Novi Sad J. Math. Vol. 48, No. 1, 2018, 143-150
https://doi.org/10.30755/NSJOM.07233

SPHERES AND CIRCLES IN THE TANGENT SPACE AT A POINT ON A RIEMANNIAN MANIFOLD WITH RESPECT TO AN INDEFINITE METRIC

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Abstract. We consider a 3-dimensional Riemannian manifold with an additional circulant structure, whose third power is the identity. This structure is compatible with the metric such that an isometry is induced in any tangent space of the manifold. Further, we consider an associated metric with the Riemannian metric, which is necessarily indefinite. We find equations of a sphere and equations of a circle, which are given with respect to the associated metric, in terms of the Riemannian metric.

AMS Mathematics Subject Classification (2010): 53B30; 53B20; 15B05; 51N20

Key words and phrases: Riemannian manifold; circulant matrix; indefinite metric tensor; sphere equation; time-like; space-like

1. Introduction

The study of Riemannian manifolds with additional structures is a very substantial topic in modern differential geometry. Some of the manifolds are equipped with a structure which satisfies an equation of the third power (for example [2, 4, 3, 10, 13]).

The models of a sphere and the relations between spheres and hyperboloids, between spheres and cones are of certain interest in pseudo-Riemannian geometry. Another current problem is the obtaining of the correspondence between a circle and other quadratic curves (for instance [1, 7, 8, 9, 11, 12]).

The object of the present paper is a 3-dimensional differentiable manifold M equipped with a Riemannian metric g and a tensor q of type $(1,1)$, whose third power is the identity and q acts as an isometry on g. The components of q form a circulant matrix with respect to some basis, i.e. q is a circulant structure. Such a manifold (M, g, q) is defined in [5]. Also, we consider an associated metric f, which is introduced in [6]. The metric f is necessarily indefinite and it determines space-like vectors, isotropic vectors and time-like vectors in the tangent space T_pM at an arbitrary point p on M.

The paper is organized as follows. In Section 2, we recall some necessary facts about the manifold (M, g, q) and about a q-basis of T_pM. Also, we consider the properties of the associated metric f determined by the condition

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$f(u,v) = g(u,qv) + g(qu,v)$. In Section 3, we obtain equations of the spheres, which are given with respect to f, in terms of g. Also we establish that every vector from an orthonormal q-basis of T_pM is an isotropic one with respect to f. In Section 4, we find equations of the circles, which are given with respect to f, in terms of g.

2. Preliminaries

Let M be a 3-dimensional Riemannian manifold equipped with an additional tensor structure q of type $(1,1)$. Let the coordinates of q with respect to some coordinate system form the circulant matrix

$$q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.1}$$

Then we have

$$q^3 = \mathrm{id}. \tag{2.2}$$

Let g be a positive definite metric on M, which satisfies the equality

$$g(qu,qv) = g(u,v), \quad u,\ v \in \mathfrak{X}M. \tag{2.3}$$

Such a manifold (M,g,q) is introduced in [5].

Further u, v, w will stand for arbitrary vectors in the tangent space T_pM.

It is well-known that the norm of every vector u is given by

$$\|u\| = \sqrt{g(u,u)}. \tag{2.4}$$

Having in mind (2.3) and (2.4), for the angle $\varphi = \angle(u,qu)$ we have

$$\cos\varphi = \frac{g(u,qu)}{g(u,u)}\ . \tag{2.5}$$

In [5], for (M,g,q) it is verified that the angle φ is in $\left[0,\frac{2\pi}{3}\right]$. If $\varphi \in \left(0,\frac{2\pi}{3}\right)$, then u form a basis $\{u, qu, q^2u\}$, which is called a q-basis of T_pM. There exists an orthonormal q-basis.

The associated metric f on (M,g,q), determined by

$$f(u,v) = g(u,qv) + g(qu,v), \tag{2.6}$$

is necessarily indefinite [6].

