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THE ADJOINT SEMIGROUP OF A Γ-SEMIGROUP

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Abstract. Given a Γ-semigroup S and a fixed $\gamma_0 \in \Gamma$, we construct a semigroup Σ_{γ_0} in such a way that there is a one to one correspondence between the set of principal one sided ideals (resp. principal quasi-ideals) of S and their counterparts in Σ_{γ_0}. This correspondence allows us to obtain several results for S without having the need to work directly with it, but working with Σ_{γ_0} instead and employing well known results of semigroup theory. For example, we obtain an analogue of the Green's theorem for Γ-semigroups as a corollary of the usual Green's theorem in semigroups. Also we prove that, if S is a Γ-semigroup and $\gamma_0 \in \Gamma$ such that S_{γ_0} is a completely simple semigroup, then for every $\gamma \in \Gamma$, S_γ is completely simple too.

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1. Introduction and preliminaries

Let S and Γ be two non empty sets. Every map from $S \times \Gamma \times S$ to S will be called a Γ-multiplication in S and is denoted by $(\cdot)_\Gamma$. The result of this multiplication for $a, b \in S$ and $\gamma \in \Gamma$ is denoted by $a\gamma b$. According to Sen and Saha [5], a Γ-semigroup S is an ordered pair $(S, (\cdot)_\Gamma)$ where S and Γ are non empty sets and $(\cdot)_\Gamma$ is a Γ-multiplication on S which satisfies the following property

$$\forall (a, b, c, \alpha, \beta) \in S^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c).$$

Here we give a few notions and present some auxiliary results that will be used throughout the paper. Some of the results regarding Γ-semigroups may be found in [4] and [5] but for the reader's convenience we have restated them below.

Let S be a Γ-semigroup and A, B subset of S. We define the set

$$A\Gamma B = \{a\gamma b | a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

For simplicity we write $a\Gamma B$ instead of $\{a\}\Gamma B$ and similarly we write $A\Gamma b$, and write $A\gamma B$ in place of $A\{\gamma\}B$.

Definition 1.1. [4] Let S be a Γ-semigroup. A non empty subset S_1 of S is said to be a Γ-subsemigroup of S if $S_1 \Gamma S_1 \subseteq S_1$.

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Definition 1.2. [4] A right [left] ideal of a Γ-semigroup S is a non empty subset $R[L]$ of S such that $R\Gamma S \subseteq R$, $[S\Gamma L \subseteq L]$.

The principal right ideal $(a)_r$ generated by an element a in a Γ-semigroup S has been defined by [4] as the smallest right ideal of S containing a, and it is proved that $(a)_r = a \cup a\Gamma S$. Dually, the principal left ideal $(a)_l$ generated by an element a exists and is given by $(a)_l = a \cup S\Gamma a$. By analogy with plain semigroups, Saha defined in [4] relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{H}$ in a Γ-semigroup S by setting

$$(a, b) \in \mathcal{R} \text{ iff } (a)_r = (b)_r,$$

$$(a, b) \in \mathcal{L} \text{ iff } (a)_l = (b)_l,$$

$$(a, b) \in \mathcal{H} \text{ iff } (a)_r = (b)_r \text{ and } (a)_l = (b)_l.$$

By analogy with the definition of quasi-dieals in plain semigroups [6] we give the following.

Definition 1.3. A quasi-ideal of a Γ-semigroup S is a non empty subset Q of S such that $Q\Gamma S \cap S\Gamma Q \subseteq Q$.

It is easy to see that the principal quasi-ideal $(a)_q$ generated by a in a Γ-semigroup S exists and is given by

$$(a)_q = a \cup (a\Gamma S \cap S\Gamma a) = (a)_r \cap (a)_l.$$

We can now define the relation $\mathcal{Q}$ in a Γ-semigroup S by setting

$$(a, b) \in \mathcal{Q} \text{ iff } (a)_q = (b)_q.$$

Similarly to plain semigroups, one can prove just as easily that relations $\mathcal{Q}$ and $\mathcal{H}$ in Γ-semigroups coincide.

