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# ATANASSOV'S INTUITIONISTIC FUZZY SET THEORY APPLIED TO QUANTALES

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**Abstract.** The main goal of this paper is to study quantales in the framework of intuitionistic fuzzy left (right) ideals. The notions of intuitionistic fuzzy left (right) ideals are introduced, and the related properties are investigated. The concept of lower (resp. upper) level cut is provided and the characterizations of intuitionistic fuzzy left (right) ideals are studied. The notions of intuitionistic fuzzy product is considered and their properties related to intuitionistic fuzzy left (right) ideals are studied. Using the notions of intuitionistic fuzzy left (right) ideals, the main theorem for a regular quantale is investigated.

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## 1. Introduction

Zadeh introduced a mathematical framework called fuzzy sets [23] which plays a significant role in many fields of real life. Fuzzy set has several advantages over a Cantorian set because it has a clear demarcation about uncertainty and vagueness. A fuzzy set is actually characterized by a membership function with the range of [0, 1]. The membership of an element in a fuzzy set is a single value between 0 and 1. However in actual practice, it may not always be true that the degree of non-membership function of an element in a fuzzy set is equal to 1 minus the membership degree because there may be some margin of hesitation. Therefore, Atanassov [2, 3] coined the concept of intuitionistic fuzzy set which is abbreviated as IFS. IFS is basically characterized by a membership and non-membership functions which has hesitation degree and is defined as 1 minus the sum of membership and non-membership degrees. Therefore, IFS is an extension of Zadeh's fuzzy set. To interpret knowledge about intuitionistic fuzzy set it became most desirable, significant and mostly permissible with addition to the degree of belongingness, non-belongess and hesitation margin [4, 5]. In terms of a membership function Szmidt and Kacpryzk [6] showed that intuitionistic fuzzy sets are pretty useful in situations when description

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of a problem by a linguistic variable given only seems too rough. Due to the flexibility of IFS in handling uncertainty, they are tool for a more human consistent reasoning under imperfectly defined facts and imprecise knowledge [19]. In medical diagnosis De et al. [10] gave the approach of an intuitionistic fuzzy sets using three steps such as; determination of diagnosis on the basis of composition of intuitionistic fuzzy relations, determination of symptoms and formulation of medical knowledge based on intuitionistic fuzzy relations. In modelling real life problems intuitionistic fuzzy set is a tool like psychological investigations, negotiation process, sale analysis, financial services, etc. At each moment of evaluation of an unknown object [20] there is a fair chance of the existence of a non-null hesitation part. On the theory and applications of intuitionistic fuzzy sets Atanassov [4, 5] carried out rigorous research. Distance measures approach are carried out in many applications of IFS. In fuzzy mathematics distance measure is an important concept between intuitionistic fuzzy sets because of its wide applications in real world, such as market prediction, pattern recognition, decision making and machine learning. The comparison of vague sets are identical with that of intuitionistic fuzzy sets have shown by Burillo and Bustine [7]. The concept of the degree of similarity between intuitionistic fuzzy sets, which may be finite or continuous, were introduced by Dengfeng and Chunfian [11] and they gave corresponding proofs of these similarity measures and the applications of the similarity measures between intuitionistic fuzzy sets to pattern recognition problems are discussed. In the field of mathematics intuitionistic fuzzy sets have many applications. This concept was applied by Davvaz et al. [8] in $H_v$-modules. The notion of an IF $H_v$-submodule of an $H_v$-module introduced by them and also studied about the related properties. Intuitionistic fuzzy ideals were proposed by Khan et al. [17, 18] in ordered semigroups.

The theory of quantales initiated by Mulvey [15] is a combination of algebraic structures and partially ordered structures. This theory was developed for the study of spectrum of $C^*$-algebras and the foundations of quantum mechanics [15]. Wang and Zhao [21] proposed the concepts of ideals and prime ideals of quantales. In [22] Yang and Xu considered the notions of quantales as universal sets and defined rough prime ideals, rough semi-prime ideals etc. and prime radicals of upper rough ideals of quantales. In addition, they showed that the set consisting of all rough ideals is a directed complete partially ordered set as well as an algebraic lattice. Fuzzy ideals and fuzzy rough ideals of quantales were introduced by Luo and Wang [14]. Mohsin Khan defined regular and intra-regular quantales in [13] and gave the basic characterizations of these notions. They further discussed the properties of these classes of quantales in the context of intuitionistic fuzzy left and right ideals.

Intuitionistic fuzzy sets have many applications in different areas of sciences. One of the application has been discussed in Davvaz et al. [9]. They used the relationship between intuitionistic fuzzy sets and the symptoms of patients and to diagnose of illness. In [16], intuitionistic fuzzy sets have been used in the electoral system. In [1], M. Akram et al. investigated some interesting properties of intuitionistic fuzzy line graphs. In [12], intuitionistic fuzzy sets have been used and introduced the concepts of intuitionistic fuzzy hyperideal extension of semihypergroup and intuitionistic fuzzy prime (semiprime) hyperideal of a semihypergroup.

In this paper, we extend the concept of intuitionistic fuzzy sets in quantales and define intuitionistic fuzzy left (resp. right) ideals. The basic properties of these notions are discussed in detail. The characterizations of regular quantales is studied by using intuitionistic fuzzy left and right ideals.

## 2. Definitions and basic results

**Definition 2.1.** A quantale is a complete lattice $Q$ with an associative binary operation (see [15]) satisfying

$$(\forall a_i, b_i, a, b) \qquad a \circ (\vee_{i\in I} b_i) = \vee_{i\in I}(a \circ b_i) \textit{ and } (\vee_{i\in I} b_i) \circ a = \vee_{i\in I}(b_i \circ a)$$

where $I$ is an index set.

Throughout this paper, the least and greatest elements of a quantale are denoted by "0" and "1". In this paper, we adopt the definition of ideals of a quantale given by Wang and Zhao [14, 22], which combined the partial order and the operator " $\circ$ " of a quantale.

**Definition 2.2.** Let $Q$ be a quantale. A non-empty subset $I \subseteq Q$ is called $left$ (resp.,$right$) *ideal* [22] of $Q$ if it satisfies the following conditions:
$(Q1)$ $(\forall a, b \in Q)\,(a, b \in I \implies a \bigvee b \in I)$.
$(Q2)$ $(\forall a, b \in Q)\,(\forall b \in I)\,(a \leq b \implies a \in I)$
$(Q3)$ $(\forall x, a \in Q)$ $(a \in I \implies a \circ x \in I$ (resp., $x \circ a \in I))$.