Now, using (2.1), (2.3) and (2.6), we establish that f satisfies the following equalities:

$$f(u,u) = 2g(u,qu), \tag{2.7}$$

$$f(u,qu) = g(u,u) + g(u,qu). \tag{2.8}$$

According to the physical terminology we give the following

Definition 2.1. Let f be the associated metric on (M,g,q). If a vector u satisfies $f(u,u)>0$ (resp. $f(u,u)<0$), then u is space-like (resp. time-like). If a nonzero vector u satisfies $f(u,u)=0$, then u is isotropic.

Next we get

Theorem 2.2. *Let f be the associated metric on (M,g,q) and φ be the angle between u and qu, with respect to g. Then the following propositions hold:*

(i) u is a space-like vector if and only if $\varphi \in \left[0, \frac{\pi}{2}\right)$;

(ii) u is an isotropic vector if and only if $\varphi = \frac{\pi}{2}$;

(iii) u is a time-like vector if and only if $\varphi \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$.

Proof. From (2.4), (2.5) and (2.7) we get $f(u,u) = 2\|u\|^2\cos\varphi$. Having in mind Definition 2.1, the proof follows. □

Obviously, taking into account (2.2) and (2.7), we have

Corollary 2.3. *If u is a space-like (isotropic or time-like) vector, then qu and q^2u are space-like (isotropic or time-like) vectors, respectively.*

3. Spheres with respect to f

Let $\{u, qu, q^2u\}$ be an orthonormal q-basis of T_pM. Let p_{xyz} be a coordinate system such that the vectors u, qu and q^2u are on the axes p_x, p_y and p_z, respectively. So p_{xyz} is an orthonormal coordinate system. If $N(x,y,z)$ is an arbitrary point with a radius vector v, then v is expressed by equality

$$v = xu + yqu + zq^2u. \tag{3.1}$$

The equation of a sphere s of a radius r centered at the origin p, with respect to the associated metric f on (M,g,q), is

$$s: \quad f(v,v) = r^2. \tag{3.2}$$

Bearing in mind that f is an indefinite metric, we have three different options for the constant r^2, which are $r^2>0$, $r^2=0$ and $r^2<0$.

We apply (3.1) into (3.2) and, using (2.7) and (2.8), we obtain the equation of s as follows

$$2xy + 2xz + 2yz = r^2. \tag{3.3}$$

We rotate the coordinate system p_{xyz} into $p_{x'y'z'}$ by substitutions:

$$\begin{aligned} x &= \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{6}}y' + \frac{1}{\sqrt{3}}z' \\ y &= \frac{\sqrt{2}}{\sqrt{3}}y' + \frac{1}{\sqrt{3}}z' \\ z &= -\frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{6}}y' + \frac{1}{\sqrt{3}}z'. \end{aligned} \tag{3.4}$$

Then (3.3) transforms into the following equation of a quadratic surface:

$$x'^2 + y'^2 - 2z'^2 = -r^2. \tag{3.5}$$

We substitute $r^2 = 0$ into (3.5) and we get an equation of a cone:

$$s_0 : \quad x'^2 + y'^2 - 2z'^2 = 0. \tag{3.6}$$

The cone is shown in Figure 1.

Let r satisfy inequality $r^2 > 0$. Then, from (3.5), we have a hyperboloid of two sheets:

$$s_1 : \quad x'^2 + y'^2 - 2z'^2 = -a^2, \qquad a^2 = r^2 > 0. \tag{3.7}$$

For example, a hyperboloid with the equation $x'^2 + y'^2 - 2z'^2 = -2$ is shown in Figure 2.

Let r satisfy inequality $r^2 < 0$. Then, from (3.5), we obtain a hyperboloid of one sheet:

$$s_2 : \quad x'^2 + y'^2 - 2z'^2 = a^2, \quad a^2 = -r^2 > 0. \tag{3.8}$$

For example, a hyperboloid with the equation $x'^2 + y'^2 - 2z'^2 = 1$ is shown in Figure 3.

