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SEMI-SLANT SUBMERSION FROM AN ALMOST PARA COSYMPLECTIC MANIFOLD

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Abstract. In this paper, we introduce semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold. We obtain some results and investigate the geometry of foliations. Finally, we obtain the necessary and sufficient conditions for a semi-slant submersion to be totally geodesic and harmonic. Also, we provide some examples of such submersions.

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1. Introduction

A differentiable map $f : (M, g_r) \to (N, g_s)$ between Riemannian manifolds (M, g_r) and (N, g_s) is called a Riemannian submersion if f_* is onto and it satisfies

$$g_s(f_*X, f_*Y) = g_r(X, Y)$$

for X, Y vector fields tangent to M. Firstly B. O' Neill [17] and A. Gray [8] studied Riemannian submersions between Riemannian manifolds. Riemannian submersions between Riemannian manifolds equipped with differentiable structures was studied by Watson in [25]. Watson also showed that the base manifold and each fiber have the same kind of structure as the total space, in most cases, see [24] and [25]. Also, there are several kinds of Riemannian submersions, like: slant submersions [21], anti-invariant submersions [23], [6], contact-complex submersions [9], [5], quaternionic submersions [12], H-slant submersions [18], semi-invariant submersions [22], H-semi-invariant submersions [19], paracontact semi-Riemannian submersions [11], locally conformal Kähler submersions [16], hemi-slant submersions [1], para-quaternionic submersions [4]. After that, there are lots of papers on this topic. Riemannian submersions have several applications in mathematical physics [14]. Indeed, Riemannian submersions have their applications in the Yang-Mills theory [2], Kaluza-Klein theory [3] and [13], supergravity and superstring theories [15], etc. Later such submersions were considered between manifolds with differentiable

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structures, see [7]. In [20], K.S. Park and Rajendra Prasad introduced semi-slant submersion from almost Hermitian manifolds onto Riemannian manifolds and in [10], Y. Gunduzalp studied semi-slant submersion from almost product Riemannian manifolds. They showed that such submersions have rich geometric properties and they are useful for investigating the geometry of the total space. We know that a semi-slant submersion is the generalized version of a slant submersion. There are some similarities and differences between slant Riemannian submersions and semi-slant Riemannian submersions. In order to such submersions be harmonic, a semi-slant submersion has much nicer form than a slant submersion. Motivated by the above, we are interested in studying a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold.

In this paper, we define a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold. The paper is organized as follows: In section two, we recall some notions needed for this paper. In section three, we give definition of a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold and we obtain some results and investigate the geometry of foliations. We also investigate the geometry of leaves of the distributions. Finally, we obtain necessary and sufficient condition for the harmonicity of semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold. Also, we provide some examples of such submersions.

2. Preliminaries

In this section, we are going to recall main definitions and several properties of an almost para-cosymplectic manifold and also semi-slant submersion and the results needed for study throughout this paper.

2.1. Almost para-cosymplectic manifold

Let M be an odd dimensional manifold. If there exist on M a $(1,1)$ type tensor field J, a vector field ξ, and 1-form η satisfying

$$J^2 = I - \eta \otimes \xi, \eta(\xi) = 1, rank J = 2n \tag{2.1}$$

$$J\xi = 0,\ \eta \otimes J = 0, \tag{2.2}$$

An almost para-contact manifold is said to be an almost para-contact metric manifold if the Riemannian metric g_r on M satisfies

$$g_r(JX, JY) = g_r(X, Y) - \eta(X)\eta(Y),\ g_r(\xi, X) = \eta(X), \tag{2.3}$$

for all $X, Y \in \Gamma(TM)$. The almost para-contact metric structure (J, ξ, η, g_r) is said to be normal if $[J, J] - 2d\eta \otimes \xi = 0$, where $[J, J]$ is Nijenhuis tensor. The fundamental 2-form Φ on M is defined by $\Phi(X, Y) = g_r(X, JY)$ for any vector fields $X, Y \in \Gamma(TM)$. Using equation (2.1) and (2.2), we have