Given a Γ-semigroup S it is obvious that to any fixed $\gamma \in \Gamma$ one can associate to S a semigroup $(S_\gamma, \circ)$ where $S_\gamma = S$ and $\circ$ is defined by setting $x \circ y = x\gamma y$ for every $x, y \in S$. A remarkable result of Sen and Saha in [5] states that *if S is a Γ-semigroup without zero and if S_γ is a group for some $\gamma \in \Gamma$, then S_γ is a group for every $\gamma \in \Gamma$.* Such a Γ-semigroup is called a Γ-group. The main puropose of our paper is to generalize Sen's and Saha's result by replacing the group condition for some S_γ with S_γ being a completely simple semigroup. Recall from [2] that for a simple semigroup S (without a zero element) the following conditions are equivalent:

(i) S is completely simple, that is, it contains a primitive idempotent;

(ii) S is completely regular, that is, every $\mathcal{H}$-class is a group;

(iii) S satisfies $\min_L$ and $\min_R$;

(iii) S contains at least a minimal left ideal and at least a minimal right ideal.

For further readings on semigroups the reader is refereed to the monograph [2].

2. The adjoint semigroup Σ_{γ_0}

In this section we will need some notions from reductions systems which can be found in [1] and [3]. In order to make the paper self contained we will give below what is necessary to make the proofs involving reduction systems easy to follow.

An abstract reduction system is a pair $(A, \to)$, where the reduction $\to$ is a binary relation on the set A. We write $a \to b$ instead of $(a, b) \in \to$. In what follows we denote by $\xrightarrow{+}$ the transitive closure of $\to$, by $\xrightarrow{*}$ the reflexive transitive closure of $\to$ and by $\xleftrightarrow{*}$ the equivalence relation generated by $\to$. We call $a \in A$ reducible if and only if there is a $b \in A$ such that $a \xrightarrow{+} b$, otherwise we call it irreducible or in the normal form. If it happens that b is unique, then we denote b by $a\downarrow$. We call a and a' *joinable* (or *resolvable*) if and only if there is c such that $a \xrightarrow{*} c \xleftarrow{*} a'$, in which case we write $a \downarrow a'$.
A reduction $\to$ is called

- *Confluent* if and only if $a \xleftarrow{*} c \xrightarrow{*} b \Longrightarrow a \downarrow b$.
- *Locally-Confluent* if and only if $a \leftarrow c \to b \Longrightarrow a \downarrow b$.
- *Terminating* (or *Noetherian*) if and only if there is no infinite descending chain $a_0 \to a_1 \to ...$
- *Convergent* if and only if it is both confluent and terminating.

The following is known as the Newman's Lemma.

Lemma 2.1. *A Noetherian system is confluent if it is locally confluent.*

An important notion is that of a complete reduction system. A reduction system $(A, \to)$ is called complete if and only if every element has a unique normal form. The following characterization of complete systems, due to Newman [3], is important because it translates the completeness in terms of confluence and termination.

Lemma 2.2. *A reduction system is complete if and only if it is Noetherian and confluent.*

This lemma is the reason why sometimes complete systems are called convergent. Combining Lemma 2.1 and Lemma 2.2, we get the following characterization.

Lemma 2.3. *A reduction system is complete if and only if it is Noetherian and locally confluent.*

To define Σ_{γ_0} we first let F be the free semigroup on the disjoint union $S \sqcup \Gamma$. Its elements are finite strings $(x_1, ..., x_n)$ where each $x_i \in S \sqcup \Gamma$. Now we define Σ_{γ_0} as the quotient semigroup of F by the congruence generated from the set of relations

$$((\gamma_1, \gamma_2), \gamma_1), ((x, \gamma, y), x\gamma y), ((x, y), x\gamma_0 y)$$

for all γ_1, γ_2 and $\gamma \in \Gamma$, all $x, y \in S$ and with $\gamma_0 \in \Gamma$ a fixed element.

