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A NEW APPROACH TO TOTALLY UMBILICAL NULL SUBMANIFOLDS

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Abstract. In this paper, we study totally umbilical r-null submanifolds, with $r \geq 1$, of a semi-Riemannian manifold using new objects called generalized Newton transformations. We prove that the results presented here are umbrella of the well-known results for totally umbilical null hypersurfaces, totally umbilical half-null submanifolds and generally, totally umbilical r-null submanifolds.

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1. Introduction

In [3] and [4], the authors initiated the study of null geometry of submanifolds in semi-Riemannian manifolds. Null submanifolds are interesting objects with numerous applications to mathematical physics and general relativity. More precisely, in general relativity, they are known to represent various types of black hole horizons (see for instance, [3] and [4] and many more references therein). Due to this fact that many other researchers are actively exploring them, for example see [5], [6], [7], [8], [9], [10] and many more. In [5], the authors study totally umbilical null submanifolds of semi-Riemannian manifolds, in which they presented interesting partial differential equations concerning such submanifolds.

Null submanifolds are endowed with a number of shape operators, which we have used in this paper to study *generalized Newton transformations* for null submanifolds (see [2] and [1] for details on these transformations). Using these transformations, we give generalized versions of some results in, [3], [4] and [5] for totally umbilical null hypersurfaces, totally umbilical half-null submanifolds and totally umbilical r-null submanifolds. The rest of paper is organized as follows. In Section 2, we present basic notions on null submanifolds needed in this paper. In Section 3, we use the concept of generalized Newton transformations for null submanifolds and in Section 4 we give general results

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on totally umbilical null submanifolds and prove that they are generalizations of many well-known results for null hypersurfaces, null-half submanifolds and r-null submanifolds.

2. Preliminaries

Let $(\overline{M}, \overline{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index ν such that $m, n \geq 1$, $1 \leq \nu \leq m+n-1$, and let (M, g) be an m-dimensional submanifold of $\overline{M}$. In case g is *degenerate* on the tangent bundle TM of M, we say that M a *null submanifold* [3]. We denote the set of smooth sections of a vector bundle Ξ by $\Gamma(\Xi)$. For a degenerate metric tensor $g = \overline{g}_{|TM}$, there exists locally a non-zero vector field $E \in \Gamma(TM)$ such that $g(E, X) = 0$, for any $X \in \Gamma(TM)$. Then, for each tangent space T_xM, $x \in M$, we have $T_xM^{\perp} = \{u \in T_x\overline{M} : \overline{g}(u, v) = 0,\ \forall\, v \in T_xM\}$, which is a degenerate n-dimensional subspace of $T_x\overline{M}$. The *radical* or *null* subspace of M is denoted by $\operatorname{Rad} T_xM$ and is given by

$$\operatorname{Rad} T_xM = \{E_x \in T_xM : g(E_x, X) = 0,\ \forall\, X \in T_xM\}.$$

Notice that $\operatorname{Rad} T_xM = T_xM \cap T_xM^{\perp}$ and its dimension depends on $x \in M$. A submanifold M of $\overline{M}$ is called r-null if the mapping $\operatorname{Rad} TM : x \longrightarrow \operatorname{Rad} T_xM$, defines a smooth distribution of rank $r > 0$, where $\operatorname{Rad} TM$ is called the radical (null) distribution on M. Let $S(TM)$ be a *screen distribution* which is a semi-Riemannian complementary distribution of $\operatorname{Rad} TM$ in TM, and is given by

$$TM = \operatorname{Rad} TM \perp S(TM). \tag{2.1}$$

Note that the distribution $S(TM)$ is not unique and canonically isomorphic to the factor vector bundle $TM/\operatorname{Rad} TM$ [3]. Choose a screen transversal bundle $S(TM^{\perp})$, which is semi-Riemannian complementary to $\operatorname{Rad} TM$ in $TM^{\perp}$. Since, for any local basis $\{E_i\}$ of $\operatorname{Rad} TM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $S(TM)^{\perp}$ such that $\overline{g}(E_i, N_j) = \delta_{ij}$, it follows that there exists a null transversal vector bundle $l\mathrm{tr}(TM)$ locally spanned by $\{N_i\}$ [3]. Let $\mathrm{tr}(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}$. Then,

$$\mathrm{tr}(TM) = l\mathrm{tr}(TM) \perp S(TM^{\perp}), \tag{2.2}$$

$$\begin{aligned} T\overline{M} &= S(TM) \perp S(TM^{\perp}) \perp \{\operatorname{Rad} TM \oplus l\mathrm{tr}(TM)\} \\ &= TM \oplus \mathrm{tr}(TM). \end{aligned} \tag{2.3}$$

We say that a null submanifold M of $\overline{M}$ is (i) r-null if $1 \leq r < min\{m, n\}$, (ii) co-isotropic if $1 \leq r = n < m$, $S(TM^{\perp}) = \{0\}$, (iii) isotropic if $1 \leq r = m < n$, $S(TM) = \{0\}$, (iv) totally null if $r = n = m$, $S(TM) = S(TM^{\perp}) = \{0\}$. The details on the above classes of submanifolds with examples are found in [3]. Consider a local quasi-orthonormal field of frames of $\overline{M}$ along M, on $\mathcal{U}$ as

$$\{E_1, \cdots, E_r, N_1, \cdots, N_r, Z_{r+1}, \cdots, Z_m, W_{1+r}, \cdots, W_n\},$$

where $\{Z_{r+1},\cdots,Z_m\}$ and $\{W_{1+r},\ldots,W_n\}$ are respectively orthogonal bases of $\Gamma(S(TM)|_{\mathcal{U}})$ and $\Gamma(S(TM^{\perp})|_{\mathcal{U}})$ and that $\epsilon_a = g(Z_a, Z_a)$ and $\epsilon_\alpha = \overline{g}(W_\alpha, W_\alpha)$ are the signatures of $\{Z_a\}$ and $\{W_\alpha\}$, respectively. The following range of indices will be used. $i,j,k \in \{1,\cdots,r\}$; $\alpha,\beta,\mu \in \{r+1,\cdots,n\}$; $a,b,c \in \{r+1,\cdots,m\}$.