A non-empty subset $I$ of a quantale $Q$ is called a $two-sided\ ideal$ or simply an *ideal* of $Q$ if it is both a left and a right ideal of $Q$.

In a quantale $Q$, for $A, B \subseteq Q$, we write $A \circ B$ to denote the set $\{a \circ b \mid a \in A, b \in B\}$ and $A \bigvee B$ to denote $\{a \bigvee b \mid a \in A, b \in B\}$. For a non-empty subset $A$ of a quantale, we write

$$(A] := \{x \in Q \mid x \leq a \text{ for some } a \in A\}.$$

If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$.

Let $Q$ be a quantale and $A \subseteq Q$. The least ideal containing $A$ is called the ideal generated by $A$, and is denoted by $\langle A\rangle_t$, where $\langle A\rangle_t = (A \cup Q \circ A \cup A \circ Q \cup Q \circ A \circ Q]$ (*see* [14]). Similarly, the least left ideal and the least right ideal of $Q$ generated by $A$, are denoted by $\langle A\rangle_l$ and $\langle A\rangle_r$ , respectively. here $\langle A\rangle_l = (A \cup Q \circ A]$ and $\langle A\rangle_r = (A \cup A \circ Q]$. If $A = \{a\}$, then the least left (resp. right and two-sided) ideal of $Q$ generated by $a$ $(a \in Q)$; respectively they are given as follows: $\langle a\rangle_l = (a \cup Q \circ a]$ (resp. $\langle a\rangle_r = (a \cup a \circ Q]$ and $\langle a\rangle_t = (a \cup Q \circ a \cup a \circ Q \cup Q \circ a \circ Q]$).

A quantale is regular [13] if for every $a \in Q$, there exists $x \in Q$ such that $a \leq a \circ x \circ a$. Equivalently,
(1) $a \in (a \circ Q \circ a]$ $\forall a \in Q$.
(2) $A \subseteq (A \circ Q \circ A]$ $\forall A \subseteq Q$.

**Lemma 2.3.** *(cf. [13]) A quantale* $(Q, \circ, \leq)$ *is regular if and only if* $\langle a\rangle_r \cap \langle a\rangle_l \subseteq (\langle a\rangle_r \circ \langle a\rangle_l]$ *for every* $a \in Q$.

A mapping $\mu : Q \longrightarrow [0,1]$, where $Q$ is an arbitrary non-empty set, is called a fuzzy set in $Q$. The complement of $\mu$, denoted by $\mu^C$, is the fuzzy set in $Q$ given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in Q$. Let $\mathbf{0}$ and $\mathbf{1}$ be fuzzy sets in $Q$ defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in Q$.

**Definition 2.4.** [10] Let $Q$ be a quantale. A fuzzy subset of $Q$ is called a fuzzy left (right) ideal of $Q$ if it satisfies the following conditions:
$(FQ1)$ $(\forall x, y \in Q)\,(x \leq y \implies \mu(x) \geq \mu(y))$,
$(FQ2)$ $(\forall x, y \in Q)\,(\mu(x \bigvee y) \geq \mu(x) \bigwedge \mu(y))$,
$(FQ3)$ $(\forall x, y \in Q)\,(\mu(x \circ y) \geq \mu(y) \quad (\geq \mu(x)))$.

A fuzzy subset $\mu$ of a quantale $Q$ is called a fuzzy ideal if it is both a fuzzy left and right ideal of $Q$. Equivalently, a fuzzy subset $\mu$ is called a fuzzy ideal of $Q$ if it satisfies conditions $(FQ1)$, $(FQ2)$ and

$$(FQ4)\,(\forall x, y \in Q)\left(\mu(x \circ y) \geq \mu(x) \bigvee \mu(y)\right).$$

## 3. Intuitionistic fuzzy ideals

After the introduction of fuzzy sets by Zadeh, several researchers have started work on the generalization of fuzzy sets. As an important generalization of the notion of fuzzy sets on a non-empty set $X$, Atanassov introduced in [2] the concept of intuitionistic fuzzy sets (briefly, IFS) defined on a non-empty set $X$ as objects having the form

$$A = \{\langle x, \mu_A(x), \lambda_A(x)\rangle \mid x \in X\},$$

where the functions $\mu_A(x) : X \longrightarrow [0,1]$ and $\lambda_A(x) : X \longrightarrow [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\lambda_A(x)$) of each element $x \in X$ to the set $A$ respectively, and $0 \leq \mu_A + \lambda_A \leq 1$ for all $x \in X$. Intuitionistic fuzzy sets have been studied in many branches of mathematics, particularly in algebra (for example see [7, 8, 10])).

Let $A$ and $B$ be two intuitionistic fuzzy sets on $X$. Then the following expressions are well known identities defined in [3].

$(IF1)$ $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$, and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$,
$(IF2)$ $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$,
$(IF3)$ $A^C = \{\langle x, \lambda_A(x), \mu_A(x)\rangle \mid x \in X\}$,
$(IF4)$ $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x))\rangle\}, \{\langle x, \max(\lambda_A(x), \lambda_B(x))\rangle\}$,
$(IF5)$ $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x))\rangle\}, \{\langle x, \min(\lambda_A(x), \lambda_B(x))\rangle\}$,
$(IF6)$ $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$,
$(IF7)$ $\Diamond A = \{\langle x, 1 - \lambda_A(x), \lambda_A(x)\rangle \mid x \in X\}$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \lambda_A)$ for intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \lambda_A(x)\rangle \mid x \in X\}$. Let $\mathbf{0}_\sim = (0, 1)$ and $\mathbf{1}_\sim = (1, 0)$ be intuitionistic fuzzy sets in $Q$.

We now, establish the intuitionistic fuzzification of the concept of ideals in a quantale and investigate some of their properties.

**Definition 3.1.** Let $Q$ be a quantale. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in $Q$ is called a left (resp. right) intuitionistic fuzzy ideal of $Q$ if it satisfies the conditions:

$(IFQ1)(\forall x, y \in Q)(x \leq y \implies \mu_A(x) \geq \mu_A(y),\ \lambda_A(x) \leq \lambda_A(y))$,
$(IFQ2)(\forall x, y \in Q)\left(\mu_A\left(x \bigvee y\right) \geq \mu_A(x) \bigwedge \mu_A(y)\right)$,
$(IFQ3)(\forall x, y \in Q)\left(\lambda_A\left(x \bigvee y\right) \leq \lambda_A(x) \bigwedge \lambda_A(y)\right)$,
$(IFQ4)(\forall x, y \in Q)\left(\mu_A(x \circ y) \geq \mu_A(y) \quad (\text{resp.}, \ \geq \mu_A(x))\right)$,
$(IFQ5)(\forall x, y \in Q)\left(\lambda_A(x \circ y) \leq \lambda_A(y) \quad (\text{resp.}, \ \leq \lambda_A(x))\right)$.