$$g_r(JX, Y) = g_r(X, JY) \tag{2.4}$$

An almost para-contact metric manifold is said to be an almost para-cosymplectic Riemannian manifold if it has a normal almost para-contact metric structure and both Φ and η are closed, i.e. $d\Phi = 0$ and $d\eta = 0$. Then, the structure equation of a para-cosymplectic manifold (M, J, ξ, η, g_r) is given by

$$(\nabla_X J)Y = 0, \tag{2.5}$$

for any vector fields $X, Y \in \Gamma(TM)$, where ∇ denotes the Levi-Civita connection of g_r on M.

Example 2.1. Let (x_i, y_i, z) be Cartesian coordinates on $\mathbb{R}^{2n+1}$ for $i = 1, ..., n$. An almost para-contact metric structure (J, ξ, η, g_r) on $\mathbb{R}^{2n+1}$ is defined as follows: $g_r = \sum_i^n ((dx_i)^2 + (dy_i)^2 + (dz)^2)$, $J = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ \delta_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\xi = \frac{\partial}{\partial z}, \eta = dz$.

We can easily show that (J, ξ, η, g_r) is an almost para-cosymplectic structure in $\mathbb{R}^{2n+1}$. The vector fields $E_i = \frac{\partial}{\partial y_i}$, $E_{n+i} = \frac{\partial}{\partial x_i}$, ξ form a J-basis for an almost para-cosymplectic structure in $\mathbb{R}^{2n+1}$. Thus, $\mathbb{R}^{2n+1}$ with an almost para-cosymplectic structure(J, ξ, η, g_r) is an almost para-cosymplectic manifold.

2.2. Riemannian submersions

Let (M, g_r) be an r-dimensional Riemannian manifold and (N, g_s) be an s dimensional Riemannian manifold. A Riemannian submersion is a smooth map $f : M \to N$ which is onto and satisfies the following axioms:
(a). f has maximal rank.
(b). The differential f_* preserves the lengths of horizontal vectors.
The fundamental $(1, 2)$ tensors $\mathcal{T}$ and $\mathcal{A}$ of a submersion on M, are defined by

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F,$$

$$\mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F,$$

for any vector fields E and F on (M, g_r). Here ∇ denotes the Levi-Civita connection of (M, g_r). These tensors are called integrability tensors for Riemannian submersions. We denote the projection morphism on the distribution $ker f_*$ and $(ker f_*)^{\perp}$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.

Now, we recall the following lemma which will be needed for later discussion.

Lemma 2.2. *Let U and V be any vertical vector fields, and X and Y horizontal vector fields. Then the tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy:*

$$\mathcal{T}_U W = \mathcal{T}_W U,$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y],$$

Now, it is easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$.

Lemma 2.3. *Let $f : (M, g_r) \rightarrow (N, g_s)$ be a Riemannian submersion. If X, Y are basic vector fields on M, then:*
(a). $g_r(X, Y) = g_s(X_*, Y_*)of$,
(b). $\mathcal{H}[X, Y]$ is basic, f-related to $[X_*, Y_*]$,
(c). $\mathcal{H}(\nabla_X Y)$ is basic vector field corresponding to $\nabla^*_{X_*} Y_*$, where ∇^* is the connection on N.
(d). for any vertical vector field V, $[X, V]$ is vertical.

If X is basic and U is vertical, then $\mathcal{H}(\nabla_U X) = \mathcal{H}(\nabla_X U) = \mathcal{A}_X U$. Now, we obtain

$$\nabla_P Q = \mathcal{T}_P Q + \bar{\nabla}_P Q \tag{2.6}$$

$$\nabla_P X = \mathcal{H}\nabla_P Q + \mathcal{T}_P X \tag{2.7}$$

$$\nabla_X P = \mathcal{A}_X P + \mathcal{V}\nabla_X P \tag{2.8}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \tag{2.9}$$

for any $P, Q \in \Gamma(kerf_*)$ and $X, Y \in \Gamma((kerf_*)^\perp)$, where $\bar{\nabla}_P Q = \mathcal{V}\nabla_P Q$.