Let P be the projection morphism of TM onto $S(TM)$. Then, the Gauss-Weingartein equations [4] of an r-null submanifold M and $S(TM)$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^l(X,Y)N_i + \sum_{\alpha=r+1}^{n} h_\alpha^s(X,Y)W_\alpha, \tag{2.4}$$

$$\overline{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{\alpha=r+1}^{n} \rho_{i\alpha}(X)W_\alpha, \tag{2.5}$$

$$\overline{\nabla}_X W_\alpha = -A_{W_\alpha}X + \sum_{i=1}^{r} \varphi_{\alpha i}(X)N_i + \sum_{\beta=r+1}^{n} \theta_{\alpha\beta}(X)W_\beta, \tag{2.6}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h_i^*(X,PY)E_i, \tag{2.7}$$

$$\nabla_X E_i = -A^*_{E_i}X - \sum_{j=1}^{r} \tau_{ji}(X)E_j, \quad \forall\, X,Y \in \Gamma(TM), \tag{2.8}$$

where ∇ and ∇^* are the induced connections on TM and $S(TM)$, respectively, h_i^l and h_α^s are symmetric bilinear forms known as *local null* and *screen fundamental* forms of TM, respectively. Also h_i^* are the *second fundamental forms* of $S(TM)$. A_{N_i}, $A^*_{E_i}$ and A_{W_α} are linear operators on TM while τ_{ij}, $\rho_{i\alpha}$, $\varphi_{\alpha i}$ and $\theta_{\alpha\beta}$ are 1-forms on TM. It is easy to see from (2.4) that

$$h_i^l(X,Y) = \overline{g}(\overline{\nabla}_X Y, E_i), \quad \forall X,Y \in \Gamma(TM), \tag{2.9}$$

from which we deduce the independence of h_i^ls on the choice of $S(TM)$. It is easy to see that ∇^* is a metric connection on $S(TM)$ while ∇ is generally not a metric connection and satisfies the relation

$$(\nabla_X g)(Y,Z) = \sum_{i=1}^{r}\{h_i^l(X,Y)\lambda_i(Z) + h_i^l(X,Z)\lambda_i(Y)\}, \tag{2.10}$$

for any $X,Y \in \Gamma(TM)$ and 1-forms λ_i given by

$$\lambda_i(X) = \overline{g}(X,N_i), \quad \forall X \in \Gamma(TM). \tag{2.11}$$

The above three local second fundamental forms are related to their shape operators by the following set of equations

$$g(A^*_{E_i}X,Y)=h^l_i(X,Y)+\sum_{j=1}^{r}h^l_j(X,E_i)\lambda_j(Y),\ \bar{g}(A^*_{E_i}X,N_j)=0, \tag{2.12}$$

$$g(A_{W_\alpha}X,Y)=\epsilon_\alpha h^s_\alpha(X,Y)+\sum_{i=1}^{r}\varphi_{\alpha i}(X)\lambda_i(Y), \tag{2.13}$$

$$\bar{g}(A_{W_\alpha}X,N_i)=\epsilon_\alpha\rho_{i\alpha}(X),\quad g(A_{N_i}X,Y)=h^*_i(X,\mathcal{P}Y), \tag{2.14}$$

for any $X,Y\in\Gamma(TM)$. Let $(M,g,S(TM),S(TM^\perp))$ be a m-dimensional r-null submanifold of a $(m+n)$-dimensional semi-Riemannian manifold $(\overline{M},\overline{g})$. Let $\overline{R}$ and R denote the curvature tensors of $\overline{\nabla}$ and ∇, respectively. The following identities are needed in this paper (see [3] or [4] for details)

$$\begin{aligned}&\overline{R}(X,Y,E,PU)=\overline{g}((\nabla_Yh^l)(X,PU)-(\nabla_Xh^l)(Y,PU),E)\\ &+\overline{g}(h^s(Y,PU),h^s(X,E))-\overline{g}(h^s(X,PU),h^s(Y,E)),\end{aligned} \tag{2.15}$$

$$\begin{aligned}&\overline{R}(X,Y,N,PU)=\overline{g}((\nabla_YA)(N,X)-(\nabla_XA)(N,Y),PU)\\ &+\overline{g}(h^s(Y,PU),D^s(X,N))-\overline{g}(h^s(X,PU),D^s(Y,N)),\end{aligned} \tag{2.16}$$

$$\begin{aligned}&\overline{R}(X,Y,W,PU)=\overline{g}((\nabla_YA)(W,X)-(\nabla_XA)(W,Y),PU)\\ &+\overline{g}(h^*(Y,PU),D^l(X,W))-\overline{g}(h^*(X,PU),D^l(Y,W)),\end{aligned} \tag{2.17}$$

$$\begin{aligned}&g(R(X,Y)PU,N)=\overline{g}((\nabla_XA)(N,Y)-(\nabla_YA)(N,X),PU)\\ &+\overline{g}(h^l(X,PU),A_NY)-\overline{g}(h^l(Y,PU),A_NX),\end{aligned} \tag{2.18}$$

$$g(R(X,Y)E,PU)=g((\nabla_YA^*)(E,X)-(\nabla_XA^*)(E,Y),PU), \tag{2.19}$$

for all $X,Y,U\in\Gamma(TM)$. A semi-Riemannian manifold of constant curvature c is called a *semi-Riemannian space form* [3, p. 41] and is denoted by $(\overline{M}(c),\overline{g})$. Then, the curvature tensor $\overline{R}$ of $\overline{M}(c)$ is given by

$$\overline{R}(X,Y)Z=c\{\overline{g}(Y,Z)X-\overline{g}(X,Z)Y\},\quad\forall\, X,Y,Z\in\Gamma(T\overline{M}). \tag{2.20}$$

A null submanifold $(M,g,S(TM),S(TM^\perp))$, of a semi-Riemannian manifold $(\overline{M},\overline{g})$ is said to be totally umbilical in $\overline{M}$ [4] if there is a smooth transversal vector field $\mathcal{H}\in\Gamma(\mathrm{tr}(TM))$, called the transversal curvature vector of M such that

$$h(X,Y)=\mathcal{H}g(X,Y), \tag{2.21}$$

for all $X,Y\in\Gamma(TM)$. Moreover, it is easy to see that M is totally umbilical in $\overline{M}$ if and only if on each coordinate neighborhood $\mathcal{U}$ there exist smooth vector fields $\mathcal{H}^l\in\Gamma(l\mathrm{tr}(TM))$ and $\mathcal{H}^s\in\Gamma(S(TM^\perp))$ and smooth functions $\mathcal{H}^l_i\in F(l\mathrm{tr}(TM))$ and $\mathcal{H}^s_\alpha\in F(S(TM^\perp))$ such that, for all $X,Y\in\Gamma(TM)$,

$$\begin{aligned}&h^l(X,Y)=\mathcal{H}^lg(X,Y),\quad h^s(X,Y)=\mathcal{H}^sg(X,Y),\\ &h^l_i(X,Y)=\mathcal{H}^l_ig(X,Y),\quad h^s_\alpha(X,Y)=\mathcal{H}^s_\alpha g(X,Y).\end{aligned} \tag{2.22}$$

3. Generalized Newton transformations

In this section, we use the notion of generalized Newton transformations of a system of endomorphisms on a r-null submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ (see [2] and [1] for more details).

Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-null submanifold of $(\overline{M}, \overline{g})$. Notice that the operators $A^*_{E_1}, \cdots, A^*_{E_r}$ are self-adjoint on $S(TM)$, and hence diagonalizable on $S(TM)$. Let $\mathbb{Z}^+(r)$ denote the set of all sequences $u = (u_1, \cdots, u_r)$, with $u_i \in \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. Then the length of u is denoted by $|u|$ and given by $|u| = \sum_{i=1}^{r} u_i$. Let us define an operator $\mathcal{A}^* \in \mathrm{End}^r(TM)$ by $\mathcal{A}^* = (A^*_{E_1}, \cdots, A^*_{E_r})$, where $\mathrm{End}^r(TM)$ is the vector space $\mathrm{End}(TM) \times \cdots \times \mathrm{End}(TM)$ (r-times). Furthermore, let $t = (t_1, \cdots, t_r) \in \mathbb{R}^r$ and set $t^u = t_1^{u_1} \cdots t_r^{u_r}$ and $t\mathcal{A}^* = \sum_{i=1}^{r} t_i A^*_{E_i}$. Then, the Newton polynomial of $\mathcal{A}^*$ is denoted by $P_{\mathcal{A}^*}$ and defined by $P_{\mathcal{A}^*} : \mathbb{R}^r \longrightarrow \mathbb{R}$, $P_{\mathcal{A}^*}(t) = \det(\mathbb{I} + t\mathcal{A}^*) = \sum_{|u| \leq p} \sigma^*_u t^u$, where the coefficients $\sigma^*_u = \sigma^*_u(\mathcal{A}^*)$ (the symmetric functions or mean curvatures) depend only on $\mathcal{A}^*$. We note that $\sigma^*_{(0,\cdots,0)} = 1$. We suppose further that $\sigma^*_u = 0$ for all $|u| > p$. Consider the functions $\varrho^{\sharp} : \mathbb{Z}^+(r) \longrightarrow \mathbb{Z}^+(r)$ and $\varrho_{\flat} : \mathbb{Z}^+(r) \longrightarrow \mathbb{Z}^+(r)$, given by $\varrho^{\sharp}(s_1, \cdots, s_r) = (s_1, \cdots s_{i-1}, s_i + 1, s_{i+1}, \cdots, s_r)$ and $\varrho_{\flat}(s_1, \cdots, s_r) = (s_1, \cdots s_{i-1}, s_i - 1, s_{i+1}, \cdots, s_r)$. We can see that $\varrho^{\sharp}$ increases the value of the i-th element by 1 and $\varrho_{\flat}$ decreases the value of i-th element by 1. It is also clear that $\varrho^{\sharp}$ is the inverse map to $\varrho_{\flat}$.

The *generalized Newton transformation* [2] of $\mathcal{A}^* = (A^*_{E_1}, \cdots, A^*_{E_r})$ is a system of endomorphisms $T^*_u = T^*_u(\mathcal{A}^*)$, $u \in \mathbb{Z}^+(r)$, satisfying the following condition. For every smooth curve $\gamma \mapsto \mathcal{A}^*(\gamma)$ in $\mathrm{End}^r(M)$ such that $\mathcal{A}^*(0) = \mathcal{A}^*$, we have $\frac{d}{d\gamma}\sigma^*_u(\gamma)_{\gamma=0} = \sum_i \mathrm{tr}(\frac{d}{d\gamma}A^*_{E_i}(\gamma)_{\gamma=0} \circ T^*_{i_{\flat}(u)})$. For a fixed system a of endomorphisms $\mathcal{A}^* = (A^*_{E_1}, \cdots, A^*_{E_r})$, the object T^*_u is unique (see [2] and [1]). However, it is important to note that T^*_u depend on the choice of chosen screen distribution $S(TM)$. This is due to the fact that the object $\mathcal{A}^* = (A^*_{E_1}, \cdots, A^*_{E_r})$ is dependent on $S(TM)$. In fact, let us consider two quasi-orthonormal frames $\{E_i, N_i, Z_a, W_\alpha\}$ and $\{E_i, N'_i, Z'_a, W'_\alpha\}$ induced on $\mathcal{U}$ by $\{S(TM), S(TM^{\perp}), F\}$ and $\{S'(TM), S'(TM^{\perp}), F'\}$, respectively. In this case, F and F' are the complementary vector bundles of $\mathrm{Rad}\, TM$ in $S(TM^{\perp})^{\perp}$ and $S'(TM^{\perp})^{\perp}$, respectively. Setting $Y = E_i$ in (5.2.20) of [4, p. 208]) and using $h^l_i(E_i, X) = 0$, we have

$$
\begin{aligned}
A^*_{E_i}X = A'^*_{E_i}X &+ \sum_{j=1}^{r}\{\sum_{\alpha,\beta=r+1}^{n} \varepsilon_\beta h'^s_\alpha(X, E_i) W^\beta_\alpha Q_{j\beta}\} E_j \\
&- \sum_{j=1}^{r} \tau_{ji}(X) E_j + \sum_{j=1}^{r} \tau'_{ji}(X) E_j, \quad \forall\, X \in \Gamma(TM),
\end{aligned} \tag{3.1}
$$

where W^β_α and $Q_{j\beta}$ are smooth functions on $\mathcal{U}$. Notice from (3.1) that the operators $A^*_{E_i}$ depend on the chosen screen distribution, $S(TM)$, and so $\mathcal{A}^*$ and T^*_u.