**Example 3.2.** Let $Q$ be a quantale.

(1) Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set in $Q$, defined by $\mu_A(x) = 0.5$ and $\lambda_A(x) = 0.5$ for all $x \in Q$. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy left (right) ideal of $Q$ and hence an intuitionistic fuzzy ideal of $Q$.

(2) Let $A = (\mu_A, \lambda_A)$ be a non-constant intuitionistic fuzzy set defined by
$\mu_A(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0.5 \text{ if } \neq 0 \end{cases}$ and $\lambda_A(x) = \begin{cases} 0 \text{ if } x = 0 \\ 0.5 \text{ if } \neq 0. \end{cases}$
Then $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy ideal of $Q$.

For all the results discussed in this paper, we only describe proofs for the intuitionistic fuzzy left ideals. For the intuitionistic fuzzy right ideals similar results hold as well.

**Proposition 3.3.** *If $\{A_i\}_{i\in\Lambda}$ is a family of intuitionistic fuzzy left (resp., right) ideals of a quantale $Q$. Then $\bigcap_{i\in\Lambda} A_i$ is an intuitionistic fuzzy left (resp., right) ideal of $Q$, where $\bigcap_{i\in\Lambda} A_i = \left(\bigcap_{i\in\Lambda} \mu_{A_i}, \bigcap_{i\in\Lambda} \lambda_{A_i}\right)$ and*

$$\bigcap_{i\in\Lambda} \mu_{A_i} = \bigwedge_{i\in\Lambda} \mu_{A_i}(x) = \inf\left\{\mu_{A_i}(x) \mid i \in \Lambda, x \in Q\right\},$$

$$\bigcup_{i\in\Lambda} \lambda_{A_i} = \bigvee_{i\in\Lambda} \lambda_{A_i}(x) = \sup\left\{\lambda_{A_i}(x) \mid i \in \Lambda, x \in Q\right\}.$$

*Proof.* For $x, y \in Q$ if $x \leq y$, then we have

$$\begin{aligned} \left(\bigcap_{i\in\Lambda} \mu_{A_i}\right)(x) &= \left(\bigwedge_{i\in\Lambda} \mu_{A_i}\right)(x) = \bigwedge_{i\in\Lambda} \left(\mu_{A_i}(x)\right) \\ &\geq \bigwedge_{i\in\Lambda} \left(\mu_{A_i}(y)\right) = \left(\bigwedge_{i\in\Lambda} \mu_{A_i}\right)(y) \\ &= \left(\bigcap_{i\in\Lambda} \mu_{A_i}\right)(y), \end{aligned}$$

and

$$\begin{aligned}\left(\bigcup_{i\in\Lambda}\lambda_{A_i}\right)(x) &= \left(\bigvee_{i\in\Lambda}\lambda_{A_i}\right)(x)=\bigvee_{i\in\Lambda}\left(\lambda_{A_i}(x)\right)\\ &\leq \bigvee_{i\in\Lambda}\left(\lambda_{A_i}(y)\right)=\left(\bigvee_{i\in\Lambda}\lambda_{A_i}\right)(y)\\ &= \left(\bigcup_{i\in\Lambda}\lambda_{A_i}\right)(y).\end{aligned}$$

For $x, y \in Q$, we have

$$\begin{aligned}&\left(\bigcap_{i\in\Lambda}\mu_{A_i}\right)(x\vee y)\\ &= \left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(x\vee y)=\bigwedge_{i\in\Lambda}\left(\mu_{A_i}(x\vee y)\right)\\ &\geq \bigwedge_{i\in\Lambda}\left(\mu_{A_i}(x)\wedge\mu_{A_i}(y)\right)=\left(\left(\bigwedge_{i\in\Lambda}\mu_{A_i}(x)\right)\wedge\left(\bigwedge_{i\in\Lambda}\mu_{A_i}(y)\right)\right)\\ &= \left(\bigcap_{i\in\Lambda}\mu_{A_i}\right)(x)\wedge\left(\bigcap_{i\in\Lambda}\mu_{A_i}\right)(y),\end{aligned}$$

and

$$\begin{aligned}&\left(\bigcup_{i\in\Lambda}\lambda_{A_i}\right)(x\vee y)\\ &= \left(\bigvee_{i\in\Lambda}\lambda_{A_i}\right)(x\vee y)=\bigvee_{i\in\Lambda}\left(\lambda_{A_i}(x\vee y)\right)\\ &\leq \bigvee_{i\in\Lambda}\left(\lambda_{A_i}(x)\vee\lambda_{A_i}(y)\right)=\left(\left(\bigvee_{i\in\Lambda}\lambda_{A_i}(x)\right)\vee\left(\bigvee_{i\in\Lambda}\lambda_{A_i}(y)\right)\right)\\ &= \left(\bigcup_{i\in\Lambda}\lambda_{A_i}\right)(x)\vee\left(\bigcup_{i\in\Lambda}\lambda_{A_i}\right)(y).\end{aligned}$$

Let $x, y \in Q$. then we have

$$\begin{aligned}\left(\bigcap_{i\in\Lambda}\mu_{A_i}\right)(x\circ y) &= \left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(x\circ y)=\bigwedge_{i\in\Lambda}\left(\mu_{A_i}(x\circ y)\right)\\ &\geq \bigwedge_{i\in\Lambda}\mu_{A_i}(y)=\left(\bigwedge_{i\in\Lambda}\mu_{A_i}(y)\right)\\ &= \left(\bigcap_{i\in\Lambda}\mu_{A_i}\right)(y),\end{aligned}$$

and

$$\begin{aligned}\left(\bigcup_{i\in\Lambda}\lambda_{A_i}\right)(x\circ y) &= \left(\bigvee_{i\in\Lambda}\lambda_{A_i}\right)(x\circ y)=\bigvee_{i\in\Lambda}\left(\lambda_{A_i}(x\circ y)\right)\\ &\le \bigvee_{i\in\Lambda}\lambda_{A_i}(y)=\left(\bigvee_{i\in\Lambda}\lambda_{A_i}(y)\right)\\ &= \left(\bigcup_{i\in\Lambda}\lambda_{A_i}\right)(y).\end{aligned}$$

Therefore, $\bigcap_{i\in\Lambda} A_i$ is an intuitionistic fuzzy left ideal of $Q$. □

