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ON BESICOVITCH-DOSS ALMOST PERIODIC SOLUTIONS OF ABSTRACT VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this note, we introduce the class of Besicovitch-Doss almost periodic functions in Banach spaces. After that, we investigate Besicovitch-Doss almost periodic properties of finite and infinite convolution products. Our results can be simply incorporated in the qualitative analysis of solutions of certain classes of abstract (degenerate) Volterra integro-differential equations in Banach spaces.

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1. Introduction and preliminaries

The notion of an almost periodic function was introduced by H. Bohr in 1925 and later generalized by many other mathematicians (see e.g. [6], [10] and [15] for further information in this direction). Let $I = \mathbb{R}$ or $I = [0, \infty)$, let X be a Banach space, and let $f : I \to X$ be continuous. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ iff $\|f(t + \tau) - f(t)\| \leq \epsilon$, $t \in I$. The set consisted of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f, \epsilon)$. It is said that $f(\cdot)$ is almost periodic, a.p. for short, iff for each $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l contains an element of $\vartheta(f, \epsilon)$.

As mentioned in the abstract, this paper is intended to be only a note. We introduce the class of Besicovitch-Doss almost periodic functions in Banach spaces and after that we analyze the Besicovitch-Doss almost periodic properties of the infinite convolution product

$$t \mapsto \int_{-\infty}^{t} R(t-s)g(s)\,ds, \quad t \in \mathbb{R} \tag{1.1}$$

and the finite convolution product

$$t \mapsto \int_{0}^{t} R(t-s)f(s)\,ds, \quad t \geq 0. \tag{1.2}$$

Keeping in mind the fact that solutions of nonhomogeneous abstract (degenerate) Volterra integro-differential equations in Banach spaces are basically

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given by variation of parameters formulae involving the integrals like (1.1) or (1.2), our results can be utilized straightforwardly in the study of existence and uniqueness of Besicovitch-Doss almost periodic solutions of certain classes of nonhomogeneous abstract Volterra integro-differential equations; see [9], [18] and [11]-[12] for more details on the subject (cf. [2] and [17] for some known results on Besicovitch almost periodic solutions of abstract differential equations). The strongly continuous operator family $(R(t))_{t>0} \subseteq L(X,Y)$ appearing in (1.1)-(1.2) is assumed to satisfy the condition

$$\int_0^{\infty} (1+s)\|R(s)\|\, ds < \infty,$$

which clearly implies the impossibility to apply our results to certain classes of fractional differential equations with Riemann-Liouville or Caputo derivatives [14]; here, Y is a Banach space, as well. But, the results obtained easily apply to a wide class of abstract (degenerate) differential equations of first order [9], [12], as well as to some abstract higher-order differential equations [5] and abstract Volterra integro-differential inclusions [14]. It is also worth noting that we introduce the class of Besicovitch-p-vanishing functions and show that this class reduces to the class consisting of all p-locally integrable X-valued functions whose Besicovitch seminorm is equal to zero.

We use the standard notation henceforth. By X and Y we denote two non-trivial complex Banach spaces. The symbol $L(X,Y)$ designates the space consisting of all continuous linear mappings from X into Y; $L(X) \equiv L(X,X)$. The norm of an element $x \in X$ is denoted by $\|x\|$. If $1 \leq p < \infty$, then by $L^p_{loc}(I : X)$ we denote the vector space consisting of all locally p-integrable functions with domain I and taking values in X; if $X = \mathbb{C}$, then we simply write $L^p_{loc}(I)$ for this space.

2. Besicovitch-Doss almost periodic functions

The main aim of this section is to consider Besicovitch-Doss almost periodic functions in Banach spaces. We also analyze the classes of vector-valued Besicovitch almost periodic functions and vector-valued Besicovitch vanishing functions.

Let $1 \leq p < \infty$, let $l > 0$, and let $f,\ g \in L^p_{loc}(I : X)$, where $I = \mathbb{R}$ or $I = [0,\infty)$. We define the Stepanov 'metric' by

$$D^p_{S_l}\big[f(\cdot), g(\cdot)\big] := \sup_{x \in I} \left[\frac{1}{l} \int_x^{x+l} \big\|f(t) - g(t)\big\|^p \, dt \right]^{1/p}.$$

Then we know that, for every two numbers $l_1,\ l_2 > 0$, there exist two positive real constants $k_1,\ k_2 > 0$ independent of $f,\ g$, such that

$$k_1 D^p_{S_{l_1}}\big[f(\cdot), g(\cdot)\big] \leq D^p_{S_{l_2}}\big[f(\cdot), g(\cdot)\big] \leq k_2 D^p_{S_{l_1}}\big[f(\cdot), g(\cdot)\big],$$

as well as that (see e.g. [3, pp. 72-73]) in the scalar-valued case there exists

$$D^p_W\big[f(\cdot),g(\cdot)\big] := \lim_{l\to\infty} D^p_{S_l}\big[f(\cdot),g(\cdot)\big] \tag{2.1}$$

in $[0,\infty]$. The distance appearing in (2.1) is called the Weyl distance of $f(\cdot)$ and $g(\cdot)$. The Stepanov and Weyl 'norm' of $f(\cdot)$ are defined by

$$\big\|f\big\|_{S^p_l} := D^p_{S_l}\big[f(\cdot),0\big] \quad \text{and} \quad \big\|f\big\|_{W^p} := D^p_W\big[f(\cdot),0\big],$$

respectively. We refer the reader to [13] for the notions of Stepanov and Weyl almost periodic functions.

