Novi Sad J. Math. Vol. 48, No. 1, 2018, 117-127
https://doi.org/10.30755/NSJOM.07047

GENERALIZED KANTOROVICH SAMPLING TYPE SERIES ON HYPERGROUPS

A. Sathish Kumar[1], **Mahmood Pourgholamhossein**[2] **and Seyyed Mohammad Tabatabaie**[34]

Abstract. In this article, we introduce and study a family of integral operators in the Kantorovich sense acting on functions defined on hypergroups. We obtain the pointwise and uniform convergence results for these operators in the setting of Orlicz spaces with respect to the modular convergence.

AMS Mathematics Subject Classification (2010): 41A35; 46E30; 43A62

Key words and phrases: Kantorovich sampling type operators; Orlicz spaces; hypergroups; pointwise convergence; uniform convergence

1. Introduction and preliminaries

The theory of generalized sampling series have been introduced by P. L. Butzer and his school [7] and [8] (see also [13, 11]). Recently, it is an attractive topic in approximation theory due to its wide range of applications, especially in signal and image processing (see [9, 10, 1]). For $w > 0$, a generalized sampling series of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$(T_w^{\varphi} f)(x) \quad = \quad \sum_{k=-\infty}^{\infty} \varphi(wx - k) f\left(\frac{k}{w}\right), \;\; x \in \mathbb{R},$$

where φ is a kernel function on $\mathbb{R}$. The Kantorovich type generalizations of approximation operators is an important subject in approximation theory and they are the method to approximate Lebesgue integrable functions. The Kantorovich type generalizations of the generalized sampling operators were introduced by P. L. Butzer and his school (see, e.g. [7, 6, 14, 19]).

In [4], the authors have introduced the sampling Kantorovich operators and studied their rate of convergence in the general settings of Orlicz spaces. Also, the nonlinear version of sampling Kantorovich operators has been studied in [12] and [20]. A Voronovskaja type theorem and their quantitative estimate in terms of modulus of smoothness for multivariate extension of Kantorovich

[1]Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur, Nagpur-440010, India, e-mail: mathsatish9@gmail.com
[2]Department of Mathematics, University of Qom, Qom, Iran,
e-mail: m-purghol@qom.ac.ir
[3]Department of Mathematics, University of Qom, Qom, Iran,
e-mail: sm.tabatabaie@qom.ac.ir
[4]Corresponding author

generalized sampling series were studied in [2]. The approximation properties of multivariate Kantorovich-Kotelnikov type operators generated by different band-limited functions are studied in [16]. In particular, the authors [16] considered a wide class of functions with discontinuous Fourier transform. They have also given the L_p-rate of convergence for these operators in terms of the classical moduli of smoothness.

Recently, Vinti and Zampogni introduced and studied the family of integral operators in the Kantorovich sense for functions acting on locally compact topological groups in [21]. They also obtain the convergence results for these operators with respect to the point-wise and uniform convergence and in the setting of Orlicz spaces with respect to the modular convergence. In this paper, we extend the main results in [21] to locally compact hypergroups. Hypergroups, as extensions of locally compact groups, were introduced in the 70's and from that time many researches have worked on harmonic analysis on hypergroups. For convenience of readers, we recall the basic concepts of locally compact hypergroups and Orlicz spaces. For more details about hypergroups we refer to [5] and [15].

1.1. Locally compact hypergroups

Let K be a locally compact Hausdorff space. We denote by $\mathcal{M}(K)$ the space of all regular complex Borel measures on K, by $\mathcal{M}^+(K)$ the space of all nonnegative measures in $\mathcal{M}(K)$, and by δ_x the Dirac measure at the point x. The support of a measure $\mu \in \mathcal{M}(K)$ is denoted by $\mathrm{supp}(\mu)$. We denote by $\mathcal{C}(K)$ the set of all complex-valued continuous functions on K and by $\mathcal{C}_c^+(K)$ the set of all nonnegative compact supported elements of $\mathcal{C}(K)$. The cone topology on $\mathcal{M}(K)$ is the weakest topology such that the mapping $\mu \mapsto \mu(K)$ and for each $f \in \mathcal{C}_c^+(K)$, the mapping $\mu \mapsto \int_K f\,d\mu$ is continuous.