Lemma 2.4. *Every element of Σ_{γ_0} can be represented by an irreducible word which has the form (γ, x, γ'), (γ, x), (x, γ), γ or x where $x \in S$ and $\gamma, \gamma' \in \Gamma$.*

Proof. To prove the lemma, we show first that the reduction system arising from the given presentation is Noetherian and confluent, and therefore any element of Σ_{γ_0} is given by a unique irreducible word from F. Secondly, we show that the irreducible words have one of the five claimed forms.
If a word w of F has the form $(u, \gamma_1, \gamma_2, v)$ where $\gamma_1, \gamma_2 \in \Gamma$, and u, v are possibly empty words of F, then w reduces to $w' = (u, \gamma_1, v)$. Now if for some $x, y \in S$ and $\gamma \in \Gamma$, the word w contains a subword of the form (x, γ, y), which is to say that $w = (u, x, \gamma, y, v)$ with u, v being possibly empty words from F, then it reduces to $w' = (u, x\gamma y, v)$. Finally, if the word w contains two adjacent letters from S, meaning that $w = (u, x, y, v)$ where u and v as before and $x, y \in S$, then it reduces to $w' = (u, x\gamma_0 y, v)$. In this way we obtain a reduction system made of the following three type of reductions:

$$\begin{array}{ccc} (u, \gamma_1, \gamma_2, v) & \to & (u, \gamma_1, v) \\ (u, x, \gamma, y, v) & \to & (u, x\gamma y, v) \\ (u, x, y, v) & \to & (u, x\gamma_0 y, v) \end{array}$$

which is length reducing and therefore Noetherian. To prove that it is confluent, from Newman's lemma, it is sufficient to show that it is locally confluent. As there are no inclusion ambiguities, we need to check only overlapping ones. There are only five such pairs:
1- $(x, y, \gamma, z) \to (x\gamma_0 y, \gamma, z)$ and $(x, y, \gamma, z) \to (x, y\gamma z)$. Both resolve to $(x\gamma_0 y\gamma z)$.
2- $(x, \gamma, y, z) \to (x, \gamma, y\gamma_0 z)$ and $(x, \gamma, y, z) \to (x\gamma y, z)$ which resolve to $(x\gamma y\gamma_0 z)$.
3- $(x, \gamma, y, \gamma', z) \to (x, \gamma, y\gamma' z)$ and $(x, \gamma, y, \gamma', z) \to (x\gamma y, \gamma', z)$ which resolve to $(x\gamma y\gamma' z)$.
4- $(x, y, z) \to (x\gamma_0 y, z)$ and $(x, y, z) \to (x, y\gamma_0 z)$, which resolve to $(x\gamma_0 y\gamma_0 z)$.
5- $(\gamma_1, \gamma_2, \gamma_3) \to (\gamma_1, \gamma_3)$ and $(\gamma_1, \gamma_2, \gamma_3) \to (\gamma_1, \gamma_2)$ which resolve to (γ_1).

To complete the proof, we need to show that the irreducible words representing elements of Σ_{γ_0} have the claimed forms. Any word which has neither a prefix nor a suffix made entirely of letters from Γ reduces to an element of S by performing reductions of types one, two and three. Otherwise, if the word is (η, U, η') where η, η' are words from the free monoid with base Γ and U has neither a prefix nor a suffix made entirely of letters from Γ, then reduce η and η' to a single letter form Γ by performing reductions of the first type, and than reduce as before U to a single letter from S. □

Lemma 2.4 shows in particular that the natural epimorphism $\mu : F \to \Sigma_{\gamma_0}$ is injective on S and Γ. In what follows we will identify the elements of Σ_{γ_0} with the irreducible words from F they are represented of written without brackets and commas, and if we want to multiply in Σ_{γ_0} two such words, we take their concatenation and then find its irreducible form. For instance, the product in Σ_{γ_0} of x with γy is $x \cdot \gamma y = x\gamma y$.