In the same paper, we have introduced the notions of Weyl-p-vanishing function and equi-Weyl-p-vanishing function as follows. It is said that $q \in L^p_{loc}([0,\infty):X)$ is Weyl-p-vanishing iff

(2.2)

$$\lim_{t\to\infty}\big\|\mathbf{q}(t,\cdot)\big\|_{W^p}=0, \text{ i.e., } \lim_{t\to\infty}\lim_{l\to\infty}\sup_{x\geq 0}\left[\frac{1}{l}\int_x^{x+l}\big\|q(t+s)\big\|^p\,ds\right]^{1/p}=0,$$

while $q \in L^p_{loc}([0,\infty):X)$ is said to be equi-Weyl-p-vanishing iff

$$\lim_{l\to\infty}\lim_{t\to\infty}\sup_{x\geq 0}\left[\frac{1}{l}\int_x^{x+l}\big\|q(t+s)\big\|^p\,ds\right]^{1/p}=0.$$

We know that any equi-Weyl-p-vanishing function $q \in L^p_{loc}([0,\infty):X)$ is already Weyl-p-vanishing. Before proceeding further, we would like to observe that there is a great number of very simple examples showing that for a function $q \in L^p_{loc}([0,\infty):X)$ the situation in which $\|\mathbf{q}(t,\cdot)\|_{W^p} \neq \|\mathbf{q}(t',\cdot)\|_{W^p}$ for all $t \neq t'$ can occur: Consider, for instance, the function $q(t) := 2^{-1}(t+1)^{(-1)/2}$, $t \geq 0$ and the case in which $p=1$; then a direct computation yields that $\|\mathbf{q}(t,\cdot)\|_{W^p} = (t+1)^{(-1)/2}$, $t \geq 0$. The situation is completely different in the case of examination of Besicovitch-p-vanishing functions, as we will see below.

The class of Besicovitch almost periodic functions extends the classes of Stepanov and Weyl almost periodic functions. There is several possible ways to introduce the notion of Besicovitch almost periodic function with values in a Banach space X; cf. also L. I. Danilov [4] for the corresponding notion in complete metric spaces.

The standard procedure goes as follows. Let $1 \leq p < \infty$. Following A. S. Besicovitch [3], for every function $f \in L^p_{loc}(\mathbb{R}:X)$, we define

$$\|f\|_{\mathcal{M}^p} := \limsup_{t\to+\infty}\left[\frac{1}{2t}\int_{-t}^{t}\|f(s)\|^p\,ds\right]^{1/p};$$

if $f \in L^p_{loc}([0,\infty):X)$, then

$$\|f\|_{\mathcal{M}^p} := \limsup_{t\to+\infty}\left[\frac{1}{t}\int_0^{t}\|f(s)\|^p\,ds\right]^{1/p}.$$

Here we use the abbreviation $\|f\|_{\mathcal{M}^p}$ because of later J. Marcinkiewicz's investigations of Besicovitch class [16].

In both cases, $\|\cdot\|_{\mathcal{M}^p}$ is a seminorm on the space $\mathcal{M}^p(I:X)$ consisting of those $L^p_{loc}(I:X)$-functions $f(\cdot)$ for which $\|f\|_{\mathcal{M}^p}<\infty$. Denote $K_p(I:X):=\{f\in\mathcal{M}^p(I:X):\|f\|_{\mathcal{M}^p}=0\}$ and

$$M_p(I:X):=\mathcal{M}^p(I:X)/K_p(I:X).$$

The seminorm $\|\cdot\|_{\mathcal{M}^p}$ on $\mathcal{M}^p(I:X)$ induces the norm $\|\cdot\|_{M^p}$ on $M^p(I:X)$ under which $M^p(I:X)$ is complete; in other words, $(M^p(I:X),\|\cdot\|_{M^p})$ is a Banach space.

Definition 2.1. Let $1\leq p<\infty$. We say that a function $f\in L^p_{loc}(I:X)$ is Besicovitch-p-almost periodic iff there exists a sequence of X-valued trigonometric polynomials converging to $f(\cdot)$ in $(M^p(I:X),\|\cdot\|_{M^p})$.

The vector space consisting of all Besicovitch-p-almost periodic functions $I\to X$ will be denoted by $B^p(I:X)$. It is well known that $B^p(I:X)$ is a closed subspace of $M^p(I:X)$ and therefore Banach space itself, when equipped with the norm $\|\cdot\|_{M^p}$.

The Besicovitch class can be also introduced in a Bohr-like manner, by using the notion of satisfactorily uniform sets (see e.g. [3] and [1, Definition 5.10, Definition 5.11]). We will not use this approach henceforth.

