Then f is an entire function of finite order.

4. Proof of the theorems

Proof of Thorem 2.9. Let z_0 be a zero of $f - a$ and $f^{(1)} - a$ with multiplicity $q(\geq 2)$, since by hypotheses each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity. Then z_0 is a zero of $f^{(1)} - a^{(1)}$ with multiplicity $q-1$. Hence z_0 is a zero of $a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)$ with multiplicity $q - 1$. Since $q \leq 2(q - 1)$, we have

$$N_{(2}(r, a; f) \leq 2N(r, 0; a - a^{(1)}) + N_A(r, a; f) = S(r, f). \quad (1)$$

Let $\lambda = \frac{f^{(1)}-a}{f-a}$ and $F = f - a$. Then by the hypotheses we get

$$\begin{aligned} N(r, 0; \lambda) + N(r, \infty; \lambda) &\leq N_A(r, 0; f - a) + N_A(r, 0; f^{(1)} - a) \\ &= S(r, f). \end{aligned} \quad (2)$$

Now

$$F^{(1)} = \lambda F + a - a^{(1)} = \lambda F + b, \quad (3)$$

where $b = a - a^{(1)}$. Also

$$\begin{aligned} F^{(2)} &= \lambda F^{(1)} + \lambda^{(1)}F + b^{(1)} \\ &= \lambda(\lambda F + b) + \lambda^{(1)}F + b^{(1)} \\ &= (\lambda^2 + d_1\lambda)F + \lambda b + b^{(1)}, \end{aligned} \quad (4)$$

where $d_1 = \frac{\lambda^{(1)}}{\lambda}$ and $T(r, d_1) = N(r, 0; \lambda) + N(r, \infty; \lambda) + S(r, \lambda) = S(r, f)$ Set

$$\tau = \frac{(a - a^{(1)})(f^{(2)} - a^{(2)}) - (a - a^{(2)})(f^{(1)} - a^{(1)})}{f - a}. \quad (5)$$

Then by the lemma of logarithmic derivative $m(r,\tau) = S(r,f)$. Now by (1) and by hypotheses we get $N(r,\tau) = S(r,f)$ and so $T(r,\tau) = S(r,f)$.

From (5) we get

$$\tau F = (a - a^{(1)})F^{(2)} - (a - a^{(2)})F^{(1)} = bF^{(2)} - (b + b^{(1)})F^{(1)}.$$

Using (3) and (4) we obtain from the above equation

$$\{b\lambda^2 + (bd_1 - b - b^{(1)})\lambda - \tau\}F = b^2(1 - \lambda). \tag{6}$$

If $b\lambda^2 + (bd_1 - b - b^{(1)})\lambda - \tau \not\equiv 0$, then from (6) we get

$$F = -\frac{b^2\lambda - b}{b\lambda^2 + (bd_1 - b - b^{(1)})\lambda - \tau}. \tag{7}$$

Then from (7) we get by Lemma 3.4, $T(r,F) = O(T(r,\lambda)) + S(r,f)$ and also $T(r,f) = T(r,F) + S(r,f) = O(T(r,\lambda)) + S(r,f)$. This implies that $S(r,f)$ is replaceable by $S(r,\lambda)$.

Also from (7) we see that F is a rational function in λ, which can be made irreducible. We put

$$F = \frac{P_l(\lambda)}{Q_{l+1}(\lambda)}, \tag{8}$$

where $P_l(\lambda)$ and $Q_{l+1}(\lambda)$ are relatively prime polynomials in λ of respective degrees l and $l+1$. Also the coefficients of the both the polynomials are small functions of λ. Without loss of generality we assume that $Q_{l+1}(\lambda)$ is a monic polynomial. We further note that the counting function of the common zeros of $P_l(\lambda)$ and $Q_{l+1}(\lambda)$, if any, is $S(r,\lambda)$, because $P_l(\lambda)$ and $Q_{l+1}(\lambda)$ are relatively prime and the coefficients are small functions of λ.

Since $N(r,\infty;F) = S(r,f) = S(r,\lambda)$, we see from (8) that $N(r,0;Q_{l+1}(\lambda))$ $= S(r,\lambda)$. Also by (2) we know that $N(r,\infty;\lambda) = S(r,f) = S(r,\lambda)$. So by Lemma 3.6 we get

$$Q_{l+1}(\lambda) = (\lambda + \frac{c}{l+1})^{l+1}, \tag{9}$$

where c is the coefficient of λ^l in $Q_{l+1}(\lambda)$.

