Novi Sad J. Math.
Vol. 47, No. 2, 2017, 173-185

COUPLED COINCIDENCE POINT RESULTS IN PARTIALLY ORDERED JS-METRIC SPACES

Narawadee Phudolsitthiphat[1 2] **and Atit Wiriyapongsanon**[3]

Abstract. In this paper, we establish a coupled coincidence point theorem for some contraction type mappings in partially ordered JS-metric spaces which generalizes the result of Kadelburg et al. (Fixed Point Theory Appl. 2015:27,2015). We also prove a coupled coincidence point theorem for α-Geraghty contraction type mappings in such spaces. Finally, suitable example is presented to support our main result.

AMS Mathematics Subject Classification (2010): 47H09; 47H10; 54H25

Key words and phrases: coupled coincidence point; α-Geraghty contraction type; partially ordered; JS-metric spaces

1. Introduction

One of the most significant results in the theory of fixed point is Banach contraction principle [2] because it can be utilized in several branches of mathematics. A large number of mathematicians have been generalizing, in many different ways, the Banach contraction principle. One of the interesting results was given by Geraghty [6] in the setting of complete metric spaces by considering an auxiliary function. Let $\mathcal{F}$ be the family of all functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the condition

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0 \qquad \text{for all} \quad t_n \in [0, +\infty).$$

Definition 1.1. [6] Let (X, d) be a metric space. A mapping $f : X \to X$ is called Geraghty contraction if there exits $\beta \in \mathcal{F}$ such that for all $x, y \in X$,

$$d(fx, fy) \leq \beta(d(x, y))d(x, y).$$

The result of Geraghty has been attracting a number of authors [1, 5, 9, 10]. In 2013, Cho, Bae and Karapinar [4] defined the notion of α-Geraghty contraction type mapping as follows:

[1]Center of Excellence in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand,
e-mail: narawadee_n@hotmail.co.th
[2]Corresponding author
[3]Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand. e-mail:r-tit_kp99@hotmail.com

Definition 1.2. [4] Let (X, d) be a metric space and $\alpha : X \times X \to \mathbb{R}$. A mapping $f : X \to X$ is called an α-Geraghty type contraction if there exits $\beta \in \mathcal{F}$ such that for all $x, y \in X$,

$$\alpha(x,y)d(fx,fy) \leq \beta(M(x,y))M(x,y),$$

where $M(x,y) = \max\{d(x,y), d(x,fx), d(y,fy)\}$.

The coupled fixed point was put into use in 1987 by Guo and Lakshmikantham [7]. Later, Bhaskar and Lakshmikantham [3] defined the concept of mixed monotone property and established the existence of a coupled fixed point under the mixed monotone property and applied to a periodic boundary valued problem. In 2008, Radenović [11] extended the results of Bhaskar and Lakshmikantham [3] by using monotone property. In 2015, Kadelburg et al. [9] have studied some coupled coincidence point results for Geraghty-type contraction mappings by using g-monotone property in complete partially ordered metric spaces. Let Θ be the family of all functions $\theta : [0, +\infty) \times [0, +\infty) \to [0, 1)$ satisfying the following conditions:

(θ_1) $\theta(s,t) = \theta(t,s)$ for all $s, t \in [0, +\infty)$,

(θ_2) for any two sequences $\{s_n\}$ and $\{t_n\}$ of nonnegative real numbers,

$$\lim_{n\to\infty} \theta(s_n, t_n) = 1 \text{ implies } \lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = 0.$$

Theorem 1.3. *[9] Let $(X, d, \preceq)$ be a complete partially ordered metric space, $F : X \times X \to X$ and $g : X \to X$. Suppose that the following conditions hold:*

(i) $F(X^2) \subseteq g(X)$,

(ii) *F has the g-monotone property,*

(iii) *there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \preceq F(y_0, x_0)$,*

(iv) *there exists $\theta \in \Theta$ such that*

$$d(F(x,y), F(u,v)) \leq \theta(d(gx,gu), d(gy,gv)) \max\{d(gx,gu), d(gy,gv)\},$$

for all $x, y, u, v \in X$ with $gx \preceq gu$ and $gy \preceq gv$ or $gx \succeq gu$ and $gy \succeq gv$,

(v) *g and F are compatible,*

(vi) *g is continuous and $g(X)$ is closed,*

(vii) *(a) F is continuous or (b) if for an increasing sequence $\{x_n\}$ in $X, x_n \to x \in X$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then, g and F have a coupled coincidence point.

A new class of generalized metric spaces is introduced by Jleli and Samet [8] (for short JS-metric spaces). The class of such metric spaces is larger than the class of standard metric spaces, b-metric spaces and dislocated metric spaces. They proved Banach contraction principle and Ćirić's fixed point theorem in such spaces. We recall the definition of a JS-metric space.