We define the Besicovitch 'distance' of functions $f,\ g\in L^p_{loc}(I:X)$ by

$$D_{B^p}[f(\cdot),g(\cdot)]:=\|f-g\|_{\mathcal{M}^p};$$

the Besicovitch 'norm' of $f\in L^p_{loc}(I:X)$ is defined by

$$\|f\|_{B^p}:=D_{B^p}[f(\cdot),0]:=\|f\|_{\mathcal{M}^p}.$$

Let us recall that, in scalar-valued case (see e.g. [3, p. 73]), we have:

$$\|f-g\|_\infty\geq D_{S^p_l}[f(\cdot),g(\cdot)]\geq D_{W^p}[f(\cdot),g(\cdot)]\geq D_{B^p}[f(\cdot),g(\cdot)], \tag{2.3}$$

for $1\leq p<\infty$, $l>0$ and $f,\ g\in L^p_{loc}(I)$, as well as that the assumption $\|f\|_{\mathcal{M}^p}=0$ does not imply $f=0$ a.e. on I.

We introduce the notion of Besicovitch-Doss-p-almost periodic function following the fundamental characterization of scalar-valued Besicovitch almost periodic functions established by R. Doss in [7]-[8] (cf. also [1, pp. 160-161] for further information on the subject):

Definition 2.2. Let $1\leq p<\infty$. It is said that $f\in L^p_{loc}(I:X)$ is Besicovitch-Doss-p-almost periodic iff the following conditions hold:

(i) (B^p-boundedness) We have $\|f\|_{\mathcal{M}^p}<\infty$.

(ii) (B^p-continuity) We have

$$\lim_{\tau\to 0}\limsup_{t\to+\infty}\left[\frac{1}{2t}\int_{-t}^{t}\|f(s+\tau)-f(s)\|^p\,ds\right]^{1/p}=0,$$

in the case that $I = \mathbb{R}$, resp.,

$$\lim_{\tau \to 0+} \limsup_{t \to +\infty} \left[\frac{1}{t} \int_0^t \| f(s+\tau) - f(s) \|^p \, ds \right]^{1/p} = 0,$$

in the case that $I = [0, \infty)$.

(iii) (Doss almost periodicity) For every $\epsilon > 0$, the set of numbers $\tau \in I$ for which

$$\limsup_{t \to +\infty} \left[\frac{1}{2t} \int_{-t}^t \| f(s+\tau) - f(s) \|^p \, ds \right]^{1/p} < \epsilon, \tag{2.4}$$

in the case that $I = \mathbb{R}$, resp.,

$$\limsup_{t \to +\infty} \left[\frac{1}{t} \int_0^t \| f(s+\tau) - f(s) \|^p \, ds \right]^{1/p} < \epsilon,$$

in the case that $I = [0, \infty)$, is relatively dense in I.

(iv) For every $\lambda \in \mathbb{R}$, we have that

$$\lim_{l \to +\infty} \limsup_{t \to +\infty} \frac{1}{l} \left[\frac{1}{2t} \int_{-t}^t \left\| \left(\int_x^{x+l} - \int_0^l \right) e^{i\lambda s} f(s) \, ds \right\|^p dx \right]^{1/p} = 0,$$

in the case that $I = \mathbb{R}$, resp.,

$$\lim_{l \to +\infty} \limsup_{t \to +\infty} \frac{1}{l} \left[\frac{1}{t} \int_0^t \left\| \left(\int_x^{x+l} - \int_0^l \right) e^{i\lambda s} f(s) \, ds \right\|^p dx \right]^{1/p} = 0,$$

in the case that $I = [0, \infty)$.

The vector space consisting of all Besicovitch-Doss-p-almost periodic functions $I \to X$ in the sense of Definition 2.2 will be denoted by $\mathrm{B}^p(I : X)$. In the case that $X = \mathbb{C}$, the fundamental result of R. Doss says that $\mathrm{B}^p(I : X) = B^p(I : X)$. But, the argumentation from [7]-[8] cannot be so easily transferred to vector-valued case. Because of that, we would like to raise the following issue:

Problem. Let $1 \leq p < \infty$, and let X be a Banach space. Is it true that $\mathrm{B}^p(I : X) = B^p(I : X)$ in the set theoretical sense?

If $f \in L^p_{loc}(\mathbb{R} : X)$, then its restriction to the non-negative real axis $f_+ \in L^p_{loc}([0, \infty) : X)$ and it is very elementary to prove that the supposition $f \in B^p(\mathbb{R} : X)$, resp., $f \in \mathrm{B}^p(\mathbb{R} : X)$ implies $f_+ \in B^p([0, \infty) : X)$, resp., $f_+ \in \mathrm{B}^p([0, \infty) : X)$; see e.g. [1, p. 153] for more details given in scalar-valued case.