Let $T^* = (T^*_u : u \in \mathbb{Z}^+(r))$ be the generalized Newton transformation of $\mathcal{A}^*$. Then for every $u \in \mathbb{Z}^+(r)$ of length greater or equal to p we have $T^*_u = 0$

(Cayley-Hamilton Theorem). Moreover, T^*_u satisfy the following recurrence relation

$$T^*_0 = \mathbb{I}, \qquad \text{where } 0 = (0, \cdots, 0),$$

$$T^*_u = \sigma^*_u \mathbb{I} - \sum_{i=1}^{r} A^*_{E_i} \circ T^*_{i_\flat(u)}, \quad \text{where } |u| \geq 1, \tag{3.2}$$

where $\mathbb{I}$ denotes the identity on M. We also have [1]:

$$\operatorname{tr}(T^*_u) = (m - r - |u|)\sigma^*_u, \quad \sum_{i=1}^{r} \operatorname{tr}(A^*_{E_i} \circ T^*_{i_\flat(u)}) = |u|\sigma^*_u, \tag{3.3}$$

$$\text{and} \quad \sum_{i,j=1}^{r} \operatorname{tr}(A^*_{E_i} \circ A^*_{E_j} \circ T^*_{j_\flat i_\flat(u)}) = -|u|\sigma^*_u + \sum_{i=1}^{r} \operatorname{tr}(A^*_{E_i})\sigma^*_{i_\flat(u)}, \tag{3.4}$$

where trace is taken with respect to $S(TM)$. If $r = 1$, i.e., M is a null hypersurface or a half-null submanifold, then $u = (u_1, 0 \cdots, 0)$ and thus $|u| = u_1$, from which $\sigma_u = S_{u_1}$ (the well-known symmetric polynomials of one shape operator). Furthermore, $\mathcal{A}^* = (A^*_E, 0, \ldots, 0)$, where E is the only section spanning $\operatorname{Rad} TM$. Thus, T^*_u in this case coincides with that already known for one shape operator (see [1]).

3.1. The operator $\mathcal{A} = (A_{N_1}, \cdots, A_{N_r})$, where $\{N_1, \cdots, N_r\} \in l\operatorname{tr}(TM)$

It is important to note that the screen local second fundamental forms h^*_i are not generally symmetric. This makes the operators A_{N_i}, for $i \in \{1, \cdots, r\}$ non-symmetric (not self-adjoint on $S(TM)$) with respect to g. However, when $S(TM)$ is integrable then it is well-known, see Theorem 2.5 of [3, p. 161], that h^*_i are symmetric and all the operators A_{N_i} become symmetric (or self-adjoint) on $S(TM)$. Moreover, each 1-form $\operatorname{tr}(\tau_{ij})$ induced by $S(TM)$ is closed, i.e., $d\operatorname{tr}(\tau_{ij}) = 0$. Thus, each operator A_{N_i} is diagonalizable on $S(TM)$. For this case, let us consider $\mathcal{A} = (A_{N_1}, \cdots, A_{N_r}) \in \operatorname{End}^r(TM)$. Then its corresponding symmetric function $\sigma_u = \sigma_u(\mathcal{A})$ and generalized Newton transformation T_u satisfy the following recurrence relation

$$T_0 = \mathbb{I}, \qquad \text{where } 0 = (0, \cdots, 0),$$

$$T_u = \sigma_u \mathbb{I} - \sum_{i=1}^{r} A_{N_i} \circ T_{i_\flat(u)}, \quad \text{where } |u| \geq 1. \tag{3.5}$$

The above objects also satisfy relations (3.3) and (3.4) in which $\{\sigma^*_u, T^*_u\}$ is replaced with $\{\sigma_u, T_u\}$, where $\mathbb{I}$ denote the identity on M.

Example 3.1. Let us consider the Minkowski spacetime manifold $(\mathbb{R}^4_1, \overline{g})$, where $\overline{g}(x, y) = -x^0y^0 + x^1y^1 + x^2y^2 + x^3y^3$, for any $x, y \in \mathbb{R}^4$. Let $\mathcal{D}$ be an open set of $\mathbb{R}^4$ and consider a smooth function $f : \mathcal{D} \longrightarrow \mathbb{R}^4$. Then $M = \{(x^0, \cdots, x^3) \in \mathbb{R}^4_1 : x^0 = f(x^1, \cdots, x^3)\}$ is called a Monge hypersurface [3]. Consider a parameterization on M as $x^0 = f(v^0, \cdots, v^3)$; $x^{d+1} =$

v^d, $d \in \{0, \cdots, 3\}$. In this case, natural frame fields on M are given by $\partial_{v^d} = f'_{x^{d+1}}\partial_{x^0} + \partial_{x^{d+1}}$, for all $d \in \{0, \cdots, 3\}$. Then it follows that $TM^{\perp}$ is spanned by $E = \partial_{x^0} + \sum_{i=1}^{3} f'_{x^i}\partial_{x^i}$. It is known [3] that M is a null hypersurface if $TM^{\perp} = \mathrm{Rad}\, TM$, which means that E must be a null vector field. Hence, M is null Monge hypersurface if f satisfies the differential equation: $\sum_{i=1}^{3} f'^{2}_{x^i} = 1$. The corresponding null transversal vector N is given by $N = \frac{1}{2}\{-\partial_{x^0} + \sum_{i=1}^{3} f'_{x^i}\partial_{x^i}\}$. Then, $S(TM) = \mathrm{span}\{X, Y\}$ where $X = f'_{x^3}\partial x^1 - f'_{x^1}\partial_{x^3}$ and $Y = f'_{x^3}\partial x^2 - f'_{x^2}\partial_{x^3}$, in which we have considered $f'_{x^3} \neq 0$ locally on M. By simple calculations we have $\overline{g}([X, Y], N) = 0$. Hence, $S(TM)$ is integrable. Now, using the fact that $\overline{g}([X, Y], N) = 0$, we have $\overline{g}(Y, \overline{\nabla}_X N) - \overline{g}(X, \overline{\nabla}_Y N) = 0$. Hence, from this last equation we have $g(A_N X, Y) = g(X, A_N Y)$, which shows that A_N is self-adjoint on $S(TM)$. The closeness of the respective 1-form τ also follows easily.

