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GLOBAL EXISTENCE FOR A STRONGLY COUPLED REACTION-DIFFUSION SYSTEMS WITH NONLINEARITIES OF EXPONENTIAL GROWTH

Saïd Benachour[1] and Belgacem Rebiai[2][3]

Abstract. The aim of this study is to construct invariant regions in which we can establish global existence of classical solutions for reaction-diffusion systems with a general full matrix of diffusion coefficients. Our techniques are based on invariant regions and Lyapunov functional methods. The nonlinear reaction term has been supposed to be of exponential growth.

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1. Introduction

In this work, we are interested in global existence of classical solutions to the following reaction-diffusion system

$$\frac{\partial u}{\partial t} - a_{11}\Delta u - a_{12}\Delta v = f(u,v) \quad \text{in} \quad (0,+\infty)\times\Omega, \tag{1.1}$$

$$\frac{\partial v}{\partial t} - a_{21}\Delta u - a_{22}\Delta v = g(u,v) \quad \text{in} \quad (0,+\infty)\times\Omega, \tag{1.2}$$

with the initial conditions:

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x) \quad \text{in} \quad \Omega, \tag{1.3}$$

and the homogeneous boundary conditions:

$$\alpha u + (1-\alpha)\frac{\partial u}{\partial \nu} = 0, \;\; \alpha v + (1-\alpha)\frac{\partial v}{\partial \nu} = 0 \;\text{ on }\; \left((0,+\infty)\times\partial\Omega\right), \tag{1.4}$$

where Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$, $\frac{\partial}{\partial\nu}$ denotes the outward normal derivative on $\partial\Omega$, α is a function of class C^1 on $\partial\Omega$ such that $0 \le \alpha \le 1$ and the diffusion terms $a_{ij},\ i,j = 1,2$ are supposed to be positive constants such that

$$(a_{12}+a_{21})^2 < 4a_{11}a_{22},$$

[1]Institut Elie Cartan , Université Henri Poincaré, F-54506 Vandœuvre-lès-Nancy Cedex, France, e-mail: said.benachour@univ-lorraine.fr
[2]Department of Mathematics and Informatics, LAMIS Laboratory, University of Tebessa, Tebessa 12000, Algeria, e-mail: brebiai@gmail.com
[3]Corresponding author

which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is positive definite. The eigenvalues λ_1 and λ_2 ($\lambda_1 < \lambda_2$) of A are positive. If we put

$$\underline{a} = \min\{a_{11}, a_{22}\} \quad \text{and} \quad \overline{a} = \max\{a_{11}, a_{22}\}$$

then, the positivity of the diffusion terms implies that

$$\lambda_1 < \underline{a} \leq \overline{a} < \lambda_2.$$

We also put

$$\begin{aligned} \Sigma_1 &= \left\{(r,s) \in \mathbb{R}^2 : \quad \mu_2 r \leq s \leq \mu_1 r\right\}, \\ \Sigma_2 &= \left\{(r,s) \in \mathbb{R}^2 : \quad \frac{1}{\mu_2} s \leq r \leq \frac{1}{\mu_1} s\right\}, \end{aligned}$$

where

$$\mu_1 = \frac{a_{21}}{a_{11} - \lambda_1} > 0 > \mu_2 = \frac{a_{21}}{a_{11} - \lambda_2},$$

if we assume without loss of generality that $a_{11} \leq a_{22}$.

We suppose:

(**A1**) f and g are continuously differentiable on $\Sigma_1 \cup \Sigma_2$,

(**A2**) $(-1)^j(f(r,s), g(r,s)) \in \Sigma_j$ and $\mu_i f(r, \mu_i r) = g(r, \mu_i r)$ for all $(r,s) \in \Sigma_i,\ i,j = 1,2\ (j \neq i)$,

(**A3**) $g(r,s) - \mu_j f(r,s) \leq (-1)^j \psi(s - \mu_j r)(g(r,s) - \mu_i f(r,s))$ for all $(r,s) \in \Sigma_i,\ i,j = 1,2\ (j \neq i)$,

where ψ is a nonnegative continuously differentiable function on $[0, +\infty)$ such that there exists a constant $\beta \geq 1$ and $\ell \geq 0$ satisfying $\lim_{\eta \to +\infty} \eta^{\beta-1} \psi(\eta) = \ell$,