Note that $\mathcal{T}$ acts on the fibers as the second fundamental form of the submersion restricted to vertical vector fields and it can be easily obtained that $\mathcal{T} = 0$ is equivalent to the condition that the fibers are totally geodesic. A Riemannian submersion is called Riemannian submersion with totally geodesic fiber if $\mathcal{T}$ vanishes identically. Now, assume that $\{e_1, \ldots, e_{r-s}\}$ is an orthonormal frame of $\Gamma(kerf_*)$. Then the horizontal vector field $H = \frac{1}{r-s}\sum_{i=1}^{r-s} \mathcal{T}_{e_i} e_i$ is called the mean curvature vector field of the fiber. Riemannian submersion is said to be minimal if $H = 0$. A Riemannian submersion is said to be Riemannian submersion with totally umbilical fibers if

$$\mathcal{T}_X Y = g_r(X, Y)H, \tag{2.10}$$

for $X, Y \in \Gamma(kerf_*)$. $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew symmetric operators on $\Gamma(TM)$ for any $E \in \Gamma(TM)$ reversing the horizontal and the vertical distributions. Using Lemma 2.2 horizontally distribution $\mathcal{H}$ is integrable if and only if $\mathcal{A} = 0$. For any $X, Y, Z \in \Gamma(TM)$ we have

$$g_r(\mathcal{T}_X Y, Z) + g_r(\mathcal{T}_X Z, Y) = 0, \tag{2.11}$$

$$g_r(\mathcal{A}_X Y, Z) + g_r(\mathcal{A}_X Z, Y) = 0. \tag{2.12}$$

Let (M, g_r) and (N, g_s) be Riemannian manifolds and suppose that $\varphi : (M, g_r) \rightarrow (N, g_s)$ is a smooth map between them. Then the differential φ_* of φ can be viewed as a section of the bundle $Hom(TM, \varphi^{-1}TN) \to M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibers $(\varphi^{-1}TN)_x = \mathcal{T}_{\varphi(x)}N, x \in M$. $Hom(TM, \varphi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form φ is given by

$$(\nabla\varphi_*)(X,Y) = \nabla^{\varphi}_X \varphi_*(Y) - \varphi_*(\nabla^M_X Y), \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$, where ∇^{φ} is the pullback connection. It is well known that second fundamental form is symmetric. If φ is a Riemannian submersion it can be easily obtained that

$$(\nabla\varphi_*)(X,Y) = 0 \tag{2.14}$$

for any $X, Y \in \Gamma((kerf_*)^{\perp})$. A smooth map $\varphi : (M, g_r) \to (N, g_s)$ is said to be harmonic if $trace(\nabla\varphi_*) = 0$. The tension field of φ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$\tau(\varphi) = div\varphi_* = \sum_{i=1}^{r} (\nabla\varphi_*)(e_i, e_i), \tag{2.15}$$

where $\{e_1, ..., e_r\}$ is the orthonormal frame of M. Then it follows that φ is harmonic if and only if $\tau(\varphi) = 0$.

3. Semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold

In this section, we give definition of a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold and we obtain some results and investigate the geometry of foliations.

Definition 3.1. Let (M, J, ξ, η, g_r) be an almost para-cosymplectic manifold and (N, g_s) Riemannian manifold. A Riemannian submersion $f : (M, J, \xi, \eta, g_r,) \to (N, g_s)$ is called a semi-slant submersion if there is a distribution $D_a \in \Gamma(kerf_*)$ such that

$$kerf_* = D_a \oplus D_b \oplus \langle\xi\rangle, \; J(D_a) = D_a,$$

and the angle $\theta = \theta(X)$ between JX and the space $(D_b)_x$ is constant for non vanishing $X \in (D_b)_x$ and $x \in M$, where D_b is the orthogonal to D_a and $\langle\xi\rangle$ in $kerf_*$. We call θ the semi-slant angle of the semi-slant submersion.