Definition 1.1. A locally compact Hausdorff space, K, together with a bilinear mapping $(\mu,\nu) \mapsto \mu * \nu$ from $\mathcal{M}(K) \times \mathcal{M}(K)$ into $\mathcal{M}(K)$ (called convolution), and an involutive homeomorphism $x \mapsto x^-$ on K (called involution) is called a hypergroup if:

(i) for each $\mu,\nu \in \mathcal{M}^+(K)$, $\mu*\nu \in \mathcal{M}^+(K)$. Also, the mapping $(\mu,\nu) \mapsto \mu*\nu$ from $\mathcal{M}^+(K) \times \mathcal{M}^+(K)$ into $\mathcal{M}^+(K)$ is continuous, where $\mathcal{M}^+(K)$ is equipped with the cone topology.

(ii) $\mathcal{M}(K)$ with $*$ is a complex associative algebra, and for all $\mu,\nu \in \mathcal{M}(K)$ we have

$$\int_K f d(\mu * \nu) = \int_K \int_K \int_K f d(\delta_x * \delta_y) d\mu(x) d\nu(y),$$

where $f \in \mathcal{C}_0(K)$;

(iii) for all $x, y \in K$, $\delta_x * \delta_y$ is a compact supported probability measure;

(iv) there exists an element $e \in K$ (called identity) such that for all $x \in K$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$;

(v) for all $x, y \in K$, $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$, where for each $\mu \in \mathcal{M}(K)$,

$$\mu^-(f) := \int_K f(x^-) d\mu(x), \qquad (f \in \mathcal{C}_0(K)).$$

Also, $e \in \operatorname{supp}(\delta_x * \delta_y)$ if and only if $x = y^-$;

(vi) the mapping $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ from $K \times K$ into $\mathbf{C}(K)$ is continuous, where $\mathbf{C}(K)$ is the space of all non-empty compact subsets of K equipped with Michael topology in which the subbasis is given by

$$\{C_U(V) : \ U \text{ and } V \text{ are open subsets of } K\},$$

where $C_U(V) := \{A \in \mathbf{C}(K) : A \cap U \neq \varnothing \text{ and } A \subseteq V\}$.

Throughout this paper, K is a hypergroup with a left invariant measure m. For each $A, B \subseteq K$ and $x \in K$, we denote

$$A * B := \bigcup_{a \in A, b \in B} \operatorname{supp}(\delta_a * \delta_b), \quad x * A := \{x\} * A \quad \text{and} \quad A * x := A * \{x\}.$$

Also, for each measurable function $f : K \to \mathbb{C}$ and $x, y \in K$, we denote

$$f(x * y) := \int_K f \, d(\delta_x * \delta_y).$$

1.2. Orlicz spaces

A non-decreasing continuous function $\phi : [0, \infty) \to [0, \infty)$ is called a *Young function* if $\phi(0) = 0$, $\phi((0, \infty)) \subseteq (0, \infty)$, and $\lim_{x \to +\infty} \phi(x) = +\infty$.

Let ϕ be a Young function and $\mathcal{B}(K)$ denote the set of all bounded Borel measurable complex-valued functions on K. Then, the Orlicz space $L^\phi(K)$ is defined by

$$L^\phi(K) := \{f \in \mathcal{B}(K) : \int_K \phi(\lambda |f(x)|) \, dm(x) < +\infty \text{ for some } \lambda > 0\}.$$

The vector space $L^\phi(K)$ equipped with the *Luxemburg norm*

$$\|f\|_\phi := \inf\{\lambda > 0 : \int_K \phi(\frac{1}{\lambda} |f(x)|) \, dm(x) < \lambda\}$$

is a Banach space.

We say that a Young function ϕ satisfies in Δ_2-condition whenever for each $f \in \mathcal{B}(K)$, $f \in L^\phi(K)$ if and only if $\int_K \phi(\lambda |f(x)|) \, dm(x) < +\infty$ for every $\lambda > 0$. If ϕ satisfies in Δ_2-condition, then for each sequence $(f_n) \subseteq L^\phi(K)$ and $f \in L^\phi(K)$, f_n converges to f with $\|.\|_\phi$-norm if and only if for some $\lambda > 0$,

$$\lim_{n \to \infty} \int_K \phi\left(\lambda |f_n(t) - f(t)|\right) \, dm(t) = 0.$$

2. Kantorovich Type Operators on Hypergroups

Let H be a hypergroup and K be a commutative hypergroup with left Haar measures m and σ, respectively. Let $\mathcal{B}$ be a (symmetric) neighborhood basis of e in K such that for each $U \in \mathcal{B}$ there exists a set $V \in \mathcal{B}$ such that $V * V \subseteq U$.