(**A4**) $g(r,s) - \mu_j f(r,s) \leq C\varphi((-1)^i(s - \mu_i r)) e^{\alpha(s - \mu_j r)^\beta}$ for all $(r,s) \in \Sigma_i,\ i,j = 1,2\ (j \neq i)$,

where $C > 0$, $\alpha > 0$, β is the same as in (**A3**) and φ is any nonnegative continuously differentiable function on $[0, +\infty)$ such that $\varphi(0) = 0$.

The initial data are assumed to be in Σ where $\Sigma = \Sigma_1$ or $\Sigma = \Sigma_2$.

The present investigation is a continuation of results obtained in [19], where we proved the global existence of classical solutions for systems of the form:

$$\begin{aligned} &\frac{\partial u}{\partial t} - a\Delta u = -f(u,v), \\ &\frac{\partial v}{\partial t} - b\Delta v = g(u,v), \end{aligned}$$

where $a > 0$, $b > 0$ and f, g are nonnegative continuously differentiable functions on $[0, +\infty) \times [0, +\infty)$ satisfying

$$f(0, \eta) = 0,\ g(\xi, \eta) \leq C\varphi(\xi)e^{\alpha\eta^{\beta}} \text{ and } g(\xi, \eta) \leq \psi(\eta)f(\xi, \eta),$$

for some constants $C > 0$, $\alpha > 0$ and $\beta \geq 1$, where φ and ψ are any nonnegative continuously differentiable functions on $[0, +\infty)$ such that

$$\varphi(0) = 0 \text{ and } \lim_{\eta \to +\infty} \eta^{\beta-1}\psi(\eta) = \ell,\ \ell \text{ is a nonnegative constant.}$$

In this study, we will treat the case of a general full matrix of diffusion coefficients and prove that if f and g satisfying (**A1**)-(**A4**), then Σ is an invariant region for problem (1.1)-(1.4). Once the invariant regions are constructed, we demonstrate that for any initial data in Σ satisfying

$$\text{(1.5)} \qquad \|\mu_i u_0 - v_0\|_\infty < \frac{8\lambda_1\lambda_2}{\alpha\beta\ell n(\lambda_1 - \lambda_2)^2},\ \ell > 0 \text{ when } \Sigma = \Sigma_i,\ i = 1, 2,$$

problem (1.1)-(1.4) is equivalent to a problem for which the global existence follows from the technique based on Lyapunov functional method (see, e.g., [2], [5], [10], [11], [13], [16] and [19]).

In [8] J. I. Kanel and M. Kirane proved the global existence of solutions for a strongly coupled reaction-diffusion system with homogeneous Neumann boundary conditions and

$$g(u, v) = -f(u, v) = uv^m, \quad m > 0 \text{ is an odd integer,}$$

under the conditions

- $0 < a_{22} - a_{11} < a_{21}$,
- $0 < a_{12} \ll 1$,
- $|a_{22} - a_{11} + a_{12} - a_{21}| < \dfrac{\gamma_1 + 1}{\gamma_1 C_p}$,

where

$$\gamma_1 = \frac{a_{22} - a_{11} - \sqrt{(a_{22} - a_{11})^2 + 4a_{12}a_{21}}}{2a_{12}}$$

and C_p is the same constant used in the following theorem:

Theorem 1.1 (**Lamberton [15]**). *Let A be the generator of an analytic semigroup in $L^2(\Omega)$ satisfying the estimate*

$$\|e^{tA}w\|_{L^p(\Omega)} \leq \|w\|_{L^p(\Omega)},$$

for $t \geq 0$, $1 \leq p \leq \infty$ and $w \in L^2(\Omega) \cap L^p(\Omega)$. Then for $0 < T < \infty$, $1 < p < \infty$ and $f \in L^p([0, T] \times \Omega) = L^p([0, T]; L^p(\Omega))$, there exists a unique solution $u \in W^{1,p}([0, T]; \Omega)$ such that

$$\frac{du}{dt}(t) - Au(t) = f(t),\ p.p.\ t \in (0, T], u(0) = 0.$$

Moreover, there exists a constant C_p (independent of T and f) such that

$$\|\frac{du}{dt}\|_{L^p([0,T]\times\Omega)} + \|Au\|_{L^p([0,T]\times\Omega)} \le C_p\|f\|_{L^p([0,T]\times\Omega)}.$$