If $c \not\equiv 0$, then by Lemma 3.7 we obtain

$$\begin{aligned} T(r,\lambda) &\leq \overline{N}(r,0;\lambda) + \overline{N}(r,\infty;\lambda) + \overline{N}(r,-\frac{c}{l+1};\lambda) + S(r,\lambda) \\ &= \overline{N}(r,0;Q_{l+1}(\lambda)) + S(r,\lambda) \\ &= S(r,\lambda), \end{aligned}$$

a contradiction. Therefore $c \equiv 0$ and we get from (8) and (9)

$$F = \frac{P_l(\lambda)}{\lambda^{l+1}}. \tag{10}$$

Differentiating (10) we obtain $F^{(1)} = d_1 \frac{\lambda P_l^{(1)}(\lambda) - (l+1)P_l(\lambda)}{\lambda^{l+1}}$. So by Lemma 3.3 we have

$$T(r, F^{(1)}) = (l + 1 - p)T(r, \lambda) + S(r, \lambda) \tag{11}$$

for some integer p, $0 \leq p \leq l$.

Again since $F^{(1)} = \lambda F + b$, where $b = a - a^{(1)} \not\equiv 0$, we get by (10) $F^{(1)} = \frac{P_l(\lambda)}{\lambda^l} + b$ and so by Lemma 3.3 we have

$$T(r, F^{(1)}) = (l - p)T(r, \lambda) + S(r, \lambda), \tag{12}$$

where p is same as in (11). Now from (11) and (12) we get $T(r, \lambda) = S(r, \lambda)$, a contradiction.

If $b\lambda^2 + (bd_1 - b - b^{(1)})\lambda - \tau \equiv 0$, then by (6) and $b \not\equiv 0$ we deduce that $\lambda \equiv 1$. But $\lambda = \frac{f^{(1)} - a}{f - a}$. Therefore, $f^{(1)} - a = f - a$, that implies $f \equiv f^{(1)}$. This proves the theorem. □

Proof of Thorem 2.10. By the first fundamental theorem we get

$$\begin{aligned} T(r, f) &= T(r, f - a) + S(r, f) \\ &= T(r, \frac{1}{f - a}) + S(r, f) \\ &= N(r, 0; f - a) + m(r, 0; f - a) + S(r, f) \\ &\leq N(r, 0; f - a) + m(r, 0; f^{(1)} - a^{(1)}) + S(r, f) \\ &= N(r, 0; f - a) + T(r, f^{(1)}) - N(r, 0; f^{(1)} - a^{(1)}) + S(r, f). \end{aligned} \tag{13}$$

Now by Lemma 3.7 we get
$T(r, f^{(1)}) \leq \overline{N}(r, 0; f^{(1)} - a) + \overline{N}(r, 0; f^{(1)} - a^{(1)}) + \overline{N}(r, \infty; f^{(1)}) + S(r, f^{(1)})$.
Then from (13) we get

$$\begin{aligned} T(r, f) \leq\ & N(r, 0; f - a) + \overline{N}(r, 0; f^{(1)} - a) + \overline{N}(r, 0; f^{(1)} - a^{(1)}) \\ & - N(r, 0; f^{(1)} - a^{(1)}) + S(r, f). \end{aligned} \tag{14}$$

Let us denote by $N_{(k}^{p}(r, 0; G)$ the counting function of zeros of G with multiplicities not less than k and a zero of multiplicity $q (\geq k)$ is counted $q - p$ times, where $p \leq k$.

Now

$$\begin{aligned} & N(r, 0; f - a) + \overline{N}(r, 0; f^{(1)} - a^{(1)}) - N(r, 0; f^{(1)} - a^{(1)}) \\ =\ & \overline{N}(r, 0; f - a) + N_{(2}^{1}(r, 0; f - a) - N_{(2}^{1}(r, 0; f^{(1)} - a^{(1)}) \\ =\ & \overline{N}(r, 0; f - a) + \overline{N}_{(2}(r, 0; f - a) + N_{(3}^{2}(r, 0; f - a) - N_{(2}^{1}(r, 0; f^{(1)} - a^{(1)}) \\ \leq\ & \overline{N}(r, 0; f - a) + N_{(2}^{1}(r, 0; f^{(1)} - a^{(1)}) - N_{(2}^{1}(r, 0; f^{(1)} - a^{(1)}) + S(r, f) \\ =\ & \overline{N}(r, 0; f - a) + S(r, f). \end{aligned}$$