It is simply said that a Besicovitch-Doss-1-almost periodic function is Besicovitch-Doss almost periodic. We continue by stating the following notion:

Definition 2.3. It is said that $q \in L^p_{loc}([0,\infty) : X)$ is Besicovitch-p-vanishing iff

$$\text{(2.5)}\quad \lim_{t\to\infty} \big\|\mathbf{q}(t,\cdot)\big\|_{\mathcal{M}^p} = 0, \text{ i.e., } \lim_{t\to+\infty}\ \limsup_{s\to+\infty} \left[\frac{1}{s}\int_0^s \big\|q(t+r)\big\|^p\,dr\right]^{1/p} = 0.$$

For any $q \in L^p_{loc}([0,\infty) : X)$, we define the function $\|q\|(\cdot) \in L^p_{loc}([0,\infty))$ as usually. Then it is clear that $q(\cdot)$ is Weyl-p-vanishing iff

$$\lim_{t\to+\infty} \big\| \|q\|(t+\cdot)\big\|_{W^p} = 0,$$

while $q(\cdot)$ is Besicovitch-p-vanishing iff

$$\lim_{t\to+\infty} \big\| \|q\|(t+\cdot)\big\|_{B^p} = 0.$$

Hence, (2.3) immediately implies that the class consisting of all Besicovitch-p-vanishing functions extends the corresponding class consisting of all Weyl-p-vanishing functions. The reader may try to construct some examples showing that this extension is strict.

As in the case of Weyl-almost periodicity, we can replace the limits in (2.5), i.e., for any $q \in L^p_{loc}([0,\infty) : X)$ we can consider the following condition

$$\text{(2.6)}\qquad \limsup_{s\to+\infty}\ \lim_{t\to+\infty} \left[\frac{1}{s}\int_0^s \big\|q(t+r)\big\|^p\,dr\right]^{1/p} = 0.$$

If (2.6) holds, then there is a positive number $s_0 > 0$ such that

$$\lim_{t\to+\infty} \int_0^s \big\|q(t+r)\big\|^p\,dr$$

exists for all $s > s_0$. Unfortunately, an equi-Weyl-p-vanishing function need not satisfy the last condition; a simple counterexample is given by the function

$$q(t) := \sum_{n=0}^{\infty} \chi_{[n^2,n^2+1]}(t), \quad t \geq 0,$$

where $\chi_A(\cdot)$ denotes the characteristic function of set A. This is the main reason why we will not consider the class consisting of p-locally integrable X-valued functions satisfying the condition (2.6).

As mentioned in the introductory part, the class of Besicovitch-p-vanishing functions is equal to the class of p-locally integrable X-valued functions whose Besicovitch seminorm is equal to zero. This basically follows from the analysis of R. Doss [7, p. 478], showing that, for every $q \in L^p_{loc}([0,\infty) : X)$, we have $\big\|\mathbf{q}(t,\cdot)\big\|_{\mathcal{M}^p} = \|q\|_{\mathcal{M}^p}$, $t \geq 0$. We will give another proof of this fact for the sake of completeness:

Proposition 2.4. *Let $1 \leq p < \infty$, and let $q \in L^p_{loc}([0,\infty) : X)$. Then $\big\|\mathbf{q}(t,\cdot)\big\|_{\mathcal{M}^p} = \|q\|_{\mathcal{M}^p}$ for all $t \geq 0$.*

Proof. Since for any non-negative function $\varphi : \mathbb{R} \to [0, \infty)$ one has

$$\limsup_{s\to+\infty} \varphi(s) = \lim_{s\to+\infty} \sup_{y\geq s} \varphi(y),$$

it suffices to show that

$$\lim_{s\to+\infty} \sup_{y\geq s} \left[\frac{1}{y} \int_t^{t+y} \|q(r)\|^p \, dr \right]^{1/p} = \lim_{s\to+\infty} \sup_{y\geq s} \left[\frac{1}{y} \int_0^{y} \|q(r)\|^p \, dr \right]^{1/p}$$

for all $t \geq 0$. Fix such a number t. Then we have

$$\begin{aligned}
&\sup_{y\geq s} \left[\frac{1}{y} \int_t^{t+y} \|q(r)\|^p \, dr \right]^{1/p} \\
&\qquad \leq \sup_{y\geq s} \left[\frac{1}{y} \int_0^{t+y} \|q(r)\|^p \, dr \right]^{1/p} \\
&\qquad \leq \sup_{y\geq s} \left[\left(\frac{1}{t+y} + \frac{t}{s(t+y)} \right) \int_0^{t+y} \|q(r)\|^p \, dr \right]^{1/p} \\
&\qquad \leq \left(1 + \frac{t}{s} \right) \sup_{y\geq s} \left[\frac{1}{y} \int_0^{y} \|q(r)\|^p \, dr \right]^{1/p},
\end{aligned}$$

showing that

$$\lim_{s\to+\infty} \sup_{y\geq s} \left[\frac{1}{y} \int_t^{t+y} \|q(r)\|^p \, dr \right]^{1/p} \leq \lim_{s\to+\infty} \sup_{y\geq s} \left[\frac{1}{y} \int_0^{y} \|q(r)\|^p \, dr \right]^{1/p}.$$

For the opposite inequality, observe that

$$\begin{aligned}
&\sup_{y\geq s} \left[\frac{1}{y} \int_t^{t+y} \|q(r)\|^p \, dr \right]^{1/p} \\
&\qquad \geq \sup_{y\geq s} \left[\frac{1}{y} \int_0^{t+y} \|q(r)\|^p \, dr - \frac{1}{y} \int_0^{t} \|q(r)\|^p \, dr \right]^{1/p} \\
&\qquad \geq \sup_{y\geq s} \left[\left(\frac{1}{y} \int_0^{t+y} \|q(r)\|^p \, dr \right)^{1/p} - \left(\frac{1}{y} \int_0^{t} \|q(r)\|^p \, dr \right)^{1/p} \right].
\end{aligned}$$