Later they improved their results in [9] where they obtained the global existence under the following assumptions

- $a_{22} < a_{11} + a_{21}$,
- $a_{12} < \varepsilon_0 = \dfrac{a_{11}a_{22}(a_{11}+a_{21}-a_{22})}{a_{11}a_{22}+a_{21}(a_{11}+a_{21}-a_{22})}$ if $a_{11} \le a_{22} < a_{11}+a_{21}$,
- $a_{12} < \min\left\{\dfrac{1}{2}(a_{11}+a_{21}), \varepsilon_0\right\}$ if $a_{22} < a_{11}$,

and

- $|F(v)| \le C_F(1+|v|^{1-\varepsilon}),\quad vF(v) \ge 0$ for all $v \in \mathbb{R}$,

where $C_F > 0$, ε is any constant such that $0 < \varepsilon < 1$ and

$$g(u,v) = -f(u,v) = uF(v).$$

On the same direction, S. Kouachi [12] has proved the global existence of solutions for two-component reaction-diffusion systems with a general full matrix of diffusion coefficients, nonhomogeneous boundary conditions and polynomial growth conditions on the nonlinear terms and he obtained in [11] the global existence of solutions for the same system with homogeneous Neumann boundary conditions and

$$g(u,v) = \rho F(u,v),\quad f(u,v) = -\sigma F(u,v),\quad \rho > 0,\quad \sigma > 0,$$

where

- $F(u,v) \le Ce^{\alpha v^{\beta}}$, $C > 0$, $\alpha > 0$, $0 < \beta \le 1$, when $-\mu_2 > \dfrac{\rho}{\sigma}$,
- $F(u,v) \le Ce^{\alpha u^{\beta}}$, $C > 0$, $\alpha > 0$, $0 < \beta \le 1$, when $-\mu_2 < \dfrac{\rho}{\sigma}$,

under these conditions

- $\|u_0 - \mu_2 v_0\|_\infty < \dfrac{-8\lambda_1\lambda_2\mu_1(\rho+\sigma\mu_2)}{\alpha n\mu_2(\rho+\sigma\mu_1)(\lambda_1-\lambda_2)^2}$, when $-\mu_2 > \dfrac{\rho}{\sigma}$,
- $\|u_0 - \mu_1 v_0\|_\infty < \dfrac{8\lambda_1\lambda_2\mu_2(\rho+\sigma\mu_1)}{\alpha n\mu_1(\rho+\sigma\mu_2)(\lambda_1-\lambda_2)^2}$, when $-\mu_2 < \dfrac{\rho}{\sigma}$.

Many chemical and biological operations are described by reaction diffusion systems with a full matrix of diffusion coefficients. The components $u(t,x)$ and $v(t,x)$ can be represent either chemical concentrations or biological population densities (see, e.g., E. L. Cussler [3] and [4]).

We note that the resolution of the problem (1.1)-(1.4) is quite more difficult. As a consequence of the blow-up examples found in [18], we can prove that there is blow-up of the solutions in finite time for such full systems even though the initial data are regular, the solutions are positive and the nonlinear terms are negative, a structure that ensured the global existence in the diagonal case.

Our goal is to understand how the results of the diagonal case extend to the nondiagonal situation without any additional assumption on the diffusion coefficients with possiblity of growth faster than exponential for the reaction terms under the assumptions (**A1**)-(**A4**). For this purpose, we construct the invariant regions in which we can demonstrate that for any initial data in this regions satisfying (1.5), problem (1.1)-(1.4) is equivalent to a problem for which the global existence follows from the same Lyapunov functional used in [19] when the reactive terms satisfies (**A1**)-(**A4**).