Let, for each $w > 0$, there exist a homeomorphism $h_w : H \to h_w(H) \subseteq K$.

We assume that for each $w > 0$, there exists a family $U_w = \{U_w(t)\}_{t \in H}$ of open nonempty subsets of K such that

- (i) $0 < m(U_w(t)) < \infty$ for each $t \in H$ and $w > 0$;
- (ii) for each $w > 0$ and $t \in H$, $h_w(t) \in U_w(t)$.
- (iii) if $B \in \mathcal{B}$, there exists a number $w_0 > 0$ such that for every $w > w_0$ we have $\{h_w(t)\} * U_w(t)^- \subseteq B$, for every $t \in H$.

Let $\{\chi_w\}_{w>0}$ be a family of measurable kernel functionals; i.e., $\chi_w : K \to \mathbb{R}$, $\chi_w \in L^1(K)$ and is bounded in a neighborhood of e $(w > 0)$. We assume that

1. the map $t \mapsto \chi_w(z * h_w(t)^-)$ belongs to $L^1(H)$ for every $z \in K$ and $w > 0$;
2. for every $w > 0$ and $z \in K$,
$$\int_H \chi_w(z * h_w(t)^-)\, d\sigma(t) = 1;$$
3. there exists a constant $M > 0$ such that
$$\sup_{z \in K} \int_H \left|\chi_w(z * h_w(t)^-)\right| \, d\sigma(t) < M < \infty;$$
for every $w > 0$; if $w > 0$, $z \in K$ and $B \in \mathcal{B}$, set
$$B_{z,w} := \{t \in H | \{z\} * \{h_w(t)^-\} \subseteq B\}.$$
Then,
$$\lim_{w \to \infty} \int_{H \setminus B_{z,w}} \left|\chi_w(z * h_w(t)^-)\right| \, d\sigma(t) = 0,$$
uniformly with respect to $z \in K$;
4. for every $\varepsilon > 0$ and compact set $E \subseteq K$, there exists a symmetric compact set $C \subseteq K$ containing e with $m(C) < \infty$ and such that
$$\int_{K \setminus C} \sigma(\mathcal{E}_w) \left|\chi_w(z * h_w(t)^-)\right| \, dm(z) < \varepsilon,$$
for every sufficiently large $w > 0$ and $h_w(t) \in E$, where
$$\mathcal{E}_w := \{t \in H : h_w(t) \in E\} \qquad (w > 0).$$

Let $w > 0$. For each $f \in \mathcal{B}(K)$ we define

$$S_w f(x) := \int_H \int_{U_w(t)} \frac{1}{m(U_w(t))} \chi_w(x * h_w(t)^-) f(u)\, dm(u)\, d\sigma(t), \qquad (x \in K).$$

Theorem 2.1. *If $f \in C_u(K)$, then*

$$\lim_{w\to\infty} \|S_w f - f\|_\infty = 0.$$

Proof. Since $f \in C_u(K)$, for each $\varepsilon > 0$, there is a compact set $U_\varepsilon \in \mathcal{B}$ such that for each $z, u \in K$, if $\mathrm{supp}(\delta_z * \delta_{u^-}) \bigcap U_\varepsilon \neq \varnothing$, then $|f(z) - f(u)| < \varepsilon$. There is a symmetric open set $V \in \mathcal{B}$ such that $V * V \subset U_\varepsilon$.