Therefore from (14) we get

$$T(r, f) \leq \overline{N}(r, 0; f - a) + \overline{N}(r, 0; f^{(1)} - a) + S(r, f). \tag{15}$$

Since

$$\begin{aligned} \overline{N}(r,0;f^{(1)}-a) &\le \overline{N}(r,0;f-a)+N_A(r,0;f^{(1)}-a) \\ &= \overline{N}(r,0;f-a)+S(r,f). \end{aligned} \tag{16}$$

Then from (15) and (16) we get

$$T(r,f) \le 2\overline{N}(r,0;f-a)+S(r,f). \tag{17}$$

Since we have

$$\lambda = \frac{f^{(1)}-a}{f-a}. \tag{18}$$

Then $f^{(1)}-a=\lambda f-\lambda a$, so

$$F^{(1)} = \lambda_1 F+\mu_1, \tag{19}$$

where $F=f-a$, $\lambda_1=\lambda$ and $\mu_1=a-a^{(1)}=b$, say. Taking the derivatives of (19) and using (19) repeatedly we get

$$F^{(k)} = \lambda_k F+\mu_k, \tag{20}$$

where $\lambda_{k+1}=\lambda_k^{(1)}+\lambda_1\lambda_k$ and $\mu_{k+1}=\mu_k^{(1)}+\mu_1\lambda_k$ for $k=1,2,\ldots$.

Now we shall prove that $T(r,\lambda)=S(r,f)$. If λ is constant, then obviously $T(r,\lambda)=S(r,f)$. So we suppose that λ is nonconstant. From the hypotheses we get

$$\begin{aligned} N(r,0;\lambda)+N(r,\infty;\lambda) &\le N_A(r,0;f-a)+N_A(r,0;f^{(1)}-a) \\ &= S(r,f). \end{aligned} \tag{21}$$

Put $k=1$ in $\lambda_{k+1}=\lambda_k^{(1)}+\lambda_1\lambda_k$ we get $\lambda_2=\lambda^2+d_1\lambda$ where $d_1=\frac{\lambda^{(1)}}{\lambda}$. Again putting $k=2$ in $\lambda_{k+1}=\lambda_k^{(1)}+\lambda_1\lambda_k$ we get $\lambda_3=\lambda_2^{(1)}+\lambda_1\lambda_2$, so $\lambda_3=\lambda^3+3d_1\lambda^2+d_2\lambda$, where $d_2=d_1^2+d_1^{(1)}$. Similarly $\lambda_4=\lambda_3^{(1)}+\lambda_1\lambda_3=\lambda^4+6d_1\lambda^3+(6d_1^2+3d_1^{(1)}+d_2)\lambda^2+(d_2^{(1)}+d_1d_2)\lambda$. Therefore, in general, we get for $k\ge 2$

$$\lambda_k=\lambda^k+\sum_{j=1}^{k-1}\alpha_j\lambda^j, \tag{22}$$

where $T(r,\alpha_j)=O(\overline{N}(r,0;\lambda)+\overline{N}(r,\infty;\lambda))+S(r,\lambda)=S(r,f)$ for $j=1,\cdots,k-1$.

Again put $k=1$ in $\mu_{k+1}=\mu_k^{(1)}+\mu_1\lambda_k$ and we get $\mu_2=\mu_1^{(1)}+\mu_1\lambda_1=b\lambda+b^{(1)}$. Also putting $k=2$ in $\mu_{k+1}=\mu_k^{(1)}+\mu_1\lambda_k$, we obtain by (22) $\mu_3=\mu_2^{(1)}+\mu_1\lambda_2=b\lambda^{(1)}+b^{(1)}\lambda+b^{(2)}+b(\lambda^2+d_1\lambda)=b\lambda^2+(b^{(1)}+bd_1+b)\lambda+b^{(2)}$. Similarly $\mu_4=b\lambda^3+(5bd_1+b^{(1)})\lambda^2+(b^{(2)}+2bd_1+2bd_1^2+2bd_1^{(1)}+b^{(1)}d_1+$

$b^{(1)})\lambda + b^{(3)}$.
Therefore, in general, for $k \geq 2$

$$\mu_k = \sum_{j=1}^{k-1} \beta_j \lambda^j + b^{(k-1)}, \tag{23}$$

where $T(r, \beta_j) = O(\overline{N}(r, 0; \lambda) + \overline{N}(r, \infty; \lambda)) + S(r, \lambda) = S(r, f)$ for $j = 1, \cdots, k-1$ and $\beta_{k-1} = b$.