Throughout this work, we denote by $\|\cdot\|_p$, $p \in [1, +\infty)$ the norm in $L^p(\Omega)$ and $\|\cdot\|_\infty$ the norm in $C(\overline{\Omega})$ or $L^\infty(\Omega)$.

2. Local existence and invariant regions

The study of local existence and uniqueness of solutions (u, v) of (1.1)-(1.4) follows from the basic existence theory for parabolic semilinear equations (see, e.g., [1], [6], [7] and [17]). As a consequence, for any initial data in $C(\overline{\Omega})$ or $L^\infty(\Omega)$ there exists a $T^* \in (0, +\infty]$ such that (1.1)-(1.4) has a unique classical solution on $[0, T^*) \times \Omega$. Furthermore, if $T^* < +\infty$, then

$$\lim_{t \uparrow T^*} \left(\|u(t)\|_\infty + \|v(t)\|_\infty\right) = +\infty.$$

Therefore, if there exists a positive constant C such that

$$\|u(t)\|_\infty + \|v(t)\|_\infty \leq C \quad \text{for all} \quad t \in [0, T^*),$$

then $T^* = +\infty$.

Since the initial conditions are in Σ, then under the assumptions (**A1**)-(**A2**), the next proposition says that the classical solution of (1.1)-(1.4) on $[0, T^*) \times \Omega$ remains in Σ for all t in $[0, T^*)$.

Proposition 2.1. *Suppose that the assumptions (**A1**)-(**A2**) are satisfied. Then for any (u_0, v_0) in Σ the classical solution (u, v) of problem* (1.1)-(1.4) *on $[0, T^*) \times \Omega$ remains in Σ for all t in $[0, T^*)$.*

Proof of Proposition 2.1. Let $(x_{i1}, x_{i2})^t$, $i = 1, 2$, be the eigenvectors of the matrix A^t associated with its eigenvalues λ_i, $i = 1, 2$. Multiplying equations (1.1) and (1.2) by x_{i1} and x_{i2} respectively and adding the resulting equations, if we put

$$z_i = x_{i1} u + x_{i2} v, \quad i = 1, 2, \quad \text{in} \quad (0, +\infty) \times \Omega, \tag{2.1}$$

and

$$(-1)^{i+j+1} F_i(z_1, z_2) = x_{i1} f(u, v) + x_{i2} g(u, v), \quad i = 1, 2, \tag{2.2}$$

for all (u,v) in $\Sigma_j,\ j=1\,\text{or}\,j=2$, we get

$$(2.3)\qquad \frac{\partial z_1}{\partial t}-\lambda_1\Delta z_1 = (-1)^j F_1(z_1,z_2)\quad \text{in}\quad (0,+\infty)\times\Omega,$$

$$(2.4)\qquad \frac{\partial z_2}{\partial t}-\lambda_2\Delta z_2 = (-1)^{j+1} F_2(z_1,z_2)\quad \text{in}\quad (0,+\infty)\times\Omega.$$

with the initial conditions:

$$(2.5)\qquad z_i(0,x)=z_i^0(x),\quad i=1,2,\quad \text{in}\quad \Omega,$$

and the homogeneous boundary conditions:

$$(2.6)\qquad \alpha z_i+(1-\alpha)\frac{\partial z_i}{\partial\nu}=0,\quad i=1,2,\quad \text{on}\quad ((0,+\infty)\times\partial\Omega).$$

Since λ_1 and λ_2 are the eigenvalues of the matrix A^t, (1.1)-(1.4) is equivalent to (2.3)-(2.6) and to prove that Σ is an invariant region for system (1.1)-(1.2) it is sufficient to prove that the region

$$(2.7)\qquad \left\{(z_1^0,z_2^0)\in\mathbb{R}^2:\quad z_i^0\geq 0,\quad i=1,2\right\}=[0,+\infty)^2$$

is invariant for (2.3)-(2.4) and there exist some constants x_{ij}, $i,j=1,2$, such that

$$(2.8)\qquad \Sigma=\left\{(u_0,v_0)\in\mathbb{R}^2:\quad z_i^0=x_{i1}u_0+x_{i2}v_0\geq 0,\quad i=1,2\right\}.$$