Since for any $y \geq s$ one has

$$\frac{1}{y} \int_0^{t} \|q(r)\|^p \, dr \leq \frac{1}{s} \int_0^{t} \|q(r)\|^p \, dr,$$

the above computation yields that

$$\lim_{s\to+\infty} \sup_{y\geq s} \left[\frac{1}{y} \int_t^{t+y} \|q(r)\|^p \, dr \right]^{1/p} \geq \lim_{s\to+\infty} \sup_{y\geq s} \left[\frac{1}{y} \int_0^{t+y} \|q(r)\|^p \, dr \right]^{1/p}.$$

The final conclusion follows by noticing that

$$\begin{aligned}
&\lim_{s\to+\infty}\sup_{y\geq s}\left[\frac{1}{y}\int_0^{t+y}\|q(r)\|^p\,dr\right]^{1/p}\\
&\geq \lim_{s\to+\infty}\sup_{y\geq s}\left[\frac{1}{t+y}\int_0^{t+y}\|q(r)\|^p\,dr\right]^{1/p}\\
&= \lim_{s\to+\infty}\sup_{y\geq s+t}\left[\frac{1}{y}\int_0^{y}\|q(r)\|^p\,dr\right]^{1/p}\\
&= \lim_{v\to+\infty}\sup_{y\geq v}\left[\frac{1}{y}\int_0^{y}\|q(r)\|^p\,dr\right]^{1/p},
\end{aligned}$$

where the last equality follows by using the substitution $v = s + t$. □

Since for any non-negative function $\varphi : \mathbb{R} \to [0,\infty)$ the equivalence relation

$$\limsup_{s\to+\infty}\varphi(s) = 0 \Leftrightarrow \lim_{s\to+\infty}\varphi(s) = 0$$

holds good, Proposition 2.4 immediately implies:

Corollary 2.5. *Let $1 \leq p < \infty$, and let $q \in L^p_{loc}([0,\infty) : X)$. Then $q(\cdot)$ is Besicovitch-p-vanishing iff $\|q\|_{\mathcal{M}^p} = 0$ iff $q \in K_p([0,\infty) : X)$ iff*

$$\lim_{s\to+\infty}\frac{1}{s}\int_0^s \big\|q(r)\big\|^p\,dr = 0.$$

Denote by $B^p_0([0,\infty) : X)$ the set consisting of Besicovitch-p-vanishing functions. Then it can be trivially shown that $B^p_0([0,\infty) : X)$ has a linear vector structure. A great number of new ‘asymptotically almost periodic function spaces’ can be defined as the sum of space $B^p_0([0,\infty) : X)$ and corresponding spaces of (Stepanov, Weyl, Doss, Hartman) almost periodic functions: such sums are not direct, in general [13]. The complete analysis is, unquestionably, without the scope of this paper and we only want to note that it is ridiculous to introduce the space of asymptotically Besicovitch almost periodic functions since the sum of space $B^p(I : X)$ and $B^p_0([0,\infty) : X)$, with the meaning clear, is again the space $B^p(I : X)$ on account of Corollary 2.5; cf. also Theorem 3.3. Now we will prove that the sum of space $\mathrm{B}^p([0,\infty) : X)$ and $B^p_0([0,\infty) : X)$ is $\mathrm{B}^p([0,\infty) : X)$, as well:

Proposition 2.6. *Let $1 \leq p < \infty$. Then we have $\mathrm{B}^p([0,\infty) : X) + B^p_0([0,\infty) : X) = \mathrm{B}^p([0,\infty) : X)$.*

Proof. It suffices to show that, for every function $q \in B^p_0([0,\infty) : X)$ and for every two real numbers $\tau \geq 0$ and $\lambda \in \mathbb{R}$ we have

$$\lim_{\tau\to0+}\limsup_{t\to+\infty}\left[\frac{1}{t}\int_0^t \|q(s+\tau)-q(s)\|^p\,ds\right]^{1/p} = 0,$$

$$\limsup_{t\to+\infty}\left[\frac{1}{t}\int_0^t \|q(s+\tau)-q(s)\|^p\,ds\right]^{1/p}=0,$$

and

$$\lim_{l\to+\infty}\limsup_{t\to+\infty}\frac{1}{l}\left[\frac{1}{t}\int_0^t\left\|\left(\int_x^{x+l}-\int_0^l\right)e^{i\lambda s}q(s)\,ds\right\|^p dx\right]^{1/p}=0.$$