Definition 1.4. [8] A JS-metric on a set X is a mapping $\mathcal{D} : X \times X \to [0, +\infty]$ satisfying the following conditions: there exists $K > 0$ such that for every $x, y \in X$:

($\mathcal{D}_1$) if $\mathcal{D}(x, y) = 0$ then $x = y$,

($\mathcal{D}_2$) $\mathcal{D}(x, y) = \mathcal{D}(y, x)$,

($\mathcal{D}_3$) if $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} \mathcal{D}(x_n, x) = 0$ then

$$\mathcal{D}(x, y) \leq K \limsup_{n\to\infty} \mathcal{D}(x_n, y).$$

Then $(X, \mathcal{D})$ is called a JS-metric space.

In such a space, convergence of sequences is defined in the usual way: a sequence $\{x_n\} \in X$ is said to $\mathcal{D}$-converge to $x \in X$ if $\lim_{n\to\infty} D(x_n, x) = 0$. Also, a sequence is said to be Cauchy (or $\mathcal{D}$-Cauchy) if $\lim_{m,n\to\infty} \mathcal{D}(x_n, x_{n+m}) = 0$. The space $(X, \mathcal{D})$ is said to be $\mathcal{D}$-complete if every $\mathcal{D}$-Cauchy sequence has a limit. It is noted in [8] that the limit of $\mathcal{D}$-convergent sequence is unique.

In this work, we prove some coupled coincidence point theorems for α-Geraghty contraction type mappings in partially ordered JS-metric spaces. Moreover, we give an example to illustrate the main result.

2. Preliminaries

In this section, we recall the useful notations. Let X be a nonempty set, $F : X \times X \to X$ and $g : X \to X$. We say that an element $(x, y) \in X \times X$ is a coupled coincidence point of g and F if

$$gx = F(x, y) \text{ and } gy = F(y, x).$$

We say that g and F are commuting if

$$gF(x, y) = F(gx, gy), \text{ for every } x, y \in X.$$

For a partial order $\preceq$, $E_\preceq = \{(x, y) \in X \times X : x \preceq y\}$ (see [8]). Now, we denote the definition of $\preceq$-g-monotone mapping which is necessary for our main results.

Definition 2.1. Let $(X, \preceq)$ be a partially ordered set, $F : X \times X \to X$ and $g : X \to X$. Then F has the $\preceq$-g-monotone property if and only if for every $x, y \in X$,

$$x_1, x_2 \in X, (gx_1, gx_2) \in E_\preceq \implies (F(x_1, y), F(x_2, y)) \in E_\preceq,$$

and

$$y_1, y_2 \in X, (gy_1, gy_2) \in E_\preceq \implies (F(x, y_1), F(x, y_2)) \in E_\preceq.$$

Definition 2.2. Let $F : X \times X \to X, g : X \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty]$. Then F and g are said to be $\alpha-$admissible if

$$\alpha((gx, gy), (gu, gv)) \geq 1 \text{ implies } \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1$$

for all $x, y, u, v \in X$.

Definition 2.3. Let $F : X \times X \to X, g : X \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty]$. Then F and g are said to be triangular $\alpha-$admissible if F and g are $\alpha-$admissible and

$$\alpha((gx, gy), (gu, gv)) \geq 1 \text{ and } \alpha((gu, gv), (F(u, v), F(v, u))) \geq 1 \text{ imply}$$
$$\alpha((gx, gy), (F(u, v), F(v, u))) \geq 1$$

for all $x, y, u, v \in X$.

3. Main results

Following Kadelburg et al. [9], let Θ' be the family of all functions $\theta : [0, +\infty] \times [0, +\infty] \to [0, 1)$ which satisfy the conditions (θ_1) and (θ_2) except the value of θ may be infinite. Now, we present our first main result as follows.

Theorem 3.1. *Let $(X, \mathcal{D}, \preceq)$ be a complete partially ordered JS-metric space, and let $F : X \times X \to X$, $g : X \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty]$. Suppose that the following conditions hold:*

(i) $F(X^2) \subseteq g(X)$,

(ii) *F is $\preceq$-g-monotone and $\mathcal{D}$-continuous,*

(iii) *g is $\mathcal{D}$-continuous, and commutes with F,*

(iv) *g and F are triangular α-admissible,*

(v) *there exists $\theta \in \Theta'$ such that*

$$\alpha((gx, gy), (gu, gv))\mathcal{D}(F(x, y), F(u, v))$$
$$\leq \theta(\mathcal{D}(gx, gu), \mathcal{D}(gy, gv))M((gx, gu), (gy, gv)),$$

where

$$M((gx, gu), (gy, gv)) = \max\{\mathcal{D}(gx, gu), \mathcal{D}(gy, gv), \mathcal{D}(gx, F(x, y)),$$
$$\mathcal{D}(gy, F(y, x)), \mathcal{D}(gu, F(u, v)), \mathcal{D}(gv, F(v, u))\},$$

for all $x, y, u, v \in X$ with $(gx, gu) \in E_{\preceq}$ and $(gy, gv) \in E_{\preceq}$,