Since $(x_{i1},x_{i2})^t$ is an eigenvector of A^t associated to the eigenvalue $\lambda_i,\ i=1,2$, we have

$$(a_{11}-\lambda_i)x_{i1}+a_{21}x_{i2}=0,\quad i=1,2.$$

Because, if we choose $-x_{12}=x_{22}=1$, we get

$$x_{i1}u_0+x_{i2}v_0\geq 0,\ i=1,2\Leftrightarrow \mu_2u_0\leq v_0\leq\mu_1u_0,$$

while if we choose $x_{12}=x_{22}=1$, we obtain

$$x_{i1}u_0+x_{i2}v_0\geq 0,\ i=1,2\Leftrightarrow \frac{1}{\mu_2}v_0\leq u_0\leq\frac{1}{\mu_1}v_0.$$

Then (2.8) is proved and (2.1) can be written as

- $z_1=\mu_1u-v$ and $z_2=-\mu_2u+v$ for all $(u,v)\in\Sigma_1$,
- $z_1=-\mu_1u+v$ and $z_2=-\mu_2u+v$ for all $(u,v)\in\Sigma_2$.

Now, to prove that the region $[0,+\infty)^2$ is invariant for (2.3)-(2.4), it suffices to show that $(-1)^{i+j+1}F_i(z_1,z_2)\geq 0$ ($j=1$ or $j=2$) for all (z_1,z_2) such that $z_i=0$ and $z_k\geq 0,\ i,k=1,2$ ($k\neq i$) thanks to the invariant regions method [20]. Using expressions (2.2), we get

- $F_1=-\mu_1f+g$ and $F_2=-\mu_2f+g$ for all $(u,v)\in\Sigma_1$,
- $F_1=-\mu_1f+g$ and $F_2=\mu_2f-g$ for all $(u,v)\in\Sigma_2$.

Since, from (**A2**), we have $F_i(z_1,z_2)=0$ for all (z_1,z_2) such that $z_i=0$ and $z_j\geq 0$ and $F_j(z_1,z_2)\geq 0$ for all (z_1,z_2) in $[0,+\infty)^2$ when $\Sigma=\Sigma_i$, $i,j=1,2$ ($j\neq i$), then we obtain $z_i(t,x)\geq 0,\ i=1,2$ for all $(t,x)\in[0,T^*)\times\Omega$. As a consequence, Σ is an invariant region for (1.1)-(1.2). □

3. Global existence

As the determinant of the linear algebraic system (2.1), with respect to variables u and v, is different from zero, to prove global existence of solutions for (1.1)-(1.4) we need to prove it for (2.3)-(2.6).

Since we can use the same way to treat the two cases relating to $\Sigma = \Sigma_1$ or $\Sigma = \Sigma_2$, we only deal with the first case.

Since, from (**A2**), we have $F_1 \geq 0$, then z_1 satisfies the maximum principle, *i.e.*,

$$\|z_1(t)\|_\infty \leq \|z_1^0\|_\infty \quad \text{for all} \quad t \in [0, T^*).$$

Based on that, the problem of global existence reduces to establish the uniform boundedness of z_2 in $[0, T^*)$. By L^p-regularity theory for parabolic operator (see, e.g., [14]) it follows that it is sufficient to derive a uniform estimate of $\|F_2(z_1, z_2)\|_p$ on $[0, T^*)$ for some $p > \frac{n}{2}$.

The main result is stated in the following key proposition.

Proposition 3.1. *Suppose that the assumptions (**A1**)-(**A4**) and the restriction* (1.5) *are fulfilled. For every classical solution* (z_1, z_2) *of* (2.3)-(2.6) *on* $[0, T^*) \times \Omega$*, define the function*

$$L : t \longmapsto \int_\Omega \left[\delta z_1 + (M - z_1)^{-\gamma} e^{\alpha p (z_2+1)^\beta}\right](x, t)dx,$$

where α*,* β*,* γ*,* δ*,* p *and* M *are positive constants such that*

$$\beta \geq 1, \ \|z_1^0\|_\infty < M < \frac{2\gamma}{\alpha\beta\ell n} \ \text{ and } \ \gamma = \frac{4\lambda_1\lambda_2}{(\lambda_1 - \lambda_2)^2}. \tag{3.1}$$

Then there exists $\delta > 0$ *and* $p > \frac{n}{2}$ *such that*

$$L \quad \text{is nonincreasing on} \quad [0, T^*). \tag{3.2}$$

Before proving this proposition we first need the following lemma.