Also, by simple calculations the eigenvalues of A_N with respect to eigenvectors E, X and Y are $k_0 = 0$, $k_1 = \frac{2f'_{x^1}f'_{x^3}f''_{x^1x^3} - (f'_{x^1})^2 f''_{x^3x^3} - (f'_{x^3})^2 f''_{x^1x^1}}{2(1-(f'_{x^2})^2)}$ and $k_2 = \frac{2f'_{x^2}f'_{x^3}f''_{x^2x^3} - (f'_{x^2})^2 f''_{x^3x^3} - (f'_{x^3})^2 f''_{x^2x^2}}{2(1-(f'_{x^1})^2)}$, respectively. Hence, $\sigma_0 = 1$ and $\sigma_u = \sigma_u(A_N) = \sigma_q(k_0, k_1, k_2)$, for $q = 1, 2$, which is the usual symmetric function of the single operator A_N. Then, it follows that $T_0 = \mathbb{I}$ and $T_q = \sigma_q(k_0, k_1, k_2)\mathbb{I} - (k_{q-1})T_{q-1} \circ A_N$, for $q = 1, 2$.

3.2. The operator $\widehat{\mathcal{A}} = (A_{W_{r+1}}, \cdots, A_{W_n})$, where $\{W_\alpha\}_{r+1 \le \alpha \le n} \in S(TM^{\perp})$

From (2.13) we see that the operators A_{W_α}, for $\alpha \in \{r+1, \cdots, n\}$ are each self-adjoint on $S(TM)$, and thus diagonalizable on $S(TM)$. Let us consider an operator $\widehat{\mathcal{A}} = (A_{W_{r+1}}, \cdots, A_{W_n}) \in \mathrm{End}^{n-r}(TM)$ and let $\widehat{\sigma}_u$, for $u \ge 1$, be its corresponding symmetric function (generalized mean curvatures). Furthermore, let $\widehat{T}_u$ denote its generalized Newton transformation. Then, $\widehat{\sigma}$ and $\widehat{T}_u$ satisfy the following recurrence relation

$$\widehat{T}_0 = \mathbb{I}, \qquad \text{where } 0 = (0, \cdots, 0),$$

$$\widehat{T}_u = \widehat{\sigma}_u \mathbb{I} - \sum_{\alpha=r+1}^{n} A_{W_\alpha} \circ \widehat{T}_{\alpha_\flat(u)}, \quad \text{where } |u| \ge 1, \tag{3.6}$$

where $\mathbb{I}$ denotes the identity on M. It is easy to show that the above objects also satisfy relations (3.3) and (3.4) in which $\{\sigma^*_u, T^*_u\}$ is replaced with $\{\widehat{\sigma}_u, \widehat{T}_u\}$ and all the sums taken within $(r+1)$ to n.

Let us consider a quasi-orthonormal basis $\{X_1, \cdots, X_m\}$ adapted to TM. Then, the divergence [3] of a $(1, p)$-tensor $\mathbf{T}$ is a $(1, p-1)$-tensor $(\mathrm{div}^{\nabla}\mathbf{T})$ given by

$$(\mathrm{div}^{\nabla}\mathbf{T})(\omega_1, \cdots, \omega_{p-1}) = \sum_{e=1}^{m} (\nabla_{X_e}\mathbf{T})(X_e, \omega_1, \cdots, \omega_{p-1}). \tag{3.7}$$

4. Main results

In this section, we give new generalized results on umbilical null submanifolds in semi-Riemannian manifolds, which brings together well-known results on null hypersurfaces, half-null submanifolds and in a more general sense, r-null submanifolds given by Duggal-Bejancu [3], Duggal-Sahin [4] and Duggal-Jin [5]. The following technical lemma is fundamental to this paper.

Lemma 4.1. *For all $X \in \Gamma(TM)$, we have*

$$\sum_{j,a} g((\nabla_{Z_a} A^*_{E_j}) T^*_{j_\flat(u)} Z_a, X) = \sum_{j,a} g((\nabla_{Z_a} A^*_{E_j}) X, T^*_{j_\flat(u)} Z_a) - \sum_{i,j=1}^{r} \mathrm{tr}(A^*_{E_i} \circ A^*_{E_j} \circ T_{j_\flat(u)}) \lambda_i(X).$$

Proof. Applying the definition of covariant derivative of a tensor we have

$$\sum_{j=1}^{r} g((\nabla_{Z_a} A^*_{E_j}) T^*_{j_\flat(u)} Z_a, X) = \sum_{j=1}^{r} g(\nabla_{Z_a} A^*_{E_j} T^*_{j_\flat(u)} Z_a, X) - \sum_{j=1}^{r} g(\nabla_{Z_a} T^*_{j_\flat(u)} Z_a, A^*_{E_j} X), \tag{4.1}$$

for any $X \in \Gamma(TM)$. Now, setting $X = Z_a$, $Y = A^*_{E_j} T^*_{j_\flat(u)} Z_a$ and $Z = X$ in (2.10), we have

$$\sum_{j=1}^{r} g(\nabla_{Z_a} A^*_{E_j} T^*_{j_\flat(u)} Z_a, X) = \sum_{j=1}^{r} Z_a(g(A^*_{E_j} T^*_{j_\flat(u)} Z_a, X)) - \sum_{j=1}^{r} g(\nabla_{Z_a} X, A^*_{E_j} T^*_{j_\flat(u)} Z_a) - \sum_{i,j=1}^{r} h^l_i(Z_a, A^*_{E_j} T^*_{j_\flat(u)} Z_a) \lambda_i(X). \tag{4.2}$$

Also, setting $X = Z_a$, $Y = T^*_{j_\flat(u)} Z_a$ and $Z = A^*_{E_j} Z_a$ in (2.10) we derive

$$\sum_{j=1}^{r} g(\nabla_{Z_a} T^*_{j_\flat(u)} Z_a, A^*_{E_j} X) = \sum_{j=1}^{r} Z_a(g(T^*_{j_\flat(u)} Z_a, A^*_{E_j} X)) - \sum_{j=1}^{r} g(\nabla_{Z_a} A^*_{E_j} X, T^*_{j_\flat(u)} Z_a). \tag{4.3}$$