(vi) *there exist $x_0, y_0 \in X$ such that*

$$(gx_0, F(x_0, y_0)), (gy_0, F(y_0, x_0)) \in E_{\preceq}, \text{ and}$$
$$\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1 \text{ and}$$
$$\alpha((gy_0, gx_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1,$$

(*vii*) *if* $\{x_n\}, \{y_n\}$ *are sequences such that*

$$\lim_{n\to\infty} \mathcal{D}(gx_n, gx_{n+1}) = 0 \text{ and } \lim_{n\to\infty} \mathcal{D}(gy_n, gy_{n+1}) = 0,$$

then $\sup\{\mathcal{D}(gx_0, gx_n), \mathcal{D}(gy_0, gy_n) : n \in \mathbb{N}\} < \infty$.

Then, g and F have a coupled coincidence point.

Proof. Let x_0, y_0 be elements in X satisfying assumption (vi). Since $F(X^2) \subseteq g(X)$, we can pick $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Since $F(X^2) \subseteq g(X)$ again, we can pick $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continue this procedure to obtain sequences $\{x_n\}, \{y_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n) \text{ for each } n \in \mathbb{N}.$$

If $gx_{n_0+1} = gx_{n_0}$ and $gy_{n_0+1} = gy_{n_0}$ for some $n_0 \in \mathbb{N}$, then (x_{n_0}, y_{n_0}) is a coupled coincidence point of g and F. Therefore, in what follows, we will assume that for each $n \in \mathbb{N}$,

$$gx_{n+1} \neq gx_n \text{ or } gy_{n+1} \neq gy_n.$$

By condition (vi),

$$(gx_0, gx_1) \in E_{\preceq} \text{ and } (gy_0, gy_1) \in E_{\preceq}.$$

Since F is $\preceq$-g-monotone,

$$(F(x_0, y_0), F(x_1, y_1)) \in E_{\preceq} \text{ and } (F(y_0, x_0), F(y_1, x_1)) \in E_{\preceq}.$$

That is,

$$(gx_1, gx_2) \in E_{\preceq} \text{ and } (gy_1, gy_2) \in E_{\preceq}.$$

Continuing this method, we get that

$$(gx_n, gx_{n+1}) \in E_{\preceq} \text{ and } (gy_n, gy_{n+1}) \in E_{\preceq} \text{ hold for all } n \in \mathbb{N}.$$

By transitivity of $\preceq$,

$$(gx_n, gx_{n+m}) \in E_{\preceq} \text{ and } (gy_n, gy_{n+m}) \in E_{\preceq} \text{ for all } n, m \in \mathbb{N}.$$

By assumption (vi),

$$\alpha((gx_0, gy_0), (gx_1, gy_1)) = \alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1.$$

Since F and g are α-admissible, we obtain

$$\alpha((gx_1, gy_1), (gx_2, gy_2)) = \alpha((F(x_0, y_0), F(y_0, x_0)), (F(x_1, y_1), F(y_1, x_1))) \geq 1.$$

Thus, by mathematical induction, we have

$$\alpha((gx_n, gy_n), (gx_{n+1}, gy_{n+1})) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Similarily,

$$\alpha((gy_n, gx_n), (gy_{n+1}, gx_{n+1})) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Since F and g are triangular α-admissible,

$$\alpha((gx_n, gy_n), (gx_{n+m}, gy_{n+m})) \geq 1 \text{ and}$$
$$\alpha((gy_n, gx_n), (gy_{n+m}, gx_{n+m})) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Now, we will show that

$$\lim_{n\to\infty} \mathcal{D}(gx_n, gx_{n+1}) = 0 \text{ and } \lim_{n\to\infty} \mathcal{D}(gy_n, gy_{n+1}) = 0.$$

By way of contradiction, suppose at least one of $\lim_{n\to\infty} \mathcal{D}(gx_n, gx_{n+1}) \neq 0$ or $\lim_{n\to\infty} \mathcal{D}(gy_n, gy_{n+1}) \neq 0$ holds. Then there exists $\varepsilon > 0$ for which we can obtain subsequence $\{n_k\}$ such that $n_k \geq k$ and

$$\epsilon \leq \max\{\mathcal{D}(gx_{n_k}, gx_{n_k+1}), \mathcal{D}(gy_{n_k}, gy_{n_k+1})\}.$$