Therefore, we state the following

Theorem 3.1. *Let f be the associated metric on (M, g, q), $\{u, qu, q^2u\}$ be an orthonormal q-basis of T_pM and p_{xyz} be a coordinate system such that $u \in p_x$, $qu \in p_y$, $q^2u \in p_z$. Then the sphere* (3.2) *has the equation* (3.5) *with respect to the coordinate system $p_{x'y'z'}$, obtained by the rotation* (3.4) *of p_{xyz}. In particular,*

(i) if $r^2 = 0$, then s is a circular double cone s_0 with (3.6)*,*

(ii) if $r^2 > 0$, then s is a circular hyperboloid of two sheets s_1 with (3.7)*,*

(iii) if $r^2 < 0$, then s is a circular hyperboloid of one sheet s_2 with (3.8)*.*

Note. We will say that the surfaces s_0, s_1, and s_2 are produced from the sphere s.

Corollary 3.2. *If the surfaces s_0, s_1 and s_2 are produced from the sphere s, then*

(i) every point on s_0 has an isotropic radius vector,

(ii) every point on s_1 has a space-like radius vector,

(iii) every point on s_2 has a time-like radius vector.

Proof. According to Definition 2.1 and due to (3.2), the statement holds. □

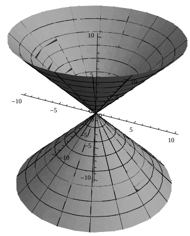

Figure 1: Cone

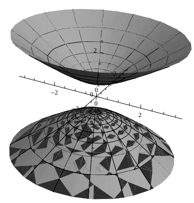

Figure 2: Hyperboloid of two sheets

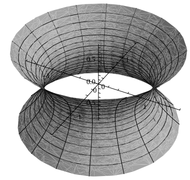

Figure 3: Hyperboloid of one sheet

Theorem 3.3. *Let f be the associated metric on (M, g, q) and $\{u, qu, q^2u\}$ be an orthonormal q-basis of T_pM. If p_{xyz} is a coordinate system such that $u \in p_x$, $qu \in p_y$ and $q^2u \in p_z$, then u, qu and q^2u are isotropic vectors and their heads lie at the circles*

$$x'^2 + y'^2 = \frac{2}{3}, \quad z' = \pm\frac{1}{\sqrt{3}}, \tag{3.9}$$

where $p_{x'y'z'}$ is the coordinate system obtained by the rotation (3.4) *of p_{xyz}.*

Proof. Bearing in mind (2.7) and Definition 2.1 we have that u, qu and q^2u are isotropic vectors with respect to f. Therefore, their heads are on the cone s_0 determined by (3.6). On the other hand, the heads of u, qu and q^2u lie at the unit sphere

$$x^2 + y^2 + z^2 = 1. \tag{3.10}$$

The system of (3.6) and (3.10) implies the intersection of a cone with a sphere. This intersection, with respect to the coordinate system $p_{x'y'z'}$, is represented by

$$x'^2 + y'^2 - 2z'^2 = 0, \quad x'^2 + y'^2 + z'^2 = 1,$$

or by the equivalent system (3.9), which is an intersection of a cylinder with a plane. The resulting curves are two circles. □

4. Circles with respect to *f*

Let α be the 2-plane spanned by vectors u and qu, $(qu \neq u)$. Then, for $\varphi = \angle(u, qu)$ we have $\varphi \in (0, \frac{2\pi}{3}]$. Supposing that $\|u\| = 1$, we define another vector w by the equality

$$w = \frac{1}{\sin\varphi}(-u\cos\varphi + qu), \quad \varphi = \angle(u, qu). \tag{4.1}$$

We construct a coordinate system p_{xy} on α, such that u is on the axes p_x and w is on the axes p_y. Using (2.4), (2.5) and (4.1), we calculate that $g(u, w) = 0$ and $g(w, w) = 1$, i.e. p_{xy} is an orthonormal coordinate system of α.