Let $f : (M, J, \xi, \eta, g_r) \to (N, g_s)$ be a semi-slant Riemannian submersion from an almost para-cosymplectic manifold onto a Riemannian manifold then there is a distribution $D_a \subset (kerf_*)$ such that

$$kerf_* = D_a \oplus D_b \oplus \langle\xi\rangle, \; J(D_a) = D_a \tag{3.1}$$

and the angle $\theta = \theta(X)$ between JX and the space $(D_b)_x$ is constant for nonzero $X \in (D_b)_x$ and $x \in M$, where D_b is the orthogonal to D_a and $\langle \xi \rangle$ in $kerf_*$.

Then for $X \in \Gamma(kerf_*)$, we have

$$X = PX + QX + \eta(X)\xi, \tag{3.2}$$

where $PX \in \Gamma(D_a)$ and $QX \in \Gamma(D_b)$.

For $X \in \Gamma(kerf_*)$, we write

$$JX = \phi X + \omega X, \tag{3.3}$$

where $\phi X \in \Gamma(kerf_*)$ and $\omega X \in \Gamma((kerf_*)^\perp)$.

For $Z \in \Gamma((kerf_*)^\perp)$, we have

$$JZ = BZ + CZ, \tag{3.4}$$

where $BZ \in \Gamma(kerf_*)$ and $CZ \in \Gamma((kerf_*)^\perp)$.

For $U \in \Gamma(TM)$, we have

$$U = \mathcal{V}U + \mathcal{H}U, \tag{3.5}$$

where $\mathcal{V}U \in \Gamma(kerf_*)$ and $\mathcal{H}U \in \Gamma((kerf_*)^\perp)$.

For $W \in \Gamma(kerf^{-1}TN)$, we have

$$W = \bar{P}W + \bar{Q}W, \tag{3.6}$$

where $\bar{P}W \in \Gamma(rangef_*)$ and $\bar{Q}W \in \Gamma((rangef_*)^\perp)$.

Then

$$(kerf_*)^\perp = \omega D_b \oplus \mu, \tag{3.7}$$

where μ is the orthogonal complement of ωD_b in $(kerf_*)^\perp$ and it is invariant under J. For $X, Y \in \Gamma(kerf_*)$. Now, we define the covariant derivative of ϕ and ω

$$(\nabla_X \phi)Y = \bar{\nabla}_X \phi Y - \phi(\bar{\nabla}_X Y),$$

$$(\nabla_X \omega)Y = \mathcal{H}\bar{\nabla}_X \omega Y - \omega(\bar{\nabla}_X Y),$$

for all $X, Y \in \Gamma(kerf_*)$, where $\bar{\nabla} = \mathcal{V}\nabla_X Y$. Then we obtain easily

Lemma 3.2. *Let (M, J, ξ, η, g_r) be an almost para-cosymplectic manifold and (N, g_s) be a Riemannian manifold. Let $f : (M, J, \xi, \eta, g_r) \to (N, g_s)$ be a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold. Then we get the following:*

$$\bar{\nabla}\phi Y - \phi\bar{\nabla}_X Y = B\mathcal{T}_X Y - \mathcal{T}_X \omega Y, \tag{3.8}$$

$$\mathcal{H}\nabla_X \omega Y - \omega\bar{\nabla}_X Y = C\mathcal{T}_X Y - \mathcal{T}_X \phi Y, \tag{3.9}$$

for $X, Y \in \Gamma(ker f_*)$.

$$\mathcal{V}\nabla_Z BW + \mathcal{A}_Z CW = B\mathcal{H}\nabla_Z W + \phi\mathcal{A}_Z W, \tag{3.10}$$

$$\mathcal{H}\nabla_Z CW + \mathcal{A}_Z BW = C\mathcal{H}\nabla_Z W + \omega\mathcal{A}_Z W, \tag{3.11}$$

for $Z, W \in \Gamma((ker f_*)^{\perp})$

$$\bar{\nabla}_X BZ + \mathcal{T}_X CZ = B\mathcal{H}\nabla_X Z + \phi\mathcal{T}_X Z, \tag{3.12}$$

$$\mathcal{H}\nabla_X BZ + \mathcal{H}\nabla_X CZ = C\mathcal{H}\nabla_X Z + \omega\mathcal{T}_X Z, \tag{3.13}$$

for $X \in \Gamma(ker f_*)$ *and* $Z \in \Gamma((ker f_*)^{\perp})$.