We call Σ_{γ_0} the *adjoint semigroup* of the given Γ-semigroup. The semigroup Σ_{γ_0} satisfies the following universal property.

Theorem 2.5. *Let S and S' be both Γ-semigroups. For every homomorphism of Γ-semigroups $\varphi : S \to S'$ identical on Γ, there is a unique homomorphism of semigroups $\phi : \Sigma_{\gamma_0} \to \Sigma'_{\gamma_0}$ identical on Γ such that $\phi\mu = \mu'\varphi$.*

Proof. Let $f : F(S \cup \Gamma) \to F(S' \cup \Gamma)$ be the homomorphism of free semigroups induced from φ. We prove that φ induces a homomorphism $\phi : \Sigma_{\gamma_0} \to \Sigma'_{\gamma_0}$. To do this we need to show that every relation that defines Σ_{γ_0} lies in the kernel of $\mu' f$ where $\mu' : F(S' \cup \Gamma) \to \Sigma'_{\gamma_0}$ is the canonical homomorphism. Indeed, for the first type of relations $((\gamma_1, \gamma_2), \gamma_1)$ we have

$$\begin{aligned} \mu' f(\gamma_1, \gamma_2) &= \mu'(\varphi(\gamma_1), \varphi(\gamma_2)) \\ &= \varphi(\gamma_1) \\ &= \gamma_1 \\ &= \mu' f(\gamma_1). \end{aligned}$$

For the second type $((x, \gamma, y), x\gamma y)$ we have

$$\begin{aligned} \mu' f(x, \gamma, y) &= \mu'(\varphi(x), \gamma, \varphi(y)) \\ &= \varphi(x)\gamma\varphi(y) \\ &= \varphi(x\gamma y) \\ &= \mu' f(x\gamma y), \end{aligned}$$

and for the last type $((x, y), x\gamma_0 y)$ we have

$$\begin{aligned} \mu' f(x, y) &= \mu'(\varphi(x), \varphi(y)) \\ &= \varphi(x)\gamma_0\varphi(y) \\ &= \varphi(x\gamma_0 y) \\ &= \mu' f(x\gamma_0 y). \end{aligned}$$

Therefore $\mu' f$ induces $\phi : \Sigma_{\gamma_0} \to \Sigma'_{\gamma_0}$ such that $\phi\mu = \mu' f$. Since φ is the restriction of f in $S \cup \Gamma$, then we derive that $\phi\mu = \mu'\varphi$. The uniqueness of ϕ with the given property follows easily from the fact any other homomorphism $\hat{\phi} : \Sigma_{\gamma_0} \to \Sigma'_{\gamma_0}$ satisfying $\hat{\phi}\mu = \mu'\varphi$ coincides with ϕ on the generators of Σ_{γ_0} and therefore equals with ϕ. $\square$

The next lemma and the subsequent proposition establish a 1-1 correspondence between principal one sided ideals and quasi ideals of S, and their counterparts of Σ_{γ_0}. This correspondence will be useful in the proof of Theorem 2.8.

Lemma 2.6. *Let $x \in S$ by an arbitrary element. The following hold true.*

(i) The principal left ideal in Σ_{γ_0} generated by x is the set $(x)_\ell^{\Sigma_{\gamma_0}} = (x)_\ell^\Gamma \cup \Gamma(x)_\ell^\Gamma$ where $(x)_\ell^\Gamma = S\Gamma x \cup \{x\}$ is the left ideal in S generated by x and $\Gamma(x)_\ell^\Gamma$ is a short notation for the set $\{\gamma y : \gamma \in \Gamma \text{ and } y \in (x)_\ell^\Gamma\}$.

(ii) *The principal right ideal in* Σ_{γ_0} *generated by* x *is the set* $(x)_r^{\Sigma_{\gamma_0}} = (x)_r^{\Gamma} \cup (x)_r^{\Gamma}\Gamma$ *where* $(x)_r^{\Gamma} = x\Gamma S \cup \{x\}$ *is the right ideal in* S *generated by* x *and* $(x)_r^{\Gamma}\Gamma$ *is the short notation for the set* $\{y\gamma : \gamma \in \Gamma$ *and* $y \in (x)_r^{\Gamma}\}$.