**Lemma 3.4.** *If $A=(\mu_A,\lambda_A)$ is an intuitionistic fuzzy left (resp., right) ideal, then so are $\Box A$ and $\Diamond A$.*

*Proof.* Assume that $A=(\mu_A,\lambda_A)$ is an intuitionistic fuzzy left ideal of $Q$. Let $x,y\in Q$ be such that $x\le y$. Then $\mu_A(x)\ge\mu_A(y)$ and we have

$$\mu_A^C(x)=1-\mu_A(x)\le 1-\mu_A(y)=\mu_A^C(y).$$

For $x,y\in Q$, we have $\mu_A(x\vee y)\ge\mu_A(x)\wedge\mu_A(y)$ and we get

$$\begin{aligned}\mu_A^C(x\vee y) &= 1-\mu_A(x\vee y)\le 1-\{\mu_A(x)\wedge\mu_A(y)\}\\ &= \{(1-\mu_A(x))\vee(1-\mu_A(y))\}=\mu_A^C(x)\vee\mu_A^C(y).\end{aligned}$$

Let $x,y\in Q$. Then $\mu_A(x\circ y)\ge\mu_A(y)$ and so

$$\begin{aligned}\mu_A^C(x\circ y) &= 1-\mu_A(x\circ y)\\ &\le 1-\mu_A(y)=\mu_A^C(y).\end{aligned}$$

Hence, $\Box A=\left(\mu_A,\mu_A^C\right)$ is an intuitionistic fuzzy left ideal of $Q$. Similarly, we can prove that $\Diamond A=\left(\lambda_A,\lambda_A^C\right)$ an intuitionistic fuzzy left ideal of $Q$. □

According to the above lemma it is not difficult to see that the following theorem is valid.

**Theorem 3.5.** *Let $(Q,\circ,\le)$ be a quantale. Then $A=(\mu_A,\lambda_A)$ is an intuitionistic fuzzy left (resp., right) ideal of $Q$ if and only if $\Box A$ and $\Diamond A$ are intuitionistic fuzzy left (resp. right) ideals of $Q$.*

For any $t\in[0,1]$ and a fuzzy set $\mu$ of $Q$, the set

$$U(\mu;t):=\{x\in Q\mid \mu(x)\ge t\}\ \ (\text{resp. } L(\mu;t):=\{x\in Q\mid \mu(x)\le t\}),$$

is called the upper (resp. lower) $t$ -level cut of $\mu$.

**Theorem 3.6.** *An intuitionistic fuzzy set $A=(\mu_A,\lambda_A)$ is an intuitionistic fuzzy left (resp. right) ideal of $Q$ if and only if for all $s,t\in[0,1]$, the level sets $U(\mu_A;t)$ and $L(\lambda_A;s)$ are either empty or left (resp. right) ideals of $Q$.*

*Proof.* Assume that all non-empty sets $U(\mu_A;t)$ and $L(\lambda_A;s)$ are left ideals of $Q$. Let $x,y\in Q$ be such that $x\leq y$. If $\mu_A(y)=0$ and $\lambda_A(y)=1$, then $\mu_A(x)\geq 0=\mu_A(y)$ and $\lambda_A(x)\leq 1=\lambda_A(y)$. If $\mu_A(y)=t_0$ and $\lambda_A(y)=s_0$, then $y\in U(\mu_A;t_0)$ and $y\in L(\lambda_A;s_0)$. Since $U(\mu_A;t_0)$ and $L(\lambda_A;s_0)$ are left ideals of $Q$ and $x\leq y$, we have $x\in U(\mu_A;t_0)$ and $x\in L(\lambda_A;s_0)$. Therefore, $\mu_A(x)=t_0=\mu_A(y)$ and $\lambda_A(x)=s_0=\lambda_A(y)$. For $x,y\in Q$. If $\mu_A(x)=t_0=\mu_A(y)$ and $\lambda_A(x)=s_0=\lambda_A(y)$, then $x,y\in U(\mu_A;t_0)$ and $x,y\in L(\lambda_A;s_0)$. Therefore,

$$\mu_A(x\vee y)=t_0=t_0\wedge t_0=\mu_A(x)\wedge\mu_A(y)$$

and

$$\lambda_A(x\vee y)=s_0=s_0\vee s_0=\lambda_A(x)\vee\lambda_A(y)$$

for all $x,y\in Q$.

Let $x,y\in Q$. If $\mu_A(y)=0$ and $\lambda_A(y)=1$, then $\mu_A(x\circ y)\geq 0=\mu_A(y)$ and $\lambda_A(x\circ y)\leq 1=\lambda_A(y)$. If $t_0=\mu_A(y)$ and $s_0=\lambda_A(y)$, then $y\in U(\mu_A;t_0)$ and $y\in L(\lambda_A;s_0)$. Hence $x\circ y\in U(\mu_A;t_0)$ and $x\circ y\in L(\lambda_A;s_0)$. Then $\mu_A(x\circ y)=t_0=\mu_A(y)$ and $\lambda_A(x\circ y)=s_0=\lambda_A(y)$. Therefore, $A=(\mu_A,\lambda_A)$ is an intuitionistic fuzzy left ideal of $Q$.

Conversely, suppose that $A=(\mu_A,\lambda_A)$ is an intuitionistic fuzzy left ideal of $Q$. Let $x,y\in Q$ be such that $x\leq y$. If $y\in U(\mu_A;t)$ and $y\in L(\lambda_A;s)$. Then $\mu_A(y)\geq t$ and $\lambda_A(y)\leq s$ and we have $\mu_A(x)\geq\mu_A(y)\geq t$ and $\lambda_A(x)\leq\lambda_A(y)\leq s$, i.e., $\mu_A(x)\geq t$ and $\lambda_A(x)\leq s$ and we have $x\in U(\mu_A;t)$ and $x\in L(\lambda_A;s)$. Let $x,y\in U(\mu_A;t)$ and $x,y\in L(\lambda_A;s)$, then $\mu_A(x)\geq t$, $\mu_A(y)\geq t$ and $\lambda_A(x)\leq s$, $\lambda_A(y)\leq s$. Hence we have

$$\mu_A(x\vee y)\geq\mu_A(x)\wedge\mu_A(y)\geq t,$$

and

$$\lambda_A(x\vee y)\leq\lambda_A(x)\vee\lambda_A(y)\leq s.$$

and $\mu_A(x\vee y)\geq t$ and $\lambda_A(x\vee y)\leq s$. Thus, $x\vee y\in U(\mu_A;t)$ and $x\vee y\in L(\lambda_A;s)$. Let $x\in Q$ and $y\in U(\mu_A;t)$, and $y\in L(\lambda_A;s)$, then $\mu_A(y)\geq t$ and $\lambda_A(y)\leq s$ and we have

$$\mu_A(x\circ y)\geq\mu_A(y)\geq t,$$

and

$$\lambda_A(x\circ y)\leq\lambda_A(y)\leq s.$$

Hence $x\circ y\in U(\mu_A;t)$ and $x\circ y\in L(\lambda_A;s)$. Therefore, $U(\mu_A;t)$ and $L(\lambda_A;s)$ are left ideals of $Q$. □