Theorem 3.3. *Let* $f : (M, J, \xi, \eta, g_r) \to (N, g_s)$ *be a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold. Then* f *is a proper semi-slant submersion if and only if there exist a constant* $\lambda \in [0, 1]$ *such that*

$$\phi^2 X = \lambda(X - \eta(X)\xi) \tag{3.14}$$

for all $X \in \Gamma(D_b)$, *where* $\lambda = cos^2\theta$.

Proof. For any non zero $X \in \Gamma(D_b)$, let $\theta(X)$ be a semi-slant angle. Then we get

$$cos\theta(X) = \frac{||\phi X||}{||JX||}. \tag{3.15}$$

Using (2.4) and (3.15), we have

$$\begin{aligned} g_r(\phi^2 X, X) &= g_r(\phi X, \phi X) \\ &= cos^2\theta(X) g_r(JX, JX) \\ &= cos^2\theta(X) g_r(J^2 X, X). \end{aligned}$$

Using equation (2.1), we obtain

$$\phi^2 X = cos^2\theta(X)(X - \eta(X)\xi).$$

Let $\lambda = cos^2\theta$, then we get

$$\phi^2 X = \lambda(X - \eta(X)\xi).$$

for all $X \in \Gamma(D_b)$.
Conversly, let there exist a constant $\lambda \in [0, 1]$, which satisfies $\phi^2 X = \lambda(X - \eta(X)\xi)$. Then using (2.4) and (3.3), we obtain

$$\begin{aligned} cos\theta(X) &= \frac{g_r(JX, \phi X)}{||JX||||\phi X||} \\ &= \frac{\lambda g_r(JX, JX)}{||JX||||\phi X||} \end{aligned}$$

Thus we get for all $X \in \Gamma(D_b)$.

$$cos\theta(X) = \frac{||JX||}{||\phi X||}\lambda. \tag{3.16}$$

Using (3.15) and (3.16), we have

$$\lambda = cos^2\theta(X).$$

It is clear that for a proper semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold the semi-slant angle $\theta(X)$ is always constant.

Proposition 3.4. *Let $f : (M, J, \xi, \eta, g_r) \to (N, g_s)$ be a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold with semi-slant angle θ. Then for any $X, Y \in \Gamma(D_b)$, we obtain*

$$g_r(\phi X, \phi Y) = cos^2\theta g_r(X, Y) \tag{3.17}$$

$$g_r(\omega X, \omega Y) = sin^2\theta g_r(X, Y) \tag{3.18}$$

Proposition 3.5. *Let $f : (M, J, \xi, \eta, g_r) \to (N, g_s)$ be a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold with semi-slant angle θ. Then we obtain*

$$\phi^2 + B\omega = I - \eta \otimes \xi \tag{3.19}$$

$$\omega\phi + C\omega = 0 \tag{3.20}$$

Proposition 3.6. *Let $f : (M, J, \xi, \eta, g_r) \to (N, g_s)$ be a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold with semi-slant angle θ. Then f is a proper semi-slant submersion if and only if there exist a constant $\delta \in [0, 1]$, such that*

$$B\omega = \delta(I - \eta \otimes \xi), \tag{3.21}$$

where $\delta = sin^2\theta$.

Proof. From Theorem 3.3 and equation (3.19), we obtain the above result.

Theorem 3.7. *Let $f : (M, J, \xi, \eta, g_r) \to (N, g_s)$ be a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold with semi-slant angle θ. If ω is parallel with respect to the connection ∇ on $ker f_*$, then*

$$\mathcal{T}_{\phi X}\phi X = \lambda(\mathcal{T}_X X - \eta(X)\mathcal{T}_X\xi), \tag{3.22}$$

for all $X \in \Gamma(ker f_*)$.