Proof. We will make the proof for (i) only since the proof for (ii) is dual to that of (i). The elements of $(x)_\ell^{\Sigma_{\gamma_0}} \setminus \{x\}$ are of the following five forms:
1- $\gamma y \cdot x$ with $\gamma \in \Gamma$ and $y \in S$. But $\gamma y \cdot x = \gamma y \gamma_0 x$ which belongs to $\Gamma(x)_\ell^{\Gamma}$.
2- $\gamma y \gamma' \cdot x$ with $\gamma, \gamma' \in \Gamma$ and $y \in S$. Again $\gamma y \gamma' \cdot x = \gamma(y\gamma' x)$ which belongs to $\Gamma(x)_\ell^{\Gamma}$.
3- $\gamma \cdot x$ with $\gamma \in \Gamma$ which obviously belongs to $\Gamma(x)_\ell^{\Gamma}$.
4- $y \cdot x$ which equals to $y\gamma_0 x$ and belongs to $(x)_\ell^{\Gamma}$.
5- $y\gamma \cdot x$ which equals to $y\gamma x$ and as before belongs to $(x)_\ell^{\Gamma}$. □

Proposition 2.7. *For every* $x \in S$, $\mathcal{Q}_x^{\Sigma_{\gamma_0}} = \mathcal{Q}_x^{\Gamma}$.

Proof. From Lemma 2.6 we see that for every $x \in S$, the quasi ideal in Σ_{γ_0} generated by x is the set $(x)_q^{\Sigma_{\gamma_0}} = (x)_\ell^{\Gamma} \cap (x)_r^{\Gamma} = (x)_q^{\Gamma}$, therefore any $y \in S$ that is contained in $\mathcal{Q}_x^{\Sigma_{\gamma_0}}$ has to be contained in $\mathcal{Q}_x^{\Gamma}$, and conversely. It remains to prove that $\mathcal{Q}_x^{\Sigma_{\gamma_0}}$ has no elements of the following four forms: $\alpha y \beta$, αy, $y\beta$ or α where $\alpha, \beta \in \Gamma$ and $y \in S$. We make the proof for the first type $\alpha y \beta$ because the proofs for the other types of words are similar. If $\alpha y \beta \in \mathcal{Q}_x^{\Sigma_{\gamma_0}}$, then $(\alpha y \beta)_r^{\Sigma_{\gamma_0}} = (x)_r^{\Sigma_{\gamma_0}}$ which is impossible since the left hand side cannot contain x. □

Theorem 2.8. *(Green Theorem) Suppose that* x, y *and* $x\gamma_0 y$ *for a certain* $\gamma_0 \in \Gamma$ *belong to the same class* $\mathcal{H}_x^{\Gamma}$. *Then,* $\mathcal{H}_x^{\Gamma}$ *is a subgroup of the semigroup* S_{γ_0}.

Proof. For the particular γ_0 stated in the theorem, we construct the semigroup Σ_{γ_0} for which we know from Proposition 2.7 that $\mathcal{H}_x^{\Gamma}$ and $\mathcal{H}_x^{\Sigma_{\gamma_0}}$ coincide. Now since $x, y, x\gamma_0 y \in \mathcal{H}_x^{\Gamma}$, we have that $x, y, x\gamma_0 y \in \mathcal{H}_x^{\Sigma_{\gamma_0}}$. But $x\gamma_0 y = xy$ in Σ_{γ_0}, hence $\mathcal{H}_x^{\Sigma_{\gamma_0}}$ satisfies the Green condition and then the Green's theorem for plain semigroups implies that $\mathcal{H}_x^{\Sigma_{\gamma_0}}$ is a group. It is now obvious that $\mathcal{H}_x^{\Gamma}$ is a subgroup of S_{γ_0}. □

3. Completely simple Γ-semigroups

In this section we will define completely simple Γ-semigroups as those Γ-semigroups without zero such that each S_γ is a completely simple semigroup. It turns out that it is sufficient to assume that only a particular S_{γ_0} is a completely simple semigroup in order that every S_γ is a completely simple semigroup. This generalizes the well known result of [5] for Γ-groups.