Lemma 3.2. *Let* (z_1, z_2) *be a solution of* (2.3)-(2.6) *on* $[0, T^*) \times \Omega$*, then under the assumptions (**A1**)-(**A4**), we have*

$$\int_\Omega F_1(z_1(x, t), z_2(x, t))dx \leq -\frac{d}{dt}\int_\Omega z_1(x, t)dx \tag{3.3}$$

and there exists $\delta_1 > 0$ *and* $p > \frac{n}{2}$ *such that*

$$\int_\Omega \left[\alpha p \beta M (z_2 + 1)^{\beta-1} F_2(z_1, z_2) - \gamma F_1(z_1, z_2)\right] e^{\alpha p (z_2+1)^\beta} dx \tag{3.4}$$

$$\leq \delta_1 \int_\Omega F_1(z_1, z_2)dx, \tag{3.5}$$

where α*,* β*,* γ *and* M *are positive constants satisfying* (3.1).

Proof of Lemma 3.2. It suffices to integrate the both sides of (2.3) satisfied by z_1 on Ω, to obtain (3.3).
Now, from (3.1), we get $\frac{n}{2} < \frac{\gamma}{\alpha\beta\ell M}$, so we can choose p such that $\frac{n}{2} < p < \frac{\gamma}{\alpha\beta\ell M}$. According to the assumption (**A3**), we have

$$\left[\alpha p\beta M(z_2+1)^{\beta-1}F_2(z_1,z_2)-\gamma F_1(z_1,z_2)\right]e^{\alpha p(z_2+1)^\beta}$$
$$\leq \left[\alpha p\beta M(z_2+1)^{\beta-1}\psi(z_2)-\gamma\right]e^{\alpha p(z_2+1)^\beta}F_1(z_1,z_2).$$

Since $\alpha p\beta\ell M < \gamma$ and $(\eta+1)^{\beta-1}\psi(\eta)$ goes to ℓ as $\eta \to +\infty$, there exists $\eta_0 > 0$ such that for all $\eta > \eta_0$, we obtain

$$\left[\alpha p\beta M(\eta+1)^{\beta-1}\psi(\eta)-\gamma\right]e^{\alpha p(\eta+1)^\beta}F_1(\xi,\eta) \leq 0.$$

On the other hand, if η is in the compact interval $[0,\eta_0]$, then the continuous function

$$\eta \longmapsto \left[\alpha p\beta M(\eta+1)^{\beta-1}\psi(\eta)-\gamma\right]e^{\alpha p(\eta+1)^\beta}$$

is bounded. Thus (3.4) immediately follows. □

Proof of Proposition 3.1. Differentiating $L(t)$ with respect to t and using the Green formula, one obtains

$$\frac{d}{dt}L(t) = \delta\frac{d}{dt}\int_\Omega z_1(x,t)dx + I + J, \tag{3.6}$$

where

$$\begin{aligned} I \;=\; & \int_{\partial\Omega}\left[\lambda_1\gamma\frac{\partial z_1}{\partial\nu}+\lambda_2\alpha p\beta(M-z_1)(z_2+1)^{\beta-1}\frac{\partial z_2}{\partial\nu}\right] \\ & (M-z_1)^{-\gamma-1}e^{\alpha p(z_2+1)^\beta}ds \\ & -\int_\Omega \left[\lambda_1\gamma(1+\gamma)|\nabla z_1|^2+\alpha p\beta\gamma(\lambda_1+\lambda_2)(M-z_1)(z_2+1)^{\beta-1}\nabla z_1\nabla z_2\right. \\ & \left.+\lambda_2\alpha p\beta(M-z_1)^2\left(\beta-1+\alpha p\beta(z_2+1)^\beta\right)(z_2+1)^{\beta-2}|\nabla z_2|^2\right] \\ & \times(M-z_1)^{-\gamma-2}e^{\alpha p(z_2+1)^\beta}dx, \end{aligned}$$

where ds denotes the $(n-1)$-dimensional surface element and

$$\begin{aligned} J \;=\; & \int_\Omega[\alpha p\beta(M-z_1)(z_2+1)^{\beta-1}F_2(z_1,z_2) \\ & -\gamma F_1(z_1,z_2)](M-z_1)^{-\gamma-1}e^{\alpha p(z_2+1)^\beta}dx. \end{aligned}$$