Proof. Since ω is parallel, then using equation (3.9) of Lemma 3.2, we have

$$\mathcal{T}_X \phi Y = C\mathcal{T}_X Y, \tag{3.23}$$

for all $X, Y \in \Gamma(kerf_*)$. Substitute $X = Y$ and $Y = X$ in equation (3.23) and use the obtained equation. Since $\mathcal{T}$ is symmetric, then we get

$$\mathcal{T}_Y \phi X = \mathcal{T}_X \phi Y, \tag{3.24}$$

taking $Y = \phi X$ in equation (3.24) and using equation (3.14), we obtain

$$\mathcal{T}_{\phi X} \phi X = cos^2\theta(\mathcal{T}_X X - \eta(X)\mathcal{T}_X \xi).$$

Let $\lambda = cos^2\theta$ in above equation, then

$$\mathcal{T}_{\phi X} \phi X = \lambda(\mathcal{T}_X X - \eta(X)\mathcal{T}_X \xi),$$

for all $X \in \Gamma(kerf_*)$.

Theorem 3.8. *Let $f : (M, J, \xi, \eta, g_r) \to (N, g_s)$ be a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold with semi-slant angle θ. Then the distribution $kerf_*$ defines a totally geodesic foliation on M if and only if*

$$\begin{aligned} g_r(\mathcal{T}_X \omega Y, BW) - g_r(\mathcal{H}(\nabla_X \omega\phi Y), W) - g_r(\mathcal{H}(\nabla_X \omega Y), CW) \\ = cos^2\theta\eta(Y) g_r(\mathcal{T}_X \xi, W), \end{aligned} \tag{3.25}$$

for $X, Y \in \Gamma(kerf_*)$ and $W \in \Gamma(kerf_*)^\perp$.

Proof. Let $X, Y \in \Gamma(kerf_*)$ and $W \in \Gamma(kerf_*)^\perp$. Using (2.3), (3.3) and (3.4), we obtain

$$\begin{aligned} g_r(\nabla_X Y, W) & \\ = \; & g_r(J\nabla_X Y, JW) \\ = \; & g_r(\nabla_X \phi Y, JW) + g_r(\nabla_X \omega Y, JW) \\ = \; & g_r(J\nabla_X \phi Y, W) + g_r(\nabla_X \omega Y, BW) + g_r(\nabla_X \omega Y, CW) \\ = \; & g_r(\nabla_X J\phi Y, W) + g_r(\nabla_X \omega Y, BW) + g_r(\nabla_X \omega Y, CW) \\ = \; & g_r(\nabla_X \phi^2 Y, W) + g_r(\nabla_X \omega\phi Y, W) + g_r(\nabla_X \omega Y, BW) + g_r(\nabla_X \omega Y, CW), \end{aligned}$$

using equations (2.7), (2.10) and (3.14), we have

$$\begin{aligned} g_r(\nabla_X Y, W) = cos^2\theta g_r(\nabla_X Y, W) - \eta(Y) cos^2\theta g_r(\nabla_X \xi, W) + g_r(\nabla_X \omega Y, BW) \\ + g_r(\nabla_X \omega Y, CW). \end{aligned} \tag{3.26}$$

Now, using equation (3.26) we get

$$\begin{aligned} g_r(\mathcal{T}_X \omega Y, BW) - g_r(\mathcal{H}(\nabla_X \omega\phi Y), W) - g_r(\mathcal{H}(\nabla_X \omega Y), CW) \\ = cos^2\theta\eta(Y) g_r(\mathcal{T}_X \xi, W), \end{aligned}$$

for $X, Y \in \Gamma(kerf_*)$ and $W \in \Gamma(kerf_*)^\perp$, which proves the theorem.