Theorem 3.1. *For a given* Γ*-semigroup* S *without zero, if for some* $\gamma_0 \in \Gamma$, S_{γ_0} *is a completely simple semigroup, then* S_γ *is a completely simple semigroup for every* $\gamma \in \Gamma$.

Proof. As in the previous theorem, we let Σ_{γ_0} be the adjoint semigroup constructed for γ_0. We proceed by first showing that $\Sigma'_{\gamma_0} = \Sigma_{\gamma_0} \setminus \Gamma$ is a completely simple semigroup without zero. To show that it is simple, we note first that from Lemma 2.4 Σ'_{γ_0} is a disjoint union of subsemigroups of the form S, γS, $S\gamma$, $\gamma S \gamma'$ for γ, γ' varying in Γ. If J is an ideal of Σ'_{γ_0} containing an $x \in S$, then $J \cap S$ is an ideal of S_{γ_0} and since S_{γ_0} is a simple semigroup, then it follows that $J \cap S = S$, hence $S \subseteq J$. For any $s_0 \in S_{\gamma_0}$,

$$S \circ s_0 \circ S = S,$$

from simplicity of S_{γ_0}. For any $\gamma \in \Gamma$, we have in Σ_{γ_0} the equality

$$\gamma S \circ s_0 \circ S = \gamma S. \tag{3.1}$$

Taking into account that $S \subseteq J$ together with the fact that $\gamma S \circ s_0 \subseteq \Sigma'_{\gamma_0}$ we derive from (3.1) that $\gamma S \subseteq J$. In the same way we get $S\gamma \subseteq J$. Further, since again $S \circ s_0 \circ S = S$, then for any $\gamma, \gamma' \in \Gamma$,

$$\gamma S \circ s_0 \circ S\gamma' = \gamma S \gamma',$$

which together with the inclusions $\gamma S, S\gamma \subseteq J$ and $s_0 \in S \subseteq J$, imply that $\gamma S \gamma' \subseteq J$.

If J contains some $x\gamma$, then, for any $y \in S$, it contains $x\gamma y$ which lies in S, and than we proceed as before. The same argument applies if J contains some γx or some $\gamma x \gamma'$. So we finally have that $J = \Sigma'_{\gamma_0}$ proving the simplicity of Σ'_{γ_0}.

Now we show that Σ'_{γ_0} contains a primitive idempotent. Let ε_0 be a primitive idempotent of S_{γ_0}. We show that $\gamma_0\varepsilon_0\gamma_0$ is a primitive idempotent of Σ'_{γ_0}. To do this, one should observe first that any idempotent which is lower than $\gamma_0\varepsilon_0\gamma_0$ in the natural order must have the form $\gamma_0\varepsilon\gamma_0$ where ε is an idempotent in S_{γ_0}. It is obvious that any idempotent ε of Σ'_{γ_0} belonging to S cannot be lower than $\gamma_0\varepsilon_0\gamma_0$ since $\varepsilon \cdot (\gamma_0\varepsilon_0\gamma_0) \neq \varepsilon$. Similar arguments apply if the idempotent is from some γS or $S\gamma$. Finally if the idempotent has the form $\gamma\varepsilon\gamma'$, then

$$\begin{aligned} \gamma\varepsilon\gamma' &= (\gamma\varepsilon\gamma') \cdot (\gamma_0\varepsilon_0\gamma_0) \\ &= \gamma(\varepsilon\gamma'\varepsilon_0)\gamma_0. \end{aligned}$$