We now take advantage of (2.6) and $\beta \geq 1$, to obtain that

$$I \leq -\int_\Omega Q(\nabla z_1,\nabla z_2)(M-z_1)^{-\gamma-2}e^{\alpha p(z_2+1)^\beta}dx,$$

where

$$\begin{aligned} &Q(\nabla z_1, \nabla z_2) \\ &= \lambda_1\gamma(1+\gamma)|\nabla z_1|^2 + \alpha p\beta\gamma(\lambda_1+\lambda_2)(M-z_1)(z_2+1)^{\beta-1}\nabla z_1\nabla z_2 \\ &\quad +\lambda_2\left(\alpha p\beta(M-z_1)(z_2+1)^{\beta-1}\right)^2|\nabla z_2|^2 \end{aligned}$$

is a quadratic form with respect to ∇z_1 and ∇z_2. The discriminant of Q is given by

$$D = \gamma\left(\alpha p\beta(M-z_1)(z_2+1)^{\beta-1}\right)^2\left[\gamma(\lambda_1-\lambda_2)^2 - 4\lambda_1\lambda_2\right].$$

From conditions (3.1) we have $Q(\nabla z_1, \nabla z_2) \geq 0$ and consequently

$$I \leq 0. \tag{3.7}$$

Concerning the term J, since $0 \leq z_1 \leq \|z_1^0\|_\infty < M$, we observe that

$$J \leq (M-\|z_1^0\|_\infty)^{-\gamma-1}\int_\Omega\left[\alpha p\beta M(z_2+1)^{\beta-1}F_2(z_1,z_2) - \gamma F_1(z_1,z_2)\right]e^{\alpha p(z_2+1)^\beta}dx.$$

Thanks to (3.4), we get $\delta_1 > 0$ such that

$$J \leq \delta_1(M-\|z_1^0\|_\infty)^{-\gamma-1}\int_\Omega F_1(z_1,z_2)dx.$$

Let $\delta = \delta_1(M-\|z_1^0\|_\infty)^{-\gamma-1}$ and using (3.3), we obtain

$$J \leq -\delta\frac{d}{dt}\int_\Omega z_1(x,t)dx. \tag{3.8}$$

From (3.6)-(3.8), we conclude that

$$\frac{d}{dt}L(t) \leq 0.$$

This concludes the proof of Proposition 3.1. □

We can now establish the main result of this article.

Theorem 3.3. *Under the assumptions (**A1**)-(**A4**), the classical solutions of* (1.1)-(1.4) *with initial data in* Σ_1 *satisfying* (1.5) *are global and uniformly bounded on* $([0,+\infty)\times\Omega)$.

Proof of Theorem 3.3. Let p be the same as in Proposition 3.1. Since $M^{-\gamma} \leq (M-\xi)^{-\gamma}$ for all $\xi \in [0, \|z_1^0\|_\infty]$, it follows that

$$\|F_2(z_1,z_2)\|_p^p = \int_\Omega |F_2(z_1,z_2)|^p dx \leq M^\gamma K^p L(t)$$

where

$$K = \max_{0\leq\xi\leq\|z_1^0\|_\infty}\varphi(\xi).$$

By Proposition 3.1, we deduce

$$\|F_2(z_1, z_2)\|_p^p \leq M^\gamma K^p L(0).$$

Consequently, $F_2(z_1(t,.), z_2(t,.))$ is uniformly bounded in $L^p(\Omega)$ for all $t \in [0, T^*)$ with $p > \frac{n}{2}$. Using the regularity results for solutions of parabolic equations in [14], we conclude that the solutions of the problem (1.1)-(1.4) are uniformly bounded on $[0, +\infty) \times \Omega$. □

Remark 3.4. When ℓ is a nonnegative constant, we can replace the restriction (1.5) by

$$\ell \|\mu_i u_0 - v_0\|_\infty < \frac{8\lambda_1\lambda_2}{\alpha\beta n(\lambda_1 - \lambda_2)^2} \quad \text{when} \quad \Sigma = \Sigma_i, \ i = 1, 2,$$

and we observe that if $\ell = 0$, then the initial conditions in Σ are given arbitrarily.