The first two equalities follow almost immediately from Proposition 2.4 and Corollary 2.5, so that we only need to prove the third equality. This follows from the next computation

$$\begin{aligned}
&\frac{1}{l}\left[\frac{1}{t}\int_0^t\left\|\left(\int_x^{x+l}-\int_0^l\right)e^{i\lambda s}q(s)\,ds\right\|^p dx\right]^{1/p}\\
&\quad\le\frac{1}{l}\left[\frac{1}{t}\int_0^t\left|\left(\int_x^{x+l}+\int_0^l\right)\|q(s)\|\,ds\right|^p dx\right]^{1/p}\\
&\quad\le\frac{1}{l}\left[\frac{1}{t}\int_0^t\left|\left(\int_x^{x+l}-\int_0^l\right)\|q(s)\|\,ds+2\int_0^l\|q(s)\|\,ds\right|^p dx\right]^{1/p}\\
&\quad\le\frac{2^{p-1/p}}{l}\left[\frac{1}{t}\int_0^t\left\{\left|\left(\int_x^{x+l}-\int_0^l\right)\|q(s)\|\,ds\right|^p+\left|2\int_0^l\|q(s)\|\,ds\right|^p\right\}dx\right]^{1/p}\\
&\quad\le\frac{2^{p-1/p}}{l}\left[\frac{1}{t}\int_0^t\left|\left(\int_x^{x+l}-\int_0^l\right)\|q(s)\|\,ds\right|^p dx\right]^{1/p}\\
&\quad+\frac{2^{p-1/p}}{l}\left[\frac{1}{t}\int_0^t\left|2\int_0^l\|q(s)\|\,ds\right|^p dx\right]^{1/p}\\
&\quad=\frac{2^{p-1/p}}{l}\left[\frac{1}{t}\int_0^t\left|\left(\int_x^{x+l}-\int_0^l\right)\|q(s)\|\,ds\right|^p dx\right]^{1/p}\\
&\quad+2\frac{2^{p-1/p}}{l}\int_0^l\|q(s)\|\,ds\\
&\quad\le\frac{2^{p-1/p}}{l}\left[\frac{1}{t}\int_0^t\left|\left(\int_x^{x+l}-\int_0^l\right)\|q(s)\|\,ds\right|^p dx\right]^{1/p}\\
&\quad+2\cdot 2^{p-1/p}\left[\frac{1}{l}\int_0^l\|q(s)\|^p\,ds\right]^{1/p},
\end{aligned}$$

where the last estimate follows from an application of an inequality appearing on p. 70 of [3]. The final conclusion follows from Corollary 2.5 and the fact that $\|q(\cdot)\|\in B_0^p([0,\infty):\mathbb{C})=\mathrm{B}^p(I:\mathbb{C})$ satisfies the fourth equality of Definition 2.2. □

3. Besicovitch-Doss almost periodic properties of convolution products

In this section, we enquire into the Besicovitch-Doss almost periodic properties of finite and infinite convolution products. Our main result reads as follows:

Theorem 3.1. *Suppose that $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that $\int_0^\infty (1+s)\|R(s)\|\,ds < \infty$. If $g : \mathbb{R} \to X$ is bounded and Besicovitch-Doss almost periodic, then the function $G(\cdot)$, given by*

$$G(t) := \int_{-\infty}^{t} R(t-s)g(s)\,ds, \quad t \in \mathbb{R}, \tag{3.1}$$

is bounded and Besicovitch-Doss almost periodic, as well.

Proof. Since $G(t) = \int_0^\infty R(s)g(t-s)\,ds$, $t \geq 0$, it is evident that, for every $t \in \mathbb{R}$, we have that $G(t)$ is well-defined and

$$\|G(t)\| \leq \|g\|_\infty \int_0^\infty \|R(s)\|\,ds.$$

In particular, $G(\cdot)$ and all its translations are both locally integrable and B^1-bounded. Now we will verify that $G(\cdot)$ is Doss almost periodic. Let a number $\epsilon > 0$ be given in advance. Then we can find a finite number $l_\epsilon > 0$ such that any subinterval I of $\mathbb{R}$ of length l_ϵ contains a number $\tau \in \mathbb{R}$ such that (2.4) holds with function $f(\cdot)$ replaced therein by $g(\cdot)$. Then, for every $t > 0$, we have:

$$\begin{aligned}
&\frac{1}{2t}\int_{-t}^{t}\big\|G(s+\tau)-G(s)\big\|\,ds \\
&\quad\leq \frac{1}{2t}\int_{-t}^{t}\int_0^\infty \big\|R(v)\big\| \cdot \big\|g(s+\tau-v)-g(s-v)\big\|\,dv\,ds \\
&\quad= \int_0^\infty \big\|R(v)\big\| \cdot \frac{1}{2t}\int_{-t}^{t}\big\|g(s+\tau-v)-g(s-v)\big\|\,ds\,dv \\
&\quad= \int_0^\infty \big\|R(v)\big\| \cdot \frac{1}{2t}\int_{-t-v}^{t-v}\big\|g(s+\tau)-g(s)\big\|\,ds\,dv \\
&\quad\leq \int_0^\infty \big\|R(v)\big\| \cdot \frac{1}{2t}\int_{-t}^{t}\big\|g(s+\tau)-g(s)\big\|\,ds\,dv \\
&\quad+ \int_0^\infty \big\|R(v)\big\| \cdot \frac{1}{2t}\int_{-t-v}^{-t}\big\|g(s+\tau)-g(s)\big\|\,ds\,dv \\
&\quad+ \int_0^\infty \big\|R(v)\big\| \cdot \frac{1}{2t}\int_{t-v}^{t}\big\|g(s+\tau)-g(s)\big\|\,ds\,dv \\
&\quad\leq \int_0^\infty \big\|R(v)\big\| \cdot \frac{1}{2t}\int_{-t}^{t}\big\|g(s+\tau)-g(s)\big\|\,ds\,dv + \frac{2\|g\|_\infty}{t}\int_0^\infty v\big\|R(v)\big\|\,dv.
\end{aligned}$$