Substituting (4.2) and (4.3) in (4.1) we get

$$\sum_{j=1}^{r} g((\nabla_{Z_a} A^*_{E_j}) T^*_{j_\flat(u)} Z_a, X) = \sum_{j=1}^{r} g((\nabla_{Z_a} A^*_{E_j}) X, T^*_{j_\flat(u)} Z_a) - \sum_{i,j=1}^{r} h^l_i(Z_a, A^*_{E_j} T^*_{j_\flat(u)} Z_a) \lambda_i(X). \tag{4.4}$$

Using (2.12) we have $h_i^l(Z_a, A^*_{E_j}T^*_{j_\flat(u)}Z_a) = g(Z_a, A^*_{E_i}A^*_{E_j}T^*_{j_\flat(u)}Z_a)$. Hence, summing (4.4) over $a \in \{r+1, \cdots, m\}$ yields

$$\sum_{j,a} g((\nabla_{Z_a}A^*_{E_j})T^*_{j_\flat(u)}Z_a, X) = \sum_{j,a} g((\nabla_{Z_a}A^*_{E_j})X, T^*_{j_\flat(u)}Z_a) - \sum_{i,j=1}^{r} \mathrm{tr}(A^*_{E_i} \circ A^*_{E_j} \circ T_{j_\flat(u)})\lambda_i(X),$$

for all $X \in \Gamma(TM)$, which proves our assertion. □

Proposition 4.2. *Let $(M, g, S(TM), S(TM^\perp))$ be an m-dimensional r-null submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then,*

$$\begin{aligned}\overline{g}(\mathrm{div}^\nabla(T^*_u), X) = &-\sum_{i=1}^{r} E_i(\sigma^*_u)\lambda_i(X) - \sum_{i=1}^{r} g((\nabla_{E_i}\sum_{j=1}^{r} A^*_{E_j} \circ T^*_{j_\flat(u)})E_i, X)\\ &- \sum_{j=1}^{r} g(\mathrm{div}^{\nabla^*}(T^*_{j_\flat(u)}), A^*_{E_j}X) + \sum_{i,j=1}^{r} \mathrm{tr}(A^*_{E_i} \circ A^*_{E_j} \circ T_{j_\flat(u)})\lambda_i(X)\\ &+ \sum_{j,a} \overline{R}(Z_a, X, E_j, T^*_{j_\flat(u)}Z_a) - \sum_{i,j=1}^{r} \mathrm{tr}(A^*_{E_i} \circ T^*_{j_\flat(u)})\tau_{ij}(X)\\ &+ \sum_{j,\alpha} \varepsilon_\alpha \mathrm{tr}(A_{W_\alpha} \circ T^*_{j_\flat(u)})\varphi_{\alpha j}(X) + \sum_{j,a}\sum_{i=1}^{r} h_i^l(X, T^*_{j_\flat(u)}Z_a)\tau_{ij}(Z_a)\\ &- \sum_{j,a}\sum_{\alpha=r+1}^{n} h^s_\alpha(X, T^*_{j_\flat(u)}Z_a)\varphi_{\alpha j}(Z_a), \qquad \forall X \in \Gamma(TM).\end{aligned}$$

Proof. Applying recurrence relation (3.2) and (3.7) we derive

$$\begin{aligned}\overline{g}(\mathrm{div}^\nabla(T^*_u), X) = PX(\sigma^*_u) &- \sum_{i=1}^{r} g((\nabla_{E_i}\sum_{j=1}^{r} A^*_{E_j} \circ T^*_{j_\flat(u)})E_i, X)\\ - \sum_{j=1}^{r} g(\mathrm{div}^{\nabla^*}(T^*_{j_\flat(u)}), A^*_{E_j}X) &- \sum_{j,a} g((\nabla_{Z_a}A^*_{E_j})T^*_{j_\flat(u)}Z_a, X). \qquad (4.5)\end{aligned}$$

Applying Lemma 4.1 to (4.5) we get

$$\begin{aligned}\overline{g}(\mathrm{div}^\nabla(T^*_u), X) = PX(\sigma^*_u) &- \sum_{i=1}^{r} g((\nabla_{E_i}\sum_{j=1}^{r} A^*_{E_j} \circ T^*_{j_\flat(u)})E_i, X)\\ - \sum_{j=1}^{r} g(\mathrm{div}^{\nabla^*}(T^*_{j_\flat(u)}), A^*_{E_j}X) &+ \sum_{i,j=1}^{r} \mathrm{tr}(A^*_{E_i} \circ A^*_{E_j} \circ T_{j_\flat(u)})\lambda_i(X)\\ - \sum_{j,a} g(T^*_{j_\flat(u)}Z_a, (\nabla_{Z_a}A^*_{E_j})X), &\qquad \forall X \in \Gamma(TM). \qquad (4.6)\end{aligned}$$

From (2.4), (2.8) and (2.15) and taking $X = Z_a$, $Y = X$, $PU = T^*_{j_\flat(u)} Z_a$, we get

$$
\begin{aligned}
&g(T^*_{j_\flat(u)} Z_a, (\nabla_{Z_a} A^*_{E_j})X) \\
&= -\overline{R}(Z_a, X, E_j, T^*_{j_\flat(u)} Z_a) + g(T^*_{j_\flat(u)} Z_a, (\nabla_X A^*_{E_j})Z_a) \\
&+ \sum_{i=1}^{r} h^l_i(Z_a, T^*_{j_\flat(u)} Z_a)\tau_{ij}(X) - \sum_{i=1}^{r} h^l_i(X, T^*_{j_\flat(u)} Z_a)\tau_{ij}(Z_a) \\
&- \sum_{\alpha=r+1}^{n} \varepsilon_\alpha h^s_\alpha(Z_a, T^*_{j_\flat(u)} Z_a)\varphi_{\alpha j}(X) + \sum_{\alpha=r+1}^{n} h^s_\alpha(X, T^*_{j_\flat(u)} Z_a)\varphi_{\alpha j}(Z_a).
\end{aligned} \tag{4.7}
$$

Finally, putting (4.7) into (4.6) and using the fact that $\sum_{j=1}^{r} \mathrm{tr}(T^*_{j_\flat(u)}(\nabla_X A^*_{E_j})) = X(\sigma^*_u)$, we get the desired result, which completes the proof. □

The following result follows immediately from Proposition 4.2.