The uniqueness of the expression of elements of Σ_{γ_0} as words with letters from $S \cup \Gamma$ (Lemma 2.4) implies that $\gamma' = \gamma_0$. In a similar fashion one can prove that $\gamma = \gamma_0$. From the assumption that $\gamma_0\varepsilon\gamma_0 \leq \gamma_0\varepsilon_0\gamma_0$, we get

$$\begin{aligned} \gamma_0\varepsilon\gamma_0 &= (\gamma_0\varepsilon\gamma_0) \cdot (\gamma_0\varepsilon_0\gamma_0) \\ &= \gamma_0(\varepsilon\gamma_0\varepsilon_0)\gamma_0 = \gamma_0\varepsilon_0\gamma_0 \qquad \text{(since } \varepsilon_0 \text{ is primitive,)} \end{aligned}$$

which similarly to above implies that $\varepsilon = \varepsilon_0$. The assertion that Σ'_{γ_0} does not have a zero is proved easily. Thus Σ'_{γ_0} is completely simple and therefore it is completely regular. The latter means that any element of Σ'_{γ_0} lies in a subgroup of Σ'_{γ_0}. Let $\gamma \in \Gamma$ be arbitrary and $\gamma x \in \gamma S$. There is a subgroup G of Σ'_{γ_0} such that $\gamma x \in \gamma S \cap G$. It follows that the unit of G must have the form $\gamma\varepsilon$ where ε is an idempotent in S_γ. Indeed, the unit cannot be an element $\varepsilon \in S$ since $\varepsilon \cdot (\gamma x) \neq \gamma x$. Also it cannot be an element $\alpha\varepsilon\beta \in \alpha S\beta$ since $(\gamma x) \cdot (\alpha\varepsilon\beta) \neq \gamma x$. Similarly one shows that the unit cannot be of the form $\varepsilon\beta \in S\beta$. Finally if the unit is $\alpha\varepsilon \in \alpha S$, then

$$\gamma x = (\alpha\varepsilon) \cdot (\gamma x) = \alpha(\varepsilon\gamma x),$$

which shows that $\alpha = \gamma$.

Taking into account that the unit of G is $\gamma\varepsilon$ we show that any $g \in G$ must have the form γz with $z \in S$, therefore $G \subseteq \gamma S$. Indeed, elements of the form $x \in S$, $\alpha y\beta \in \alpha S\beta$ and $y\beta \in S\beta$ are excluded since

$$x \neq (\gamma\varepsilon) \cdot x, \alpha y\beta \neq (\alpha y\beta) \cdot (\gamma\varepsilon) \text{ and } y\beta \neq (\gamma\varepsilon) \cdot (y\beta).$$

The remaining elements are $\alpha y \in \alpha S$. For such elements we have

$$\alpha y = (\gamma\varepsilon) \cdot (\alpha y) = \gamma(\varepsilon\alpha y),$$

whence $\alpha = \gamma$ and so $G \subseteq \gamma S$ as claimed. This shows that any element $\gamma x \in \gamma S$ is contained in a subgroup G of γS, whence γS is completely regular. Using this it is easy to show that S_γ is completely regular too. For this it is enough to observe that γS and S_γ are isomorphic under the map

$$\phi : \gamma S \to S_\gamma \text{ such that } \gamma x \mapsto x.$$

To complete the proof, we show that S_γ is a simple semigroup under the assumption that S_{γ_0} is simple. To this end we show that any ideal I of S_γ is an ideal of S_{γ_0} too. Indeed, let $x \in I$ and $s \in S_\gamma$ arbitrary elements. Denote by ε_x^γ the unit of the subgroup of S_γ containing x, then

$$s \circ x = s\gamma_0 x = s\gamma_0(\varepsilon_x^\gamma \gamma x) = (s\gamma_0\varepsilon_x^\gamma)\gamma x \in I,$$

showing that I is a left ideal of S_{γ_0}. In a similar fashion with above one can show that I is a right ideal concluding the proof. □

We may now redefine a completely simple Γ semigroup as a Γ semigroup S having the property that there exists $\gamma_0 \in \Gamma$ such that S_{γ_0} is a completely simple semigroup.

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