Let

$$\Psi = \frac{(a - a^{(n+1)})(f^{(n)} - a^{(n)}) - (a - a^{(n)})(f^{(n+1)} - a^{(n+1)})}{f - a}. \tag{24}$$

Then clearly $m(r, \Psi) = S(r, f)$. Now by (1) and by hypotheses we get $N(r, \Psi) \leq N_{(2}(r, a; f) + N_{A \cup B}(r, a; f) + S(r, f) = S(r, f)$ and so $T(r, \Psi) = S(r, f)$. Using (20),(22),(23) and (24) we get $\Psi F + (a - a^{(n)})F^{(n+1)} + (a^{(n+1)} - a)F^{(n)} \equiv 0$ i.e.,

$$\begin{aligned} \{\Psi + (a - a^{(n)})\lambda^{n+1} + (a^{(n+1)} - a + \alpha_n a - \alpha_n a^{(n)})\lambda^n \\ + (a^{(n+1)} - a^{(n)}) \sum_{j=1}^{n-1} \alpha_j \lambda^j\} F \\ + b^{(n-1)}(a^{(n+1)} - a) + b^{(n)}(a - a^{(n)}) + (a - a^{(n)})\beta_n \lambda^n \\ + (a^{(n+1)} - a^{(n)}) \sum_{j=1}^{n-1} \beta_j \lambda^j \equiv 0. \end{aligned}$$

Let
$\Delta_1 = b^{(n-1)}(a^{(n+1)} - a + b^{(n)}(a - a^{(n)}) + (a - a^{(n)})\beta_n \lambda^n$
$+ (a^{(n+1)} - a^{(n)}) \sum_{j=1}^{n-1} \beta_j \lambda^j$ and
$\Delta_2 = \Psi + (a - a^{(n)})\lambda^{n+1} + (a^{(n+1)} - a + \alpha_n a - \alpha_n a^{(n)})\lambda^n$
$+ (a^{(n+1)} - a^{(n)}) \sum_{j=1}^{n-1} \alpha_j \lambda^j$. Then

$$\Delta_2 F + \Delta_1 \equiv 0 \tag{25}$$

If $\Delta_2 \equiv 0$
i.e.,
$$\Psi + (a - a^{(n)})\lambda^{n+1} + (a^{(n+1)} - a + \alpha_n a - \alpha_n a^{(n)})\lambda^n + (a^{(n+1)} - a^{(n)}) \sum_{j=1}^{n-1} \alpha_j \lambda^j \equiv 0,$$
then by Lemma 3.2 we get $m(r, \lambda) = S(r, f)$. If $a - a^{(n)} \equiv 0$, then we can show that the coefficient of $\lambda^{(n)} = a^{(n+1)} - a + \alpha_n a - \alpha_n a^{(n)} = (a^{(n)})^{(1)} - a + \alpha_n(a - a^{(n)}) = a^{(1)} - a \neq 0$(since by hypothesis $a \not\equiv a^{(1)}$), then also we can apply Lemma3.2 and we get $m(r, \lambda) = S(r, f)$. Therefore by (21) we have $T(r, \lambda) = S(r, f)$.

Next suppose that

$$\Delta_2 \not\equiv 0.$$

Then from (25) we get

$$F = -\frac{\Delta_1}{\Delta_2}. \tag{26}$$

Following the similar argument of the Theorem 2.9 and using (26) we can show that $T(r, \lambda) = S(r, \lambda)$, a contradiction. Therefore we establish that $T(r, \lambda) = S(r, f)$.