**Corollary 3.7.** *Let $I$ be a left (resp. right) ideal of a quantale $Q$. If the fuzzy sets $\mu$ and $\lambda$ are defined on $Q$ by*

$$\mu(x)=\begin{cases} a_0 \text{ if } x\in I\\ a_1 \text{ if } x\in Q\setminus I\end{cases}$$

*and*

$$\lambda(x) = \begin{cases} b_0 \text{ if } x \in I \\ b_1 \text{ if } x \in Q \setminus I \end{cases}$$

*where* $0 \le a_1 \le a_0$, $0 \le b_0 \le b_1$ *and* $a_i + b_i \le 1$ *for* $i = 0, 1$. *Then* $A = (\mu, \lambda)$ *is an intuitionistic fuzzy left (resp. right) ideal of* $Q$ *and* $U(\mu; a_0) = I = L(\lambda; b_0)$.

**Corollary 3.8.** *Let* $\chi_I$ *be the characteristic function of a left (resp. right) ideal* $I$ *of* $Q$. *Then* $I = (\chi_I, \chi_I^C)$ *is a left (resp. right) fuzzy deal of* $H$.

Let $A$ be a non-empty subset of a quantale $Q$, we denote by $A_x = \{x \in Q \mid x \le y \circ z\}$.
Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy sets in a quantale $Q$. Then the product of $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$, denoted by $A \circ B = (\mu_{A \circ B}, \lambda_{A \circ B})$, and is defined by

$$\mu_{A \circ B}(x) = \begin{cases} \bigvee_{(y,z) \in A_x} \{\mu_A(y) \wedge \mu_B(z)\} & \text{If } A_x \ne \emptyset \\ 0 & \text{If } A_x = \emptyset. \end{cases}$$

$$\lambda_{A \circ B}(x) = \begin{cases} \bigwedge_{(y,z) \in A_x} \{\lambda_A(y) \vee \lambda_B(z)\} & \text{If } A_x \ne \emptyset \\ 0 & \text{If } A_x = \emptyset. \end{cases}$$

**Theorem 3.9.** *Let* $(Q, \circ, \le)$ *be a quantale and* $A = (\mu_A, \lambda_A)$ *is an intuitionistic fuzzy set in* $Q$. *Then the following are equivalent:*

(1) $A$ *is an intuitionistic fuzzy left ideal of* $Q$.

(2) $1_\sim \circ A \subseteq A$, *i.e.*, $\mu_{1 \circ A} \le \mu_A$, $\lambda_{0 \circ A} \ge \lambda_A$.

(2.1) $(\forall x, y \in Q)(x \le y \implies \mu_A(x) \ge \mu_A(y)$, $\lambda_A(x) \le \lambda_A(y))$.

(2.2) $(\forall x, y \in Q)(\mu_A(x \vee y) \ge \mu_A(x) \wedge \mu_A(y)$, $\lambda_A(x \vee y) \le \lambda_A(x) \vee \lambda_A(y))$.

*Proof.* Assume that (1) holds. Let $x \in Q$. If $A_x \ne \emptyset$, then

$$\begin{aligned} \mu_{1 \circ A}(x) &= \bigvee_{(y,z) \ge A_x} \{\mu_1(y) \wedge \mu_A(z)\} \\ &\le \bigvee_{(y,z) \in A_x} \{1 \wedge \mu_A(y \circ z)\} \\ &\le \bigvee_{(y,z) \in A_x} \{\mu_A(x)\} = \mu_A(x), \end{aligned}$$

and

$$\begin{aligned} \lambda_{0 \circ A}(x) &= \bigwedge_{(y,z) \in A_x} \{\lambda_0(y) \vee \lambda_A(z)\} \\ &\ge \bigwedge_{(y,z) \in A_x} \{0 \vee \lambda_A(y \circ z)\} \\ &\ge \bigwedge_{(y,z) \in A_x} \{\lambda_A(x)\} = \lambda_A(x). \end{aligned}$$

If $A_x = \emptyset$, then, obviously, $\mu_{1\circ A}(x) = 0 \leq \mu_A(x)$ and $\lambda_{0\circ A}(x) = 1 \geq \lambda_A(x)$. Hence $\mu_{1\circ A} \leq \mu_A$ and $\lambda_{0\circ A} \geq \lambda_A$, i.e., $1_\sim \circ A \subseteq A$. Since $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy left ideal of $Q$, so conditions (2.1) and (2.2) are valid.

Conversely, suppose that (2.1) and (2.2) hold. Let $y, z$ be any elements of $Q$, put $x = y \circ z$. Then by hypothesis

$$\begin{aligned} \mu_A(y \circ z) &\geq \mu_{1\circ A}(x) = \bigvee_{(a,b)\in A_x} \{\mu_1(a) \wedge \mu_A(b)\} \\ &\geq \mu_1(y) \wedge \mu_A(z) = 1 \wedge \mu_A(z) = \mu_A(z), \end{aligned}$$

and

$$\begin{aligned} \lambda_A(y \circ z) &\leq \lambda_{0\circ A}(x) = \bigwedge_{(a,b)\in A_x} \{\lambda_0(a) \vee \lambda_A(b)\} \\ &\leq \lambda_0(y) \vee \lambda_A(z) = 0 \vee \lambda_A(z) = \lambda_A(z). \end{aligned}$$

This means that $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy left ideal of $Q$. □

By right dual of Theorem 3.9, we have the following:

**Theorem 3.10.** *Let $(Q, \circ, \leq)$ be a quantale and $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy set in $Q$. Then the following are equivalent:*

(1) *$A$ is an intuitionistic fuzzy right ideal of $Q$.*

(2) *$A \circ 1_\sim \subseteq A$, i.e., $\mu_{A\circ 1} \leq \mu_A$, $\lambda_{A\circ 0} \geq \lambda_A$.*

(2.1) $(\forall x, y \in Q)(x \leq y \implies \mu_A(x) \geq \mu_A(y)$, $\lambda_A(x) \leq \lambda_A(y))$.