There exists $w_0 > 0$ such that for all $w > w_0$, $\{h_w(t)\} * U_w(t)^- \subseteq V$, for every $t \in H$. Put

$$B_{z,w} := \{t \in H | \{z\} * \{h_w(t)^-\} \subseteq V\}.$$

For each $t \in B_{z,w}$ and $u \in U_w(t)$ we have

$$\mathrm{supp}(\delta_z * \delta_{u^-}) \subseteq \{z\} * \{h_w(t)^-\} * \{h_w(t)\} * U_w(t) \subseteq V * V \subseteq U_\varepsilon,$$

and so, $|f(z) - f(u)| < \varepsilon$. Thus, for each $w > w_0$,

$$\begin{aligned}
|S_w f(z) - f(z)| &= \left| \int_H \int_{U_w(t)} \chi_w(z * h_w(t)^-)(f(u) - f(z))\, dm(u)\, d\sigma(t) \right| \\
&\leq \int_H \int_{U_w(t)} |\chi_w(z * h_w(t)^-)|\, |f(u) - f(z)|\, dm(u)\, d\sigma(t) \\
&= \int_{B_{z,w}} \int_{U_w(t)} |\chi_w(z * h_w(t)^-)|\, |f(u) - f(z)|\, dm(u)\, d\sigma(t) \\
&+ \int_{H \setminus B_{z,w}} \int_{U_w(t)} |\chi_w(z * h_w(t)^-)|\, |f(u) - f(z)|\, dm(u)\, d\sigma(t) \\
&\leq \varepsilon \int_{B_{z,w}} |\chi_w(z * h_w(t)^-)|\, d\sigma(t) \\
&+ \|f\|_\infty \int_{H \setminus B_{z,w}} |\chi_w(z * h_w(t)^-)|\, d\sigma(t) \\
&\leq \varepsilon M + \|f\|_\infty \int_{H \setminus B_{z,w}} |\chi_w(z * h_w(t)^-)|\, d\sigma(t) \to 0
\end{aligned}$$

uniformly with respect to z, as $w \to \infty$. □

Theorem 2.2. *Let $\phi : \mathbb{R}_0 \to \mathbb{R}_0$ be a convex Young function and $f \in C_c(K)$. Then*

$$\lim_{w\to\infty} \|S_w f - f\|_\phi = 0.$$

Proof. Let $\lambda > 0$, and for each $w > 0$ and $z \in K$, put $g_w(z) := \phi(\lambda|S_w f(z) - f(z)|)$. We show that

$$\int_K g_w(z)\, dm(z) \to 0,$$

as $w \to \infty$.

By Theorem 2.1, $\lim_{w\to\infty} g_w(z) = 0$, uniformly with respect to z.

Let $E_1 := \mathrm{supp}(f)$. Choose a symmetric relatively compact set $B \in \mathcal{B}$, and put $E := B^- * E_1$. So, E is compact, and there is $w_0 > 0$ such that for each $w > w_0$, $h_w(t) * U_w(t)^- \subseteq B$. Hence, $U_w(t) * h_w(t)^- \subseteq B^- = B$. This implies that if $h_w(t) \notin E$, then

$$U_w(t) \subseteq U_w(t) * h_w(t)^- * h_w(t) \subseteq B * h_w(t) \subseteq B * (K - E).$$

But,

$$\begin{aligned}[B * (K - E)] \bigcap E_1 &= \left[B^- * (K - E)\right] \bigcap E_1 \subseteq B^- * \left[(K - E) \bigcap (B^- * E_1)\right] \\ &= \varnothing.\end{aligned}$$

So, $U_w(t) \bigcap E_1 = \varnothing$, for each $w > w_0$, and we have (for $w > w_0$)

$$\int_{U_w(t)} f(u)\, dm(u) = 0.$$

Now, fix $\lambda > 0$, $\varepsilon > 0$ and put

$$\mathcal{E}_w := \{t \in H | h_w(t) \in E\} \quad (w > 0)$$

. There exists a symmetric compact set $C \subseteq K$ containing e such that

$$\int_{K\setminus C} \sigma(\mathcal{E}_w) \left|\chi_w(z * h_w(t)^-)\right| \, dm(z) < \varepsilon,$$

for every sufficiently large $w > 0$ and $h_w(t) \in E$. We have

$$\begin{aligned}|S_w f(z) - f(z)| &= \left| \int_H \int_{U_w(t)} \frac{1}{m(U_w(t))} \chi_w(z * h_w(t)^-) f(u)\, dm(u)\, d\sigma(t) - f(z) \right| \\ &= \left| \int_{\mathcal{E}_w} \int_{U_w(t)} \frac{1}{m(U_w(t))} \chi_w(z * h_w(t)^-) f(u)\, dm(u)\, d\sigma(t) - f(z) \right| \\ &\leq 2\|f\|_\infty \int_{\mathcal{E}_w} |\chi_w(z * h_w(t)^-)|\, d\sigma(t)\end{aligned}$$