Theorem 4.3. *Let $(M, g, S(TM), S(TM^\perp))$ be an m-dimensional totally umbilical r-null submanifold of an $(m+n)$-dimensional semi-Riemannian manifold of constant curvature $(\overline{M}(c), \overline{g})$. Then, the generalized mean curvature functions $\sigma^*_u :\ u \geq 1$ of $\mathcal{A}^* = (A^*_{E_1}, \cdots, A^*_{E_r})$ satisfy the following partial differential equations*

$$
E_k(\sigma^*_u) - \sum_{j=1}^{r} \mathrm{tr}(A^*_{E_k} \circ A^*_{E_j} \circ T^*_{j_\flat(u)}) + \sum_{i,j=1}^{r} \mathrm{tr}(A^*_{E_i} \circ T^*_{j_\flat(u)})\tau_{ij}(E_k) = 0,
$$

for all $k \in \{1, 2, \cdots, n\}$.

Proof. When M is totally umbilical, then we have $\varphi_{\alpha j}(X) = \varphi_{\alpha j}(Z_a) = 0$, $h^l_i(X, T^*_{j_\flat(u)} Z_a) = 0$ and $h^s_\alpha(X, T^*_{j_\flat(u)} Z_a) = 0$ for any $X \in \Gamma(\mathrm{Rad}\, TM)$. Setting $X = E_k$ in Proposition 4.2 and using (2.20), then $\mathrm{div}^\nabla(T^*_u)$ belongs to $TM^\perp$, the result follows. □

Theorem 4.4. *Let $(M, g, S(TM), S(TM^\perp))$ be an m-dimensional totally umbilical r-null submanifold of an $(m+n)$-dimensional semi-Riemannian manifold of constant curvature $(\overline{M}(c), \overline{g})$. Then, the generalized mean curvature functions $\widehat{\sigma}^*_u :\ u \geq 1$ of $\widehat{\mathcal{A}} = (A_{W_{r+1}}, \cdots, A_{W_n})$ satisfy the following partial differential equations*

$$
E_j(\widehat{\sigma}_u) - \sum_{\alpha} \mathrm{tr}(A^*_{E_j} \circ A_{W_\alpha} \circ \widehat{T}_{\alpha_\flat(u)}) + \sum_{\alpha,\beta} \mathrm{tr}(A_{W_\alpha} \circ \widehat{T}_{\alpha_\flat(u)})\theta_{\alpha\beta}(E_j) = 0,
$$

for all $j \in \{1, 2, \cdots, n\}$.

Proof. By considering (3.6), (2.17), (2.10) and (2.14) we have by the method of Lemma 4.1 and Proposition 4.2 that

$$
\begin{aligned}
\overline{g}(\mathrm{div}^{\nabla}(\widehat{T}_u), X) = &-\sum_{i=1}^{r} E_i(\widehat{\sigma}_u)\lambda_i(X) - \sum_{i=1}^{r} g((\nabla_{E_i} \sum_{\alpha=r+1}^{n} A_{W_\alpha} \circ \widehat{T}_{\alpha_\flat(u)})E_i, X) \\
&- \sum_{\alpha=r+1}^{n} g(\mathrm{div}^{\nabla^*}(\widehat{T}_{\alpha_\flat(u)}), A_{W_\alpha}X) + \sum_{i,\alpha} \mathrm{tr}(A^*_{E_i} \circ A_{W_\alpha} \circ \widehat{T}_{\alpha_\flat(u)})\lambda_i(X) \\
&+ \sum_{\alpha,a} \overline{R}(Z_a, X, W_\alpha, \widehat{T}_{\alpha_\flat(u)}Z_a) - \sum_{\alpha,\beta} \mathrm{tr}(A_{W_\beta} \circ \widehat{T}_{\alpha_\flat(u)})\theta_{\alpha\beta}(X) \\
&+ \sum_{a}\sum_{\alpha,\beta} g(A_{W_\beta}X, \widehat{T}_{\alpha_\flat(u)}Z_a)\theta_{\alpha\beta}(Z_a) - \sum_{\alpha,a}\sum_{i} h^*_i(X, \widehat{T}_{\alpha_\flat(u)}Z_a)\varphi_{\alpha i}(Z_a) \\
&+ \sum_{i,\alpha} \mathrm{tr}(A_{N_i} \circ \widehat{T}_{\alpha_\flat(u)})\varphi_{\alpha i}(X), \qquad (4.8)
\end{aligned}
$$

for any $X \in \Gamma(TM)$.The result follows by taking $X = E_j$ in (4.8) and using the umbilicity of M. □

In Theorem 5.2 of [3, p. 108] the authors showed that for a totally umbilical null hypersurface of a $(m+2)$-dimensional semi-Riemannian manifold of constant curvature $(\overline{M}(c), \overline{g})$, the function ρ such that $A^*_E PX = \rho PX$, where $X \in \Gamma(TM)$, satisfies the following differential equation $E(\rho) - \rho^2 + \rho\tau(E) = 0$. The above differential equation also holds for a half null submanifold (see Theorem 4.3.3 of [4, p. 170]). For the case of a totally umbilical null submanifold, Duggal-Jin [5, p. 60] showed, in Theorem 4.2 therein, that the functions $\mathcal{H}^l_i$ and $\mathcal{H}^s_\alpha$ in (2.22) satisfy the differential equations $E_j(\mathcal{H}^l_i) - \mathcal{H}^l_i\mathcal{H}^l_j + \sum_{k=1}^{r}\mathcal{H}^l_k\tau_{ki}(E_j) = 0$ and $E_j(\mathcal{H}^s_\alpha) - \mathcal{H}^s_\alpha\mathcal{H}^l_j + \sum_{\beta=r+1}^{n}\mathcal{H}^s_\beta\theta_{\beta\alpha}(E_j) = 0$. Hence, we can say that Theorems 4.3 and 4.4 are generalizations of all the above mentioned results.

Let $(M, g, S(TM), S(TM^\perp))$ be an r-null submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, the screen distribution $S(TM)$ is said to be totally umbilical in M [3] if there is a smooth vector field $\mathbf{K}$ of $\mathrm{Rad}\,TM$ on M, such that $h^*(X, PY) = g(X, PY)\mathbf{K}$, for any $X, Y \in \Gamma(TM)$. Moreover, $S(TM)$ is totally umbilical, if and only if, on any coordinate neighborhood $\mathcal{U} \subset M$, there exist smooth functions $\mathcal{K}_i$ such that $h^*_i(X, PY) = \mathcal{K}_i g(X, PY)$, for any $X, Y \in \Gamma(TM)$. It is also easy to see that for an umbilical $S(TM)$ one gets $P(A_{N_i}X) = \mathcal{K}_i PX$, $h^*(E, PX) = 0$, $\forall\, X \in \Gamma(TM)$, where $E \in \Gamma(\mathrm{Rad}\,TM)$.