(2.2) $(\forall x, y \in Q)(\mu_A(x \vee y) \geq \mu_A(x) \wedge \mu_A(y)$, $\lambda_A(x \vee y) \leq \lambda_A(x) \vee \lambda_A(y))$.

From Theorem 3.9 and 3.10 we have the following corollary:

**Theorem 3.11.** *Let $(Q, \circ, \leq)$ be a quantale and $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy set in $Q$. Then the following are equivalent:*

(1) *$A$ is an intuitionistic fuzzy ideal of $Q$.*

(2) *$1_\sim \circ A \subseteq A$, $A \circ 1_\sim \subseteq A$.*

(2.1) $(\forall x, y \in Q)(x \leq y \implies \mu_A(x) \geq \mu_A(y)$, $\lambda_A(x) \leq \lambda_A(y))$.

(2.2) $(\forall x, y \in Q)(\mu_A(x \vee y) \geq \mu_A(x) \wedge \mu_A(y)$, $\lambda_A(x \vee y) \leq \lambda_A(x) \vee \lambda_A(y))$.

**Lemma 3.12.** *Let $A = (\mu_A, \lambda_A)$, $B = (\mu_B, \lambda_B)$, $C = (\mu_C, \lambda_C)$ and $D = (\mu_D, \lambda_D)$ be IFSs in $Q$. If $A \subseteq C$ and $B \subseteq D$, then $A \circ B \subseteq C \circ D$.*

*Proof.* For any $x \in Q$, if $A_x = \emptyset$, then the result obviously holds. Assume that $A_x \neq \emptyset$, then

$$\begin{aligned} \mu_{A\circ B}(x) &= \bigvee_{(y,z)\geq A_x} \{\mu_A(y) \wedge \mu_B(z)\} \\ &\leq \bigvee_{(y,z)\in A_x} \{\mu_C(y) \wedge \mu_D(z)\} \\ &= \mu_{C\circ D}(x), \end{aligned}$$

and

$$\begin{aligned}\lambda_{A\circ B}(x) &= \bigwedge_{(y,z)\in A_x}\{\lambda_A(y)\vee\lambda_B(z)\}\\ &\geq \bigwedge_{(y,z)\in A_x}\{\lambda_C(y)\vee\lambda_D(z)\}\\ &= \lambda_{C\circ D}(x).\end{aligned}$$

Hence $\mu_{A\circ B}\leq\mu_{C\circ D}$ and $\lambda_{A\circ B}\geq\lambda_{C\circ D}$, i.e., $A\circ B\subseteq C\circ D$. □

**Proposition 3.13.** *Let $A=(\mu_A,\lambda_A)$ be an intuitionistic fuzzy right ideal of $Q$ and $B=(\mu_B,\lambda_B)$ be an intuitionistic fuzzy left ideal of $Q$, respectively. Then*

$$A\circ B\subseteq A\cap B.$$

*Proof.* Let $x\in Q$, if $A_x=\emptyset$, then the result holds. Suppose that $A_x\neq\emptyset$, then

$$\mu_{A\circ B}(x)=\bigvee_{(y,z)\in A_x}\{\mu_A(y)\wedge\mu_B(z)\}$$

$$\lambda_{A\circ B}(x)=\bigwedge_{(y,z)\in A_x}\{\lambda_A(y)\vee\lambda_B(z)\}$$

Since $x\leq y\circ z$ and $A=(\mu_A,\lambda_A)$ is an intuitionistic fuzzy right ideal and $B=(\mu_B,\lambda_B)$ an intuitionistic fuzzy left ideal of $Q$, we have

$$\mu_A(x)\geq\mu_A(y\circ z)\geq\mu_A(y),\ \lambda_A(x)\leq\lambda_A(y\circ z)\leq\lambda_A(y)$$

and

$$\mu_B(x)\geq\mu_B(y\circ z)\geq\mu_B(z),\ \lambda_B(x)\leq\lambda_B(y\circ z)\leq\lambda_B(z)$$

Hence

$$\begin{aligned}\mu_{A\circ B}(x) &= \bigvee_{(y,z)\in A_x}\{\mu_A(y)\wedge\mu_B(z)\}\leq\bigvee_{(y,z)\in A_x}\{\mu_A(x)\wedge\mu_B(x)\}\\ &= \bigvee_{(y,z)\in A_x}(\mu_A\wedge\mu_B)(x)=(\mu_A\wedge\mu_B)(x),\end{aligned}$$

and

$$\begin{aligned}\lambda_{A\circ B}(x) &= \bigwedge_{(y,z)\in A_x}\{\lambda_A(y)\vee\lambda_B(z)\}\geq\bigwedge_{(y,z)\in A_x}\{\lambda_A(x)\vee\lambda_B(x)\}\\ &= \bigwedge_{(y,z)\in A_x}(\lambda_A\vee\lambda_B)(x)=(\lambda_A\vee\lambda_B)(x).\end{aligned}$$

Therefore, $A\circ B\subseteq A\cap B$. □

**Proposition 3.14.** *Let $A=(\mu_A,\lambda_A)$ and $B=(\mu_B,\lambda_B)$ be intuitionistic fuzzy sets. If $Q$ is regular and $A=(\mu_A,\lambda_A)$ is an intuitionistic fuzzy right ideal of $Q$, then*

$$A\cap B\subseteq A\circ B.$$

*Proof.* Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy right ideal of $Q$. Let $a \in Q$. Then there exists $x \in Q$ such that $a \leq (a \circ x) \circ a$, since $Q$ is regular. Then $(a \circ x, a) \in A$; that is $A_a \neq \emptyset$ and so

$$\begin{aligned} \mu_{A \circ B}(x) &= \bigvee_{(y,z) \in A_x} \{\mu_A(y) \wedge \mu_B(z)\} \\ &\geq \{\mu_A(a \circ x) \wedge \mu_B(a)\} \geq \mu_A(a) \wedge \mu_B(a) \\ &\geq (\mu_A \wedge \mu_B)(a), \end{aligned}$$

and

$$\begin{aligned} \lambda_{A \circ B}(x) &= \bigwedge_{(y,z) \in A_x} \{\lambda_A(y) \vee \lambda_B(z)\} \\ &\leq \lambda_A(a \circ x) \vee \lambda_B(a) \leq \lambda_A(a) \vee \lambda_B(a) \\ &\leq (\lambda_A \vee \lambda_B)(a). \end{aligned}$$

Hence $\mu_A \wedge \mu_B \leq \mu_{A \circ B} = \mu_A \circ \mu_B$ and $\lambda_A \vee \lambda_B \geq \lambda_{A \circ B} = \lambda_A \circ \lambda_B$. Therefore, $A \cap B \subseteq A \circ B$. □