There exists a constant $M > 0$ such that

$$\sup_{z\in K} \int_H \left|\chi_w(z * h_w(t)^-)\right| \, d\sigma(t) < M < \infty,$$

for every $w > 0$. Setting

$$k := \frac{1}{M}\int_{\mathcal{E}_w} |\chi_w(z * h_w(t)^-)|\, d\sigma(t),$$

since ϕ is convex we have

$$\begin{aligned}\phi(2\lambda\|f\|_\infty \int_{\mathcal{E}_w} |\chi_w(z * h_w(t)^-)|\, d\sigma(t)) &= \phi(kM2\lambda\|f\|_\infty + (1-k)0)\\ &\le k\phi(M2\lambda\|f\|_\infty) + (1-k)\phi(0)\\ &= \phi(M2\lambda\|f\|_\infty)\frac{1}{M}\int_{\mathcal{E}_w} |\chi_w(z * h_w(t)^-)|\, d\sigma(t).\end{aligned}$$

So,

$$\begin{aligned}\int_{K\setminus C} g_w(z)\, dm(z) &= \int_{K\setminus C} \phi(\lambda|S_w f(z) - f(z)|)\, dm(z)\\ &\le \int_{K\setminus C} \phi\left(2\lambda\|f\|_\infty \int_{\mathcal{E}_w} |\chi_w(z * h_w(t)^-)|\, d\sigma(t)\right) dm(z)\end{aligned}$$

$$\begin{aligned}&\le \int_{K\setminus C} \frac{1}{\sigma(\mathcal{E}_w)M}\int_{\mathcal{E}_w} \sigma(\mathcal{E}_w)\left(\phi(2\lambda M\|f\|_\infty)|\chi_w(z * h_w(t)^-)|\, d\sigma(t)\right) dm(z)\\ &\le \frac{1}{\sigma(\mathcal{E}_w)M}\int_{\mathcal{E}_w}\left(\phi(2\lambda M\|f\|_\infty)\int_{K\setminus C} \sigma(\mathcal{E}_w)|\chi_w(z * h_w(t)^-)|\, dm(z)\right) d\sigma(t)\\ &\le \frac{\varepsilon}{\sigma(\mathcal{E}_w)M}\int_{\mathcal{E}_w} \phi(2\lambda M\|f\|_\infty)\, d\sigma(t) = \frac{\varepsilon\phi(2\lambda M\|f\|_\infty)}{M} < \infty.\end{aligned}$$

Since C is compact, we have $m(C) < \infty$. Also, for every measurable set $A \subseteq C$ we have

$$\int_A \phi(\lambda|S_w f(z) - f(z)|)\, dm(z) \le \int_A \phi(2\lambda M\|f\|_\infty)\, d\sigma(t) = \phi(2\lambda M\|f\|_\infty)m(A).$$

So, for fixed $\varepsilon > 0$, it suffices to take $\delta < \frac{\varepsilon}{\phi(2\lambda M\|f\|_\infty)}$, to obtain

$$\int_A \phi(\lambda|S_w f(z) - f(z)|)\, dm(z) \le \varepsilon,$$

for every measurable set $A \subseteq C$ with $m(A) < \delta$. Finally, applying the Vitali convergence Theorem on C for family $(g_w)_{w>0}$, the proof will be completed. $\square$

For a Young function $\phi : K \to \mathbb{C}$ we denote

$$I_\phi^K(f) := \int_K \phi(|f(t)|)\, dm(t),$$

where $f : K \to \mathbb{C}$ is a measurable bounded function.

Under the following condition, we prove the next result. We assume that there exists a vector subspace $\mathcal{Y} \subseteq L^{\phi}(K)$ such that for every $g \in \mathcal{Y}$,

$$\lim_{w\to\infty} \sup \|\chi_w\|_{L^1(K)} I_\phi^H \left(\int_{B_w(.)} g(z)dm(z) \right) \leq C I_\phi^K(g),$$

for some $C > 0$.

Theorem 2.3. *Let ϕ be a convex ϕ-function and $f \in \mathcal{Y}$. Then, for some $\lambda > 0$ such that*

$$I_\phi^K(\lambda S_w f) \leq \frac{C}{M} I_\phi^K(\lambda M f),$$

for sufficiently large $w > 0$.