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References

[1] Amann, H. Dynamic theory of quasilinear parabolic equations. I. Abstract evolution equations. *Nonlinear Anal. 12*, 9 (1988), 895–919.

[2] Barabanova, A. On the global existence of solutions of a reaction-diffusion equation with exponential nonlinearity. *Proc. Amer. Math. Soc. 122*, 3 (1994), 827–831.

[3] Cussler, E. L. *Multicomponent diffusion, Chemical Engineering Monographs, Vol 3.* Elsevier Publishing Scientific Company, Amsterdam, 1976.

[4] Cussler, E. L. *Diffusion, Mass Transfer in Fluid Systems, Second Edition.* Cambridge University Press, 1997.

[5] Haraux, A., and Youkana, A. On a result of K. Masuda concerning reaction-diffusion equations. *Tohoku Math. J. (2) 40*, 1 (1988), 159–163.

[6] Henry, D. *Geometric theory of semilinear parabolic equations*, vol. 840 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin-New York, 1981.

[7] Hollis, S. L., Martin, Jr., R. H., and Pierre, M. Global existence and boundedness in reaction-diffusion systems. *SIAM J. Math. Anal. 18*, 3 (1987), 744–761.

[8] Kanel, J. I., and Kirane, M. Pointwise a priori bounds for a strongly coupled system of reaction-diffusion equations with a balance law. *Math. Methods Appl. Sci. 21*, 13 (1998), 1227–1232.

[9] Kanel, J. I., Kirane, M., and Tatar, N.-E. Pointwise a priori bounds for a strongly coupled system of reaction-diffusion equations. *Int. J. Differ. Equ. Appl. 1*, 1 (2000), 77–97.

[10] Kirane, M., and Kouachi, S. Global solutions to a system of strongly coupled reaction-diffusion equations. *Nonlinear Anal. 26*, 8 (1996), 1387–1396.

[11] Kouachi, S. Global existence of solutions in invariant regions for reaction-diffusion systems with a balance law and a full matrix of diffusion coefficients. *Electron. J. Qual. Theory Differ. Equ.* (2003), No. 4, 10.

[12] Kouachi, S. Invariant regions and global existence of solutions for reaction-diffusion systems with a full matrix of diffusion coefficients and nonhomogeneous boundary conditions. *Georgian Math. J. 11*, 2 (2004), 349–359.

[13] Kouachi, S., and Youkana, A. Global existence for a class of reaction-diffusion systems. *Bull. Polish Acad. Sci. Math. 49*, 3 (2001), 303–308.

[14] Ladyženskaja, O. A., Solonnikov, V. A., and Ural'ceva, N. N. *Linear and quasilinear equations of parabolic type.* Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[15] Lamberton, D. équations d'évolution linéaires associées à des semi-groupes de contractions dans les espaces L^p. *J. Funct. Anal. 72*, 2 (1987), 252–262.

[16] Masuda, K. On the global existence and asymptotic behavior of solutions of reaction-diffusion equations. *Hokkaido Math. J. 12*, 3 (1983), 360–370.

[17] Pazy, A. *Semigroups of linear operators and applications to partial differential equations*, vol. 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[18] Pierre, M., and Schmitt, D. Blowup in reaction-diffusion systems with dissipation of mass. *SIAM J. Math. Anal. 28*, 2 (1997), 259–269.

[19] Rebiai, B., and Benachour, S. d. Global classical solutions for reaction-diffusion systems with nonlinearities of exponential growth. *J. Evol. Equ. 10*, 3 (2010), 511–527.

[20] Smoller, J. *Shock waves and reaction-diffusion equations*, vol. 258 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]*. Springer-Verlag, New York-Berlin, 1983.

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