Theorem 4.5. *Let $(M, g, S(TM), S(TM^\perp))$ be either an r-null or a co-isotropic submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ of constant curvature c, with a totally umbilical screen distribution $S(TM)$. If M is also totally umbilical, then the generalized mean curvatures σ_u; $u \geq 1$ of $\mathcal{A} = (A_{N_1}, \cdots, A_{N_r})$*

on $S(TM)$ are a solution of the following partial differential equations

$$E_k(\sigma_u) - \sum_{j=1}^{r} \mathrm{tr}(A^*_{E_k} \circ A_{N_j} \circ T_{j_\flat(u)}) - \sum_{i,j=1}^{r} \mathrm{tr}(A_{N_i} \circ T_{j_\flat(u)})\tau_{ij}(E_k)$$
$$- c\sum_{j=1}^{r} \mathrm{tr}(T_{j_\flat(u)}) = 0,$$

for all $k \in \{1, 2, \cdots, n\}$.

Proof. By the method of Lemma 4.1 and Proposition 4.2 with recurrence (3.5) and (2.16) we derive

$$\begin{aligned}\overline{g}(\mathrm{div}^{\nabla}(T_u), X) = &-\sum_{i=1}^{r} E_i(\sigma_u)\lambda_i(X) - \sum_{i=1}^{r} g((\nabla_{E_i}\sum_{j=1}^{r} A_{N_j} \circ T_{j_\flat(u)})E_i, X)\\
&- \sum_{j=1}^{r} g(\mathrm{div}^{\nabla^*}(T_{j_\flat(u)}), A_{N_j}X) + \sum_{i,j=1}^{r} \mathrm{tr}(A^*_{E_i} \circ A_{N_j} \circ T_{j_\flat(u)})\lambda_i(X)\\
&+ \sum_{j,a} \overline{R}(Z_a, X, N_j, T_{j_\flat(u)}Z_a) + \sum_{i,j=1}^{r} \mathrm{tr}(A_{N_i} \circ T_{j_\flat(u)})\tau_{ij}(X)\\
&+ \sum_{j,\alpha} \varepsilon_\alpha \mathrm{tr}(A_{W_\alpha} \circ T_{j_\flat(u)})\rho_{\alpha j}(X) - \sum_{j,a}\sum_{i=1}^{r} h^*_i(X, T_{j_\flat(u)}Z_a)\tau_{ij}(Z_a)\\
&- \sum_{j,a} \overline{g}(h^s(X, T_{j_\flat(u)}Z_a), D^s(Z_a, N)), \qquad \forall\, X \in \Gamma(TM).\end{aligned}$$

Now, replacing X in the above equation with E_k and using the facts $\overline{M}$ is of constant sectional curvature, M and $S(TM)$ are totally umbilical we get the desired result. Hence the proof. □

Corollary 4.6. *Under the hypothesis of Theorem 4.5, ∇ on M is a metric connection if, and only if, the mean curvature functions σ_u; $u \geq 1$ are a solution of the following partial differential equations*

$$E_k(\sigma_u) - \sum_{i,j=1}^{r} \mathrm{tr}(A_{N_i} \circ T_{j_\flat(u)})\tau_{ij}(E_k) - c\sum_{j=1}^{r} \mathrm{tr}(T_{j_\flat(u)}) = 0,$$

for all $k \in \{1, 2, \cdots, n\}$.

Notice that Theorem 4.5 and Corollary 4.6 are generalizations of Theorem 4.4 and Corollary 2 of [5], respectively. Let $x \in M$ and E be a null vector of $T_x\overline{M}$. A plane Π of $T_x\overline{M}$ is called a null plane directed by E if it contains E, $\overline{g}(E, W) = 0$ for any $W \in \Pi$ and there exists $W_0 \in \Pi$ such that $\overline{g}(W_0, W_0) \neq 0$. Then, from [3, p. 95], we define the null sectional curvature of Π with respect to E and $\overline{\nabla}$ as the real number

$$\overline{K}_E(\Pi) = \frac{\overline{R}(W, E, E, W)}{g(W, W)}, \tag{4.9}$$

where W is an arbitrary non-null vector in Π. Similarly, we define the null sectional curvature $K_E(\Pi)$ of the null plane Π of the tangent space T_xM with respect to E and ∇ as the real number

$$K_E(\Pi) = \frac{R(W,E,E,W)}{g(W,W)}. \tag{4.10}$$

Using the fact that both the null sectional curvatures in (4.9) and (4.10) are independent of $W \in \Pi$, we derive from (2.15) and (2.19) that

$$\begin{aligned}\overline{K}_E(\Pi_j) &= E_j(\sigma_u^*) - \sum_{j=1}^{r} \operatorname{tr}(A_{E_j}^{*2} \circ T_{j_\flat(u)}^*) \\ &\quad + \sum_{i,j=1}^{r} \operatorname{tr}(A_{E_i}^* \circ T_{j_\flat(u)}^*)\tau_{ij}(E_j) = K_E(\Pi_j).\end{aligned} \tag{4.11}$$

Therefore, we have the following theorem.

Theorem 4.7. *Let $(M, g, S(TM), S(TM^\perp))$ be either an r-null or a co-isotropic submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then, both the null sectional curvature $K_E(\Pi_j)$ and $\overline{K}_E(\Pi_j)$ vanish, if and only if, $\sigma_u^*: u \geq 1$ of $\mathcal{A}^* = (A_{E_1}^*, \cdots, A_{E_r}^*)$ is a solution of the partial differential equations*

$$E_j(\sigma_u^*) - \sum_{j=1}^{r} \operatorname{tr}(A_{E_j}^{*2} \circ T_{j_\flat(u)}^*) + \sum_{i,j=1}^{r} \operatorname{tr}(A_{E_i}^* \circ T_{j_\flat(u)}^*)\tau_{ij}(E_j) = 0.$$

Proof. The proof follows immediately from (4.11). □

Notice that Theorem 4.7 generalizes Theorem 4.7 of [5].

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