Proof. Let $\lambda > 0$ be such that the quantity $I_\phi^K(\lambda M f) < \infty$. Then, we have

$$\begin{aligned} I_\phi^K(\lambda S_w f) &= \int_K \phi(\lambda |S_w f(z)|)\, dm(z) \\ &\leq \int_K \phi\left(\lambda \int_H |\chi_w(z * h_w(t)^-)| \left(\left| \int_{B_w(t)} f(u)dm(u) \right| \right) d\sigma(t) \right) dm(z) \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{M} \int_K \left(\int_H \phi\left(\lambda M \left| \int_{B_w(t)} f(u)dm(u) \right| \right) |\chi_w(z * h_w(t)^-)| d\sigma(t) \right) dm(z) \\ &\leq \frac{1}{M} \left(\int_H \phi\left(\lambda M \left| \int_{B_w(t)} f(u)dm(u) \right| \right) \left(\int_K |\chi_w(z * h_w(t)^-)| d\sigma(t) \right) \right) dm(z) \\ &\leq \frac{1}{M} \|\chi_w\|_{L^1(K)} I_\phi^H \left(\lambda M \left| \int_{B_w(t)} f(u)dm(u) \right| \right) \leq \frac{C}{M} I_\phi^K(\lambda M f), \end{aligned}$$

for sufficiently large $w > 0$. This completes the proof. □

Next, we prove the convergence result in the Orlicz space $L^{\phi}(K)$.

Theorem 2.4. *Let ϕ be a convex ϕ-function and $f \in \mathcal{Y}$. Then there is some $\lambda > 0$, we have*

$$I_\phi^K[\lambda(S_w f - f)] = 0.$$

Proof. Let $f \in L^{\phi}(K)$. By a density result (see, e.g.,[3]), there exists $\mu > 0$ such that for every $\epsilon > 0$ there is a function $g \in C_u(K)$ such that $I_\phi^K(\mu(f - g)) < \epsilon$. Choose $\lambda > 0$ such that $3\lambda(1 + M) < \mu$.
Now, we have

$$\begin{aligned} I_\phi^K[\lambda(S_w f - f)] &\leq I_\phi^K[3\lambda(S_w f - S_w g)] + I_\phi^K[3\lambda(S_w g - g)] + I_\phi^K[3\lambda(f - g)] \\ &\leq \left(1 + \frac{C}{M}\right) \epsilon + I_\phi^K[3\lambda(S_w g - g)], \end{aligned}$$

and the proof follows from the Theorem 2.1, since ϵ is arbitrarily chosen. □

3. Examples

3.1. Orlicz spaces

Assume that the function ϕ_p is defined by

$$\phi_p(u) := |u|^p(c + |\log|u||), \quad (u \in \mathbb{R}).$$

Then, ϕ_p is a Young function for $c \geq \frac{2p-1}{p(p-1)}$. Therefore, $L^{\phi_p}(\mathbb{R})$ is an Orlicz space. Suppose that $p \geq 2$, and the function M_p is defined by

$$M_p(u) := \frac{|u|^p}{\log(e + |u|)}, \quad (u \in \mathbb{R}).$$

Then, M_p is a Young function with Δ_2-condition. Also, for each $1 < p < \infty$, if the function N_p is defined by $N_p(u) := |u|^p/p$, then the Orlicz space $L^{N_p}(\mathbb{R})$ is same as the usual Lebesgue space $L^p(\mathbb{R})$. See [17, 18] for more examples.

3.2. Kantorovich type operator

Ler H and K be the real line. Put $h_w(t) = t/w$, and

$$U_w(t) := \left(\frac{t-1}{w}, \frac{t+1}{w}\right), \quad (w > 0).$$

Fix a function $\chi \in L^1(\mathbb{R})$, and for each $w > 0$ define $\chi_w(t) := \chi(wt)$. Then by [21], $\{\chi_w\}_{w>0}$ are kernel functions and we have

$$S_w f(z) = \frac{w}{2} \int_{\mathbb{R}} \int_{\frac{t-1}{w}}^{\frac{t+1}{w}} \chi(wz - t) f(u) du\, dt.$$

Acknowledgement

We would like to thank the referee for very good reading and useful comments.