Since $T(r, \lambda) = S(r, f)$, we see that $T(r, \lambda_k) + T(r, \mu_k) = S(r, f)$ for $k = 1, 2, \ldots$, where λ_k and μ_k are defined in (20). Let z_0 be a zero of $F = f - a$ such that $z_0 \notin A \cup B$. For $k = n$ we have from (20) $F^{(n)} = \lambda_n F + \mu_n$ and so,

$$f^{(n)} - a^{(n)} = \lambda_n(f - a) + \mu_n. \tag{27}$$

Now at the point z_0 we get $f^{(n)}(z_0) - a^{(n)}(z_0) = \lambda_n(z_0)(f - a)(z_0) + \mu_n(z_0)$ then by hypotheses we get, $a(z_0) = a^{(n)}(z_0) + \mu_n(z_0)$. If $a(z) \not\equiv a^{(n)}(z) + \mu_n(z)$, we get

$$\begin{aligned} \overline{N}(r, a; f) &\leq N_{A\cup B}(r, 0; f - a) + N(r, 0; a - a^{(n)} - \mu_n) + S(r, f) \\ &= S(r, f). \end{aligned}$$

Which contradicts (17).
Therefore

$$a(z) \equiv a^{(n)}(z) + \mu_n(z). \tag{28}$$

Again differentiate (27) and we get $f^{(n+1)} - a^{(n+1)} = \lambda_n^{(1)}(f - a) + \lambda_n(f^{(1)} - a^{(1)}) + \mu_n^{(1)}$. Now at the point z_0 we get $f^{(n+1)}(z_0) - a^{(n+1)}(z_0) = \lambda_n^{(1)}(z_0)(f - a)(z_0) + \lambda_n(z_0)(f^{(1)} - a^{(1)})(z_0) + \mu_n^{(1)}(z_0)$ then by hypotheses $a(z_0) = a^{(n+1)}(z_0) + \lambda_n(z_0)(a(z_0) - a^{(1)}(z_0)) + \mu_n^{(1)}(z_0)$. If $a(z) \not\equiv a^{(n+1)}(z) + \lambda_n(z)(a(z) - a^{(1)}(z)) + \mu_n^{(1)}(z)$, we get

$$\begin{aligned} \overline{N}(r, a; f) &\leq N_{A\cup B}(r, 0; f - a) + N(r, 0; a - a^{(n+1)} - \lambda_n(a - a^{(1)}) - \mu_n^{(1)}) \\ &+ S(r, f) \\ &= S(r, f) \end{aligned}$$

which contradicts (17).
Therefore

$$a(z) \equiv a^{(n+1)}(z) + \lambda_n(z)(a(z) - a^{(1)}(z)) + \mu_n^{(1)}(z). \tag{29}$$

Differentiate (28) and we get

$$a^{(1)}(z) \equiv a^{(n+1)}(z) + \mu_n^{(1)}(z). \tag{30}$$

From (29) and (30) we get $a(z) - a^{(1)}(z) = \lambda_n(z)(a(z) - a^{(1)}(z))$ since $a(z) - a^{(1)}(z) \not\equiv 0$ then from the above we get $\lambda_n(z) \equiv 1$. Putting the value of $\lambda_n(z) \equiv 1$ and $\mu_n(z) = a(z) - a^{(n)}(z)$ in (27) we get

$$f \equiv f^{(n)}. \tag{31}$$

Equation (3.22) can be written in the form

$$\lambda_k = \lambda^k + P_{k-1}[\lambda] \tag{32}$$

where $P_{k-1}[\lambda]$ is a differential polynomial in λ with constant coefficients having degree at most $k-1$ and weight at most k. Also we note that each term of $P_{k-1}[\lambda]$ contains some derivative of λ.

Let (32) be true. Then

$$\begin{aligned} \lambda_{k+1} &= \lambda_k^{(1)} + \lambda_1\lambda_k \\ &= (\lambda^k + P_{k-1}[\lambda])^{(1)} + \lambda(\lambda^k + P_{k-1}[\lambda]) \\ &= \lambda^{k+1} + k\lambda^{k-1}\lambda^{(1)} + (P_{k-1}[\lambda])^{(1)} + \lambda P_{k-1}[\lambda] \\ &= \lambda^{k+1} + P_k[\lambda], \end{aligned}$$

noting that differentiation does not increase the degree of a differential polynomial but increases its weight by 1. So (32) is verified by mathematical induction.