In a similar way, we can prove that

**Proposition 3.15.** *Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy sets. If $Q$ is regular and $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy left ideal of $Q$, then*

$$A \cap B \subseteq A \circ B.$$

**Corollary 3.16.** *Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy right and left ideals of $Q$, respectively. If $Q$ is regular, then*

$$A \cap B \subseteq A \circ B.$$

**Proposition 3.17.** *Let $\chi_A = \left(\mu_{\chi_A}, \lambda_{\chi_A}\right)$ and $\chi_B = \left(\mu_{\chi_B}, \lambda_{\chi_B}\right)$ be the characteristic functions of non-empty subsets $A$ and $B$, respectively. Then the following properties hold:*

(1) $A \subseteq B$ *if and only if* $\chi_A \subseteq \chi_B$.
(2) $\chi_A \cap \chi_B = \chi_{A \cap B}$.
(3) $\chi_A \cup \chi_B = \chi_{A \cup B}$.
(4) $\chi_A \circ \chi_B = \chi_{(A \circ B]}$.

*Proof.* (1) Let $A$ and $B$ be non-empty subsets of $Q$ such that $A \subseteq B$. If $\mu_{\chi_A}(a) = 0 \leq \mu_{\chi_B}(a)$ and $\lambda_{\chi_A}(a) = 1 \geq \lambda_{\chi_B}(a)$, then clearly $\chi_A(a) = (0, 1) \subseteq \chi_B(a)$. If $\mu_{\chi_A}(a) = 1$ and $\lambda_{\chi_A}(a) = 0$, then $a \in A$ and hence $a \in B$ since $A \subseteq B$. Thus $\mu_{\chi_B}(a) = 1$ and $\lambda_{\chi_B}(a) = 0$, hence $\chi_A(a) = (1, 0) \subseteq \chi_B(a)$. Therefore, $\chi_A \subseteq \chi_B$.

Conversely, assume that $\chi_A \subseteq \chi_B$. Let $a \in A$, then $\mu_{\chi_A}(a) = 1$ and $\lambda_{\chi_A}(a) = 0$. Since $\chi_A(a) \subseteq \chi_B(a)$ for all $a \in Q$, we have $\mu_{\chi_B}(a) = 1$ and $\lambda_{\chi_B}(a) = 0$. Thus $a \in B$ and so $A \subseteq B$.

(2) Let $a \in A \cap B$, then $a \in A$ and $a \in B$. Thus we have $(\chi_A \cap \chi_B)(a) = \chi_A(a) \cap \chi_B(a) = (1,0) = \chi_{A\cap B}(a)$. If $a \notin A \cap B$, then $a \notin A$ and $a \notin B$. Hence we have $(\chi_A \cap \chi_B)(a) = \chi_A(a) \cap \chi_B(a) = (0,1) = \chi_{A\cap B}(a)$. Therefore, $\chi_A \cap \chi_B = \chi_{A\cap B}$.

(3) Similar to part (2).

(4) Follows from [14, Proposition 10]. □

**Lemma 3.18.** *If $A = (\mu_A, \lambda_A)$ (resp. $B = (\mu_B, \lambda_B)$) is an intuitionistic fuzzy right ideal (resp. intuitionistic fuzzy left ideal) of $Q$. Then*

(1) $A \circ 1_\sim \subseteq A$ *(resp.* $1_\sim \circ B \subseteq B$*).*

(2) $A \circ A \subseteq A$ *(resp.* $B \circ B \subseteq B$*).*

*Proof.* Assume that $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy right ideal of $Q$ and let $x \in Q$. If $A_x = \emptyset$, $(A \circ 1_\sim)(x) \subseteq A(x)$. Assume that $A_x \neq \emptyset$ and let $y, z \in Q$ be such that $(y,z) \in A_x$. Then $x \leq y \circ z$ and so $\mu_A(x) \geq \mu_A(y \circ z) \geq \mu_A(y)$ and $\lambda_A(x) \leq \lambda_A(y \circ z) \leq \lambda_A(y)$. Hence $\mu_A(y) \wedge \mu_1(z) = \mu_A(y) \wedge 1 = \mu_A(y)$ and $\lambda_A(y) \vee \lambda_0(y) = \lambda_A(y) \vee 0 = \lambda_A(y)$ for all $(y,z) \in A_x$. Therefore,

$$\mu_{A\circ 1}(x) = \bigvee_{(y,z)\in A_x} \{\mu_A(y) \wedge \mu_1(z)\} \leq \mu_A(x),$$

$$\lambda_{A\circ 0}(x) = \bigwedge_{(y,z)\in A_x} \{\lambda_A(y) \vee \lambda_0(y)\} \geq \lambda_A(x).$$

Consequently, $\mu_{A\circ 1} \leq \mu_A$ and $\lambda_{A\circ 0} \geq \lambda_A$. Hence $(A \circ 1_\sim)(x) \subseteq A(x)$. Similarly, $(1_\sim \circ B)(x) \subseteq B(x)$ for every intuitionistic fuzzy left ideal $B = (\mu_B, \lambda_B)$ of $Q$.

(2) Note that $A \subseteq 1_\sim$ and $A \subseteq A$. Using Theorem 3.11 and part (1) we have $A \circ A \subseteq A \circ 1_\sim \subseteq A$. In a similar way we obtain $B \circ B \subseteq 1_\sim \circ B \subseteq B$ for every intuitionistic fuzzy left ideal $B = (\mu_B, \lambda_B)$ of $Q$. □

We now provide a characterization theorem for regular quantales.

**Theorem 3.19.** *A quantale $Q$ is regular if and only if for every intuitionistic fuzzy right ideal $A = (\mu_A, \lambda_A)$ and every intuitionistic fuzzy left ideal $B = (\mu_B, \lambda_B)$ of $Q$, we have*

$$A \cap B = A \circ B.$$

*Proof.* Assume that $Q$ is regular. Using Corollary 3.15, we have $A \cap B = A \circ B$.