Theorem 3.9. *Let $f : (M, J, \xi, \eta, g_r) \to (N, g_s)$ be a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold with semi-slant angle θ. Then the distribution $(ker f_*)^\perp$ defines totally geodesic foliation on M if and only if*

$$\phi(\mathcal{A}_X CY + \mathcal{V}\nabla_X BY) + B(\mathcal{A}_X BY + \mathcal{H}\nabla_X CY) = 0, \tag{3.27}$$

for $X, Y \in \Gamma(ker f_*)^\perp$.

Proof. Let $V \in \Gamma(ker f_*)$ and $X, Y \in \Gamma(ker f_*)^\perp$. Using (2.3) and (3.4), we get

$$\begin{aligned} g_r(\nabla_X Y, V) &= g_r(\nabla_X JY, JV) \\ &= g_r(\nabla_X BY + \nabla_X CY, JV), \end{aligned}$$

from equation (2.9) and (2.10), we have

$$\begin{aligned} g_r(\nabla_X Y, V) &= g_r(\mathcal{A}_X BY + \mathcal{V}\nabla_X BY + \mathcal{A}_X CY + \mathcal{H}\nabla_X CY), JV) \\ &= g_r(J\mathcal{A}_X BY + J\mathcal{V}\nabla_X BY + J\mathcal{A}_X CY + J\mathcal{H}\nabla_X CY), V) \\ &= g_r(B\mathcal{A}_X BY + C\mathcal{A}_X BY + \phi\mathcal{V}\nabla_X BY + \omega\mathcal{V}\nabla_X BY \\ &+ \phi\mathcal{A}_X CY + \omega\mathcal{A}_X CY + B\mathcal{H}\nabla_X CY) + C\mathcal{H}\nabla_X CY), V) \\ &= g_r(\phi(A_X CY + \mathcal{V}\nabla_X BY) + \omega(A_X CY + \mathcal{V}\nabla_X BY) \\ &+ B(A_X BY + \mathcal{H}\nabla_X CY) + C(A_X BY + \mathcal{H}\nabla_X CY)), V). \end{aligned}$$

So, we have from above

$$\phi(A_X CY + \mathcal{V}\nabla_X BY) + B(A_X BY + \mathcal{H}\nabla_X CY) = 0,$$

for $X, Y \in \Gamma(ker f_*)^\perp$ and $V \in \Gamma(ker f_*)$, which proves the assertion.

4. Harmonicity of semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold

In this section, we deal with the harmonicity of semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold. Also, we provide some examples of such submersions.

Theorem 4.1. *Let $f : (M, J, \xi, \eta, g_r) \to (N, g_s)$ be a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold with semi-slant angle θ. Then f is a harmonic map if and only if*

$$trace(\nabla f_*) = 0 \tag{4.1}$$

on D_b.

Proof. Since f be a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold. So we can choose a local orthonormal frame $\{e_1, e_2, ..., e_{2p}\}$ of D_a and a local orthonormal frame $\{v_1, v_2, ..., v_q\}$ of D_b. The vector field ξ is a horizontal vector field and orthogonal to D_a and D_b. So a local orthonormal frame of $ker f_*$ is $\{v_1, v_2, ..., v_q, e_1, e_2, ..., e_{2p}, \xi\}$ such that for $1 \leq i \leq p, q$

$$e_{2i-1} = Je_{2i},$$

and

$$J\xi = 0$$

Since

$$f_*(\nabla_{Je_{2i-1}} Je_{2i-1}) = -f_*(\nabla_{e_{2i}} e_{2i}), \; f_* \nabla_\xi \xi = 0$$

$$trace(\nabla f_*) = 0 \Leftrightarrow \sum_{i=1}^{r} f_*(\nabla_{e_i} e_i) = 0$$

i.e. f_* is harmonic map on D_b.