Let $N(x, y)$ be a point on α and its radius vector is denoted by v. Then v is expressed by the equality

$$v = xu + yw. \tag{4.2}$$

A circle k in α of a radius r centered at the origin p, with respect to the associated metric f on (M, g, q), is determined by

$$k: \quad f(v, v) = r^2. \tag{4.3}$$

The options for r are as follows: $r^2 > 0$, $r^2 = 0$ or $r^2 < 0$.

Theorem 4.1. *Let f be the associated metric on (M, g, q) and $\alpha = \{u, qu\}$ be an arbitrary 2-plane in T_pM. Let the vector w be defined by* (4.1) *and p_{xy} be a coordinate system such that $u \in p_x$, $w \in p_y$. Then the equation of the circle* (4.3) *in α is given by*

$$(\cos\varphi)x^2 + \frac{(1 - \cos\varphi)(1 + 2\cos\varphi)}{\sin\varphi}xy - \frac{\cos^2\varphi}{1 + \cos\varphi}y^2 = \frac{r^2}{2}\,, \tag{4.4}$$

where $\varphi \in (0, \frac{2\pi}{3}]$.

Proof. From (4.2) and (4.3) we get

$$x^2 f(u, u) + 2xyf(u, w) + y^2 f(w, w) = r^2. \tag{4.5}$$

On the other hand, using (2.6), (2.7), (2.8) and (4.1), we calculate

$$f(u,u) = 2\cos\varphi, \quad f(w,w) = -\frac{2\cos^2\varphi}{1+\cos\varphi},$$
$$f(u,w) = \frac{1}{\sin\varphi}(1+\cos\varphi - 2\cos^2\varphi).$$

Then, from (4.5) we obtain (4.4). □

Due to (4.4) we get the following

Corollary 4.2. *The discriminant D of the left side of* (4.4) *is the function*

$$D = \frac{1+3\cos\varphi}{1-\cos\varphi}. \tag{4.6}$$

Corollary 4.3. *The curve* (4.3) *is a hyperbola if and only if φ belongs to the interval $\big(0, \arccos(-\frac{1}{3})\big)$. If $\varphi = \frac{\pi}{2}$, then the hyperbola has the equation $xy = \frac{r^2}{2}$.*

Proof. The condition for (4.4) to be an equation of a hyperbola is $D > 0$. Then the proof follows from (4.6). □

Corollary 4.4. *The curve* (4.3) *has the equation*

$$(\sqrt{2}x - y)^2 = -3r^2 \tag{4.7}$$

if and only if $\varphi = \arccos(-\frac{1}{3})$. In particular,

(i) if $r^2 > 0$, then k hasn't got real points;

(ii) if $r^2 = 0$, then k is a straight line with the equation $y = \sqrt{2}x$;

(iii) if $r^2 < 0$, then k separates into two parallel lines with the equations $\sqrt{2}x - y = \pm\sqrt{-3r^2}$.

Proof. We substitute $\varphi = \arccos(-\frac{1}{3})$ into (4.4) and we obtain (4.7). □

Corollary 4.5. *The curve* (4.3) *is an ellipse if and only if φ belongs to the interval $\big(\arccos(-\frac{1}{3}), \frac{2\pi}{3}\big)$.*

Proof. The condition for (4.4) to be an equation of an ellipse is $D < 0$. Then from (4.6) the proof follows. □

Corollary 4.6. *The curve* (4.3) *has the equation*

$$x^2 + y^2 = -r^2 \tag{4.8}$$

if and only if $\varphi = \frac{2\pi}{3}$. In particular,

(i) if $r^2 > 0$, then k hasn't got real points;

(ii) if $r^2 = 0$, then k is the origin p;

(iii) if $r^2 < 0$, then k is a circle.

Proof. We substitute $\varphi = \frac{2\pi}{3}$ into (4.4) and we get (4.8). □

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Received by the editors December 12, 2017
First published online March 28, 2018