Since $\lambda_n = 1$, we get from (32) for $k = n$

$$\lambda^n + P_{n-1}[\lambda] \equiv 1. \tag{33}$$

By hypotheses we see that λ has no simple pole. If z_1 is a pole of λ with multiplicity $p(\geq 2)$, then z_1 is a pole of $P_{n-1}[\lambda]$ with multiplicity not exceeding $(n-1)p + 1$. Since $np > (n-1)p + 1$, it follows that z_1 is a pole of the left hand side of (33) with multiplicity np, which is impossible. So λ is an entire function. If λ is transcendental, then by Lemma 3.5 we get from (33) that $T(r, \lambda) = S(r, \lambda)$, a contradiction. If λ is a polynomial of degree $d(\geq 1)$, then the left hand side of (33) is a polynomial of degree nd, which is also a contradiction. Therefore λ is a constant. Hence from (32) we obtain $\lambda_k = \lambda^k$ for $k = 1, 2, \ldots$. Since $\lambda_n \equiv 1$, so $\lambda^n \equiv 1$.

We suppose that $\lambda \not\equiv 1$. Since $f \equiv f^{(n)}$ then by Lemma 3.1 we get $T(r, f) = O(r)$ but we have $T(r, a) = S(r, f) = o(T(r, f)) = o(r)$. Since λ is a constant, by a simple calculation we get $\mu_k = \sum\limits_{j=0}^{k-1} b^{(k-1-j)}\lambda^j$ for $k = 1, 2, \ldots$. Put $\mu_n = \sum\limits_{j=0}^{n-1} b^{(n-1-j)}\lambda^j$ in (28) we get

$$a - a^{(n)} = \sum_{j=0}^{n-1} b^{(n-1-j)}\lambda^j. \tag{34}$$

From (34) we get

$$(1-\lambda)a^{(n-1)} + (\lambda - \lambda^2)a^{(n-2)} + \cdots + (\lambda^{n-1} - 1)a \equiv 0. \tag{35}$$

Since $\lambda \not\equiv 1$, by applying the conclusion of Lemma 3.8 to (35) we conclude that $a = a(z)$ is an entire function of finite order. But we have $T(r, a) = o(r)$, by Lemma 3.1 we observe that $a = a(z)$ is a polynomial of degree q, say. Now from (18) we get

$$f^{(1)} = \lambda f + (1-\lambda)a. \tag{36}$$

Differentiating (36) $q+1$ times, we get $f^{(q+2)} = \lambda f^{(q+1)}$ and so

$$f^{(q+1)} = ce^{\lambda z}, \tag{37}$$

where $c(\neq 0)$ is a constant.

If $q+1 > n$ and since $f^{(n)} \equiv f$ then from (37) we have $f^{(q+1)} = (f^{(n)})^{(q+1-n)} = ce^{\lambda z}$, and so $f^{(q+1-n)} \equiv ce^{\lambda z}$. This implies $(f^{(n)})^{(q+1-2n)} = ce^{\lambda z}$ which, in turn, implies $f^{(q+1-2n)} \equiv ce^{\lambda z}$ where $c(\neq 0)$ is a constant. This process will continue untill $q+1-rn < n$ where r is a positive integer. Suppose $q+1-rn = m(< n)$ then $f^{(m)} = ce^{\lambda z}$. Now differentiate this $(n-m)$ times. We get $f^{(n)} = \lambda^{(n-m)}ce^{\lambda z}$. Since $f^{(n)} \equiv f$ we get $f = \lambda^{(n-m)}ce^{\lambda z}$ and $f^{(1)} = \lambda^{(n-m+1)}ce^{\lambda z}$. Put these values of f and $f^{(1)}$ in (36), we get $\lambda^{(n-m+1)}ce^{\lambda z} = \lambda^{(n-m+1)}ce^{\lambda z} + (1-\lambda)a$, which is impossible as $\lambda \neq 1$ and $a \neq 0$.

If $q+1 < n$ then differentiate (37) $n-q-1$ times and we get $f^{(n)} = \lambda^{(n-q-1)}ce^{\lambda z}$ and since $f^{(n)} \equiv f$ we get $f = \lambda^{(n-q-1)}ce^{\lambda z}$. Then from (36) we get $\lambda^{(n-q)}ce^{\lambda z} = \lambda^{(n-q)}ce^{\lambda z} + (1-\lambda)a$, which is impossible as $\lambda \neq 1$ and $a \neq 0$.

Last of all, if $q+1 = n$ then from (37), we get $f^{(n)} = ce^{\lambda z}$ and since $f^{(n)} \equiv f$ we get $f = ce^{\lambda z}$. Then from (36) we arrive at a contradiction. Hence $\lambda \equiv 1$ and from (18) we get $f^{(1)} - a = f - a$ implies $f \equiv f^{(1)}$. This completes the proof of the theorem.

□

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