Conversely, suppose that for every intuitionistic fuzzy right ideal $A = (\mu_A, \lambda_A)$ and every intuitionistic fuzzy left ideal $B = (\mu_B, \lambda_B)$ of $Q$, we have $A \cap B = A \circ B$. Let $a \in Q$. To prove that $Q$ is regular, by Lemma 2.3, it is enough to prove that

$$\langle a\rangle_r \cap \langle a\rangle_l \subseteq (\langle a\rangle_r \circ \langle a\rangle_l] \text{ for every } a \in Q.$$

Let $b \in \langle a\rangle_r \cap \langle a\rangle_l$, then $b \in \langle a\rangle_r$ and $b \in \langle a\rangle_l$. Since $\langle a\rangle_r$ (*resp.*, $\langle a\rangle_l$) is the right (resp., left) ideal of $Q$, it is follows from Corollary 3.8, that $\chi_{\langle a\rangle_r} =$

$\left(\mu_{\chi_{\langle a\rangle_r}}, \lambda_{\chi_{\langle a\rangle_r}}\right)$ $\left(resp.,\ \chi_{\langle a\rangle_l} = \left(\mu_{\chi_{\langle a\rangle_l}}, \lambda_{\chi_{\langle a\rangle_l}}\right)\right)$ is an intuitionistic fuzzy right (resp., left) ideal of $Q$. By hypothesis, we have

$$\mu_{\chi_{\langle a\rangle_r} \cap \chi_{\langle a\rangle l}} = \mu_{\chi_{\langle a\rangle_r} \circ \chi_{\langle a\rangle l}} \text{ and } \lambda_{\chi_{\langle a\rangle_r} \cup \chi_{\langle a\rangle l}} = \lambda_{\chi_{\langle a\rangle_r} \circ \chi_{\langle a\rangle l}},$$

since $b \in \langle a\rangle_r \cap \langle a\rangle_l$, we have $\mu_{\chi_{\langle a\rangle_r} \cap \chi_{\langle a\rangle l}}(b) = 1$ and $\lambda_{\chi_{\langle a\rangle_r} \cup \chi_{\langle a\rangle l}}(b) = 0$. Hence

$$\begin{aligned}(1,0) &= \left(\mu_{\chi_{\langle a\rangle_r} \cap \chi_{\langle a\rangle l}}(b), \lambda_{\chi_{\langle a\rangle_r} \cup \chi_{\langle a\rangle l}}(b)\right) \\ &= \left(\mu_{\chi_{\langle a\rangle_r} \circ \chi_{\langle a\rangle l}}(b), \lambda_{\chi_{\langle a\rangle_r} \circ \chi_{\langle a\rangle l}}(b)\right) \\ &= \left(\mu_{(\chi_{\langle a\rangle_r} \circ \chi_{\langle a\rangle l}]}, \lambda_{(\chi_{\langle a\rangle_r} \circ \chi_{\langle a\rangle l}]}\right)(b).\end{aligned}$$

Therefore, $\left(\mu_{(\chi_{\langle a\rangle_r} \circ \chi_{\langle a\rangle l}]}, \lambda_{(\chi_{\langle a\rangle_r} \circ \chi_{\langle a\rangle l}]}\right)(b) = (1,0)$, and so $b \in (\chi_{\langle a\rangle_r} \cap \chi_{\langle a\rangle_l}]$. Hence $Q$ is regular by (Lemma 2.3). □

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## References

[1] Akram, M., Parvathi, R., Properties of intuitionistic fuzzy line graphs. Notes on Intuitionistic Fuzzy Sets 18(3) (2012), 52–60.

[2] Atanassov, K.T., Intuitionistic fuzzy sets. Fuzzy Sets and Systems 20 (1986), 87–96.

[3] Atanassov, K.T., New operations de. . . defined over the intuitionistic fuzzy sets. Fuzzy Sets and Systems 61 (1994), 137-142.

[4] Atanassov, K.T., Intuitionistic fuzzy sets, theory and applications. Studies in Fuzziness and Soft Computing, Heidelberg. Physica-Verlag 35, 1999.

[5] Atanassov, K.T., On Intuitionistic fuzzy sets. Springer, 2012.

[6] Atanassov, K.T., Gargov, G., Interval-valued intuitionistic fuzzy sets. Fuzzy sets and systems 31 (1989), 343–349.

[7] Burrilo, P., Bustince, H., Vague sets are intuitionistic fuzzy sets. Fuzzy Sets and Systems 79 (1996), 403–405.

[8] Davvaz, B., Dudek, W.A., Jun, Y.B., Intuitionistic fuzzy Hv-submodules. Inform. Sci. 176 (2006), 285–300.

[9] Davvaz, B., Sadrabadi, E.H., An application of intuitionistic fuzzy sets in medicine. Int. J. Biomath. 9 (2016), ID 1650037, 15 pages.

[10] DE, S. K., Biswas, R., Roy, A.R., An application of intuitionistic fuzzy sets in medical diagnosis. Fuzzy Sets and Systems 117 (2001), 209–213.

[11] Dengfeng, L., Chunan, C., New similarity measures of intuitionistic fuzzy sets and applications to pattern recognitions. Pattern Reconit. Lett. Vol. 23 (2002), 221–225.

[12] Hila, K., Kikina, L., Davvaz, B., Intuitionistic Fuzzy Hyperideal Extensions of Semihypergroups. Thai Journal of Mathematics 13(2) (2015), 291–305.

[13] Khan, M., Characterizations of quantales by intuitionistic fuzzy ideals. M.Phil Dissertation, Department of Mathematics, Abdul Wali Khan University, Mardan, KP, Pakistan, 2016.

[14] Luo, Q., Wang, G., Roughness and fuzziness in quantales. Inform. Sci. 271 (2014), 14–30.

[15] Mulvey, C.J., Rend. Circ. Mat. Palermo 2(12) (1986), 99–104.

[16] Rencová, M., An Example of Applications of Intuitionistic Fuzzy Sets to Sociometry. Sofia, Bulgarian Academy of Sciences: Cybernetics and Inform. Tech. 9(2), 2009.

[17] Shabir, M., Khan, A., Intuitionistic fuzzy interior ideals in ordered semigroups. J. Appl. Math. & Inform. 27(5-6) (2009), 1447–1457.

[18] Shabir, M., Khan, A., Ordered semigroups characterized by their intuitionistic fuzzy generalized bi-ideals. Iranian Journal of Fuzzy Systems 7(2) (2010), 55–69.

[19] Szmidt, E., Kacprzyk, Intuitionistic fuzzy sets in some medical applications, Note on IFS 7(4) (2001), 58–64.

[20] Szmidt, E., Kacprzyk, J., Medical diagnostic reasoning using a similarity measure for intuitionistic fuzzy sets 10(4) (2004), 61–69.

[21] Wang, S.Q., Zhao, B., Prime ideal and weakly prime ideal of quantale. Fuzzy Syst. Math. Vol. 19(1) (2005), 78–81. (in Chinese)

[22] Yang, L.Y., Xu, L.S., Roughness in quantales. Inform. Sci. Vol. 220 (2013), 568–579.

[23] Zadeh, L.A., Fuzzy sets. Inform. Control 8, (1965), 338–353.

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