References

[1] Asdrubali, F., Baldinelli, G., Bianchi, F., Costarelli, D., Rotili, A., Seracini, M., and Vinti, G. Detection of thermal bridges from thermographic images by means of image processing approximation algorithms. *Appl. Math. Comput. 317* (2018), 160–171.

[2] Bardaro, C., and Mantellini, I. Asymptotic formulae for multivariate Kantorovich type generalized sampling series. *Acta Math. Sin. (Engl. Ser.) 27*, 7 (2011), 1247–1258.

[3] Bardaro, C., and Vinti, G. A general approach to the convergence theorems of generalized sampling series. *Appl. Anal. 64*, 3-4 (1997), 203–217.

[4] Bardaro, C., Vinti, G., Butzer, P. L., and Stens, R. L. Kantorovich-type generalized sampling series in the setting of Orlicz spaces. *Sampl. Theory Signal Image Process. 6*, 1 (2007), 29–52.

[5] Bloom, W. R., and Heyer, H. *Harmonic analysis of probability measures on hypergroups*, vol. 20 of *De Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin, 1995.

[6] Butzer, P. L., and Nessel, R. J. *Fourier analysis and approximation.* Academic Press, New York-London, 1971. Volume 1: One-dimensional theory, Pure and Applied Mathematics, Vol. 40.

[7] Butzer, P. L., and Stens, R. L. Sampling theory for not necessarily band-limited functions: a historical overview. *SIAM Rev. 34*, 1 (1992), 40–53.

[8] Butzer, P. L., and Stens, R. L. Linear prediction by samples from the past. In *Advanced topics in Shannon sampling and interpolation theory*, Springer Texts Electrical Engrg. Springer, New York, 1993, pp. 157–183.

[9] Cluni, F., Costarelli, D., Minotti, A. M., and Vinti, G. Applications of sampling Kantorovich operators to thermographic images for seismic engineering. *J. Comput. Anal. Appl. 19*, 4 (2015), 602–617.

[10] Cluni, F., Costarelli, D., Minotti, A. M., and Vinti, G. Enhancement of thermographic images as tool for structural analysis in earthquake engineering. *NDT and E International 70* (2015), 60–72.

[11] Costarelli, D., Minotti, A. M., and Vinti, G. Approximation of discontinuous signals by sampling Kantorovich series. *J. Math. Anal. Appl. 450*, 2 (2017), 1083–1103.

[12] Costarelli, D., and Vinti, G. Approximation by nonlinear multivariate sampling Kantorovich type operators and applications to image processing. *Numer. Funct. Anal. Optim. 34*, 8 (2013), 819–844.

[13] Costarelli, D., and Vinti, G. Degree of approximation for nonlinear multivariate sampling Kantorovich operators on some functions spaces. *Numer. Funct. Anal. Optim. 36*, 8 (2015), 964–990.

[14] Donnini, C., and Vinti, G. Approximating by means of Kantorovich generalized sampling operators in Musielak-Orlicz spaces. *PanAmer. Math. J. 18*, 2 (2008), 1–18.

[15] Jewett, R. I. Spaces with an abstract convolution of measures. *Advances in Math. 18*, 1 (1975), 1–101.

[16] Kolomoitsev, Y., and Skopina, M. Approximation by multivariate Kantorovich-Kotelnikov operators. *J. Math. Anal. Appl. 456*, 1 (2017), 195–213.

[17] Rao, M. M., and Ren, Z. D. *Theory of Orlicz spaces*, vol. 146 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker, Inc., New York, 1991.

[18] Rao, M. M., and Ren, Z. D. *Applications of Orlicz spaces*, vol. 250 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker, Inc., New York, 2002.

[19] Vinti, G. A general approximation result for nonlinear integral operators and applications to signal processing. *Appl. Anal. 79*, 1-2 (2001), 217–238.

[20] Vinti, G., and Zampogni, L. Approximation by means of nonlinear Kantorovich sampling type operators in Orlicz spaces. *J. Approx. Theory 161*, 2 (2009), 511–528.

[21] Vinti, G., and Zampogni, L. Approximation results for a general class of Kantorovich type operators. *Adv. Nonlinear Stud. 14*, 4 (2014), 991–1011.

Received by the editors November 9, 2017
First published online January 9, 2018