The Doss almost periodicity of $G(\cdot)$ immediately follows from the last inequality of previous calculation, the Doss almost periodicity of $g(\cdot)$ and the fact that $\int_0^\infty v\|R(v)\|\,dv < \infty$. We proceed by proving that the fourth condition in Definition 2.2 holds true for $G(\cdot)$. Let $\lambda \in \mathbb{R}$ and $\epsilon > 0$ be fixed. Then we know that

$$\exists l_0(\epsilon) > 0 \quad \forall l \geq l_0(\epsilon) \quad \exists t_l > 0 \quad \forall t > t_l :$$
$$\frac{1}{l}\frac{1}{2t}\int_{-t}^{t}\left\|\left(\int_x^{x+l} - \int_0^l\right) e^{i\lambda s} g(s)\,ds\right\| dx < \epsilon/5. \tag{3.2}$$

Furthermore,

$$\begin{aligned}
&\frac{1}{l}\frac{1}{2t}\int_{-t}^{t}\left\|\left(\int_x^{x+l} - \int_0^l\right) e^{i\lambda s} G(s)\,ds\right\| dx \\
&= \frac{1}{l}\frac{1}{2t}\int_{-t}^{t}\left\|\left(\int_x^{x+l} - \int_0^l\right) e^{i\lambda s} \int_0^\infty R(v) g(s-v)\,dv\,ds\right\| dx \\
&= \frac{1}{l}\frac{1}{2t}\int_{-t}^{t}\left\|\int_0^\infty\left(\int_x^{x+l} - \int_0^l\right) e^{i\lambda s} R(v) g(s-v)\,ds\,dv\right\| dx \\
&= \frac{1}{l}\frac{1}{2t}\int_{-t}^{t}\left\|\int_0^\infty R(v)\left(\int_x^{x+l} - \int_0^l\right) e^{i\lambda s} g(s-v)\,ds\,dv\right\| dx \\
&\leq \frac{1}{l}\frac{1}{2t}\int_{-t}^{t}\int_0^\infty \|R(v)\| \cdot \left\|\left(\int_x^{x+l} - \int_0^l\right) e^{i\lambda s} g(s-v)\,ds\right\| dv\,dx \\
&= \frac{1}{l}\frac{1}{2t}\int_{-t}^{t}\int_0^\infty \|R(v)\| \cdot \left\|\left(\int_x^{x+l} - \int_0^l\right) e^{i\lambda (s-v)} g(s-v)\,ds\right\| dv\,dx \\
&= \frac{1}{l}\frac{1}{2t}\int_{-t}^{t}\int_0^\infty \|R(v)\| \cdot \left\|\left(\int_{x-v}^{x+l-v} - \int_{-v}^{l-v}\right) e^{i\lambda s} g(s)\,ds\right\| dv\,dx \\
&= \frac{1}{l}\frac{1}{2t}\int_{-t}^{t}\int_0^\infty \|R(v)\| \cdot \left\|\left(\int_x^{x+l} - \int_0^l\right)\right. \\
&\left.+\left(-\int_{x+l}^{x-v+l} + \int_{x-v}^{x} + \int_{-v+l}^{l} - \int_{-v}^{0}\right) e^{i\lambda s} g(s)\,ds\right\| dv\,dx \\
&\leq \frac{1}{l}\frac{1}{2t}\int_{-t}^{t}\int_0^\infty \|R(v)\| \cdot \left\|\left(\int_x^{x+l} - \int_0^l\right) e^{i\lambda s} g(s)\,ds\right\| dv\,dx \\
&+ \frac{1}{l}\frac{1}{2t}\int_{-t}^{t}\int_0^\infty \|R(v)\| \cdot 4\|g\|_\infty v\,dv\,dx
\end{aligned}$$

$$\leq \frac{1}{l}\frac{1}{2t}\int_{-t}^{t}\int_{0}^{\infty}\|R(v)\|\cdot\left\|\left(\int_{x}^{x+l}-\int_{0}^{l}\right)e^{i\lambda s}g(s)\,ds\right\|\,dv\,dx$$
$$+\frac{4\|g\|_{\infty}}{l}\int_{0}^{\infty}v\|R(v)\|\,dv,\quad t>0.$$

Setting $l_0(\epsilon)' := \max\{l_0(\epsilon), 5\epsilon^{-1}\|g\|_{\infty}\int_0^{\infty} v\|R(v)\,dv\}$, we have that (3.2) holds for any $l \geq l_0(\epsilon)'$ and the same $t_l > 0$. The B^1-continuity of $G(\cdot)$ follows from the calculation used in proving the Doss almost periodicity of $G(\cdot)$ and an elementary argumentation. □

Remark 3.2. As in the case of Weyl almost periodicity, it is not clear how to prove an analogue of Theorem 2.2 in the case that $p > 1$. It is also worth noting that the boundedness of function $g(\cdot)$ is essential in our analysis.