Theorem 4.2. *Let $f : (M, J, \xi, \eta, g_r) \to (N, g_s)$ be a semi-slant submersion with totally umbilical fibers from an almost para-cosymplectic manifold onto a Riemannian manifold with semi-slant angle θ. Then $H \in \Gamma(\omega D_b)$.*

Proof. For $X, Y \in \Gamma D_b$ and $W \in \Gamma\mu$, we obtain

$$J\nabla_X Y = J\hat{\nabla}_X Y + J\mathcal{T}_X Y,$$

$$(\nabla_X J)Y - \nabla_X JY = \phi\hat{\nabla}_X Y + \omega\hat{\nabla}_X Y + B\mathcal{T}_X Y + C\mathcal{T}_X Y,$$

so that

$$g_r(\mathcal{T}_X \phi Y, W) = g_r(C\mathcal{T}_X Y, W).$$

By the above, we get

$$g_r(\phi\mathcal{T}_X Y, W) = g_r(X, \phi Y)g(H, W)$$

$$g_r(X, \phi Y)g_r(H, W) = -g_r(\mathcal{T}_X Y, \phi W)$$

$$g_r(X, \phi Y)g_r(H, W) = -g_r(X, Y)g_r(H, \phi W), \tag{4.2}$$

replacing X by Y and Y by X in above equation, we get

$$g_r(Y, \phi X)g_r(H, W) = -g_r(Y, X)g_r(H, \phi W), \tag{4.3}$$

by equation (4.2) and (4.3), we obtain

$$g_r(X, Y)g_r(H, \phi W) = 0, \tag{4.4}$$

Since $||X|| \neq 0$, hence

$$H \in \Gamma(\omega D_b).$$

Examples.

Example 4.3. Let f be a semi-invaraint submersion from an almost para-cosymplectic manifold (M, J, ξ, η, g_r) onto a Riemannian manifold (N, g_s). Then the map f is a semi-slant submersion with the semi-slant angle $\theta = \frac{\pi}{2}$.

Example 4.4. Let f be a slant submersion from an almost para-cosymplectic manifold (M, J, ξ, η, g_r) onto a Riemannian manifold (N, g_s). Then the map f is a semi-slant submersion with $D_b = ker f_*$.

Example 4.5. Define the map $f : \mathbb{R}^7 \to \mathbb{R}^2$ by

$$f(x_1, x_2, ..., x_7) = (x_3 sin\alpha - x_5 cos\alpha, x_6),$$

where $\alpha \in (0, \frac{\pi}{2})$. Then the map f is a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold such that
$D_a = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \rangle$, $D_b = \langle \frac{\partial}{\partial x_4}, cos\alpha \frac{\partial}{\partial x_3} + sin\alpha \frac{\partial}{\partial x_5} \rangle$ and $\xi = \frac{\partial}{\partial x_7}$
with the semi-slant angle $\theta = \alpha$.

Example 4.6. Define the map $f : \mathbb{R}^{11} \to \mathbb{R}^4$ by

$$f(x_1, x_2, ..., x_{11}) = (\frac{x_3 - x_5}{\sqrt{2}}, x_6, \frac{x_7 - x_9}{\sqrt{2}}, x_8).$$

Then the map f is a semi-slant submersion from an almost para-cosymplectic manifold onto a Riemannian manifold such that
$D_a = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \rangle$, $D_b = \langle \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_{10}} \rangle$ and $\xi = \frac{\partial}{\partial x_{11}}$
with the semi-slant angle $\theta = \frac{\pi}{4}$.

Example 4.7. Define the map $f : \mathbb{R}^9 \to \mathbb{R}^4$ by

$$f(x_1, x_2, \ldots, x_9) = (x_1, x_2, x_3 cos\alpha - x_5 sin\alpha, x_4 sin\beta - x_6 cos\beta)$$

where α and β are constant functions. Then the map f is a semi-slant submersion from an almost para-cosymplectic manifold M onto a Riemannian manifold N such that
$D_a = \langle \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \rangle$, $D_b = \langle sin\alpha \frac{\partial}{\partial x_3} + cos\alpha \frac{\partial}{\partial x_5}, cos\beta \frac{\partial}{\partial x_4} + sin\beta \frac{\partial}{\partial x_6} \rangle$ and $\xi = \frac{\partial}{\partial x_9}$
with the semi-slant angle $cos\theta = |sin(\alpha + \beta)|$.

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