For any strongly continuous operator family $(R(t))_{t>0} \subseteq L(X,Y)$ satisfying $\int_0^{\infty}(1+s)\|R(s)\|\,ds < \infty$ and for any function $q \in L^1_{loc}([0,\infty):X)$, the following condition holds:

$$\lim_{t\to+\infty}\limsup_{l\to+\infty}\frac{1}{l}\int_0^t\left[\int_t^{t+l}\|R(s-r)\|\,ds\right]\|q(r)\|\,dr=0, \tag{3.3}$$

because for any $t > 0$ we have

$$\lim_{l\to+\infty}\frac{1}{l}\int_0^t\left[\int_t^{t+l}\|R(s-r)\|\,ds\right]\|q(r)\|\,dr=0; \tag{3.4}$$

this follows from the estimate

$$\frac{1}{l}\int_0^t\left[\int_t^{t+l}\|R(s-r)\|\,ds\right]\|q(r)\|\,dr$$
$$\leq\left[\int_0^{\infty}(1+s)\|R(s)\|\,ds\right]\frac{1}{l}\int_0^t\|q(r)\|\,dr,\quad t,\ l>0.$$

Arguing similarly as in the proof of [13, Proposition 5.3], we can deduce the following result:

Theorem 3.3. *Suppose that $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that $\int_0^{\infty}(1+s)\|R(s)\|\,ds < \infty$. If $g : \mathbb{R} \to X$ is bounded and Besicovitch-Doss almost periodic, as well as $q \in L^1_{loc}([0,\infty):X)$ is Besicovitch-1-vanishing, then the validity of condition* (3.3) *implies that the function $F(\cdot)$, given by*

$$F(t) := \int_0^t R(t-s)[g(s)+q(s)]\,ds,\quad t\geq 0, \tag{3.5}$$

is Besicovitch-Doss almost periodic, as well.

We leave to the interested reader the analysis of various Besicovitch and Besicovitch-Doss almost periodic properties of degenerate (a, k)-regularized C-resolvent families (cf. [13] for the notion and corresponding results in the case of Weyl almost periodicity).

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References

[1] Andres, J., Bersani, A.M., Grande, R.F., Hierarchy of almost-periodic function spaces. Rend. Mat. Appl. 26 (2006), 121-188.

[2] Ayachi, M., Lassoued, D., On the existence of Besicovitch almost periodic solutions for a class of neutral delay differential equations. Facta Univ. Ser. Math. Inform. 29 (2014), 131–144.

[3] Besicovitch, A.S., Almost Periodic Functions. New York: Dover Publications Inc., 1954.

[4] Danilov, L.I., On Besicovitch almost periodic selections of multivalued maps. Preprint, `http://arxiv.org/abs/math/0503293v1`.

[5] de Andrade, B., Lizama, C., Existence of asymptotically almost periodic solutions for damped wave equations. J. Math. Anal. Appl. 382 (2011), 761–771.

[6] Diagana, T., Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces. New York: Springer-Verlag, 2013.

[7] Doss, R., On generalized almost periodic functions. Annals of Math. 59 (1954), 477-489.

[8] Doss, R., On generalized almost periodic functions-II. J. London Math. Soc. 37 (1962), 133-140.

[9] Favini, A., Yagi, A., Degenerate Differential Equations in Banach Spaces. New York: Chapman and Hall/CRC Pure and Applied Mathematics, 1998.

[10] Guérékata, G.M.N', Almost Automorphic and Almost Periodic Functions in Abstract Spaces. Dordrecht: Kluwer Acad. Publ., 2001.

[11] Kostić, M., Abstract Volterra Integro-Differential Equations. Boca Raton, Fl.: CRC Press, 2015.

[12] Kostić, M., Abstract Degenerate Volterra Integro-Differential Equations: Linear Theory and Applications. Book Manuscript, 2016.

[13] Kostić, M., Weyl-almost periodic solutions and asymptotically Weyl-almost periodic solutions of abstract Volterra integro-differential equations. Preprint, `https://www.researchgate.net/publication/318420392`.

[14] Kostić, M., Abstract Volterra Integro-Differential Equations: Almost Periodicity and Asymptotically Almost Periodic Properties of Solutions. Book Manuscript, 2017.

[15] Levitan, M., Zhikov, V.V., Almost Periodic Functions and Differential Equations. London: Cambridge Univ. Press, 1982.

[16] Marcinkiewicz, J., Une remarque sur les espaces de M. Besicovitch. C. R. Acad. Sc. Paris 208 (1939), 57–159.

[17] Pankov, A.A., Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equations. Dordrecht: Kluwer Acad. Publ., 1990.

[18] Prüss, J., Evolutionary Integral Equations and Applications. Monogr. Math., vol. 87, Boston: Birkhäuser, 1993.

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