Consider

$$\begin{aligned}
&\mathcal{D}(gx_{n_k}, gx_{n_k+1})\\
&= \mathcal{D}(F(x_{n_k-1}, y_{n_k-1}), F(x_{n_k}, y_{n_k}))\\
&\leq \alpha((gx_{n_k-1}, gy_{n_k-1}), (gx_{n_k}, gy_{n_k}))\mathcal{D}(F(x_{n_k-1}, y_{n_k-1}), F(x_{n_k}, y_{n_k}))\\
&\leq \theta(\mathcal{D}(gx_{n_k-1}, gx_{n_k}), \mathcal{D}(gy_{n_k-1}, gy_{n_k}))M((gx_{n_k-1}, gx_{n_k}), (gy_{n_k-1}, gy_{n_k}))
\end{aligned}$$
(3.1)

and

$$\begin{aligned}
&\mathcal{D}(gy_{n_k}, gy_{n_k+1})\\
&= \mathcal{D}(F(y_{n_k-1}, x_{n_k-1}), F(y_{n_k}, x_{n_k}))\\
&\leq \alpha((gy_{n_k-1}, gx_{n_k-1}), (gy_{n_k}, gx_{n_k}))\mathcal{D}(F(y_{n_k-1}, x_{n_k-1}), F(y_{n_k}, x_{n_k}))\\
&\leq \theta(\mathcal{D}(gy_{n_k-1}, gy_{n_k}), \mathcal{D}(gx_{n_k-1}, gx_{n_k}))M((gy_{n_k-1}, gy_{n_k}), \mathcal{D}(gx_{n_k-1}, gx_{n_k})).
\end{aligned}$$
(3.2)

Since $\theta(s,t) \in [0,1)$ for all $s,t \in [0,+\infty]$,

$$\begin{aligned}
M((gx_{n_k-1}, gx_{n_k}), (gy_{n_k-1}, gy_{n_k})) &= M((gy_{n_k-1}, gy_{n_k}), \mathcal{D}(gx_{n_k-1}, gx_{n_k}))\\
&= \max\{\mathcal{D}(gx_{n_k-1}, gx_{n_k}), \mathcal{D}(gy_{n_k-1}, gy_{n_k})\}.
\end{aligned}$$
(3.3)

From (3.1), (3.2) and (3.3),

$$\begin{aligned}
&\max\{\mathcal{D}(gx_{n_k}, gx_{n_k+1}), \mathcal{D}(gy_{n_k}, gy_{n_k+1})\}\\
&\leq \theta(\mathcal{D}(gx_{n_k-1}, gx_{n_k}), \mathcal{D}(gy_{n_k-1}, gy_{n_k}))\max\{\mathcal{D}(gx_{n_k-1}, gx_{n_k}), \mathcal{D}(gy_{n_k-1}, gy_{n_k})\}.
\end{aligned}$$

Continuing this process, we get that

$$\max\{\mathcal{D}(gx_{n_k}, gx_{n_k+1}), \mathcal{D}(gy_{n_k}, gy_{n_k+1})\}$$
$$\leq \prod_{i=1}^{n_k} \theta(\mathcal{D}(gx_{n_k-i}, gx_{n_k+1-i}), \mathcal{D}(gy_{n_k-i}, gy_{n_k+1-i})) \max\{\mathcal{D}(gx_0, gx_1), \mathcal{D}(gy_0, gy_1)\}.$$

We choose i_k such that

$$\theta(\mathcal{D}(gx_{n_k-i_k}, gx_{n_k+1-i_k}), \mathcal{D}(gy_{n_k-i_k}, gy_{n_k+1-i_k}))$$
$$= \max_{1\leq i\leq n_k} \{\theta(\mathcal{D}(gx_{n_k-i}, gx_{n_k+1-i}), \mathcal{D}(gy_{n_k-i}, gy_{n_k+1-i}))\}.$$

Define $\eta := \limsup_{k\to\infty}\{\theta(\mathcal{D}(gx_{n_k-i_k}, gx_{n_k+1-i_k}), \mathcal{D}(gy_{n_k-i_k}, gy_{n_k+1-i_k}))\}$.
If $\eta < 1$, then

$$\lim_{k\to\infty} \max\{\mathcal{D}(gx_{n_k}, gx_{n_k+1}), \mathcal{D}(gy_{n_k}, gy_{n_k+1})\} = 0,$$

which contradicts the assumption.
If $\eta = 1$, by passing to a subsequence, then we may assume that

$$\lim_{k\to\infty} \theta(\mathcal{D}(gx_{n_k-i_k}, gx_{n_k+1-i_k}), \mathcal{D}(gy_{n_k-i_k}, gy_{n_k+1-i_k})) = 1.$$

Since $\theta \in \Theta'$, we have

$$\lim_{k\to\infty} \mathcal{D}(gx_{n_k-i_k}, gx_{n_k+1-i_k}) = 0 \text{ and } \lim_{k\to\infty} \mathcal{D}(gy_{n_k-i_k}, gy_{n_k+1-i_k}) = 0.$$

That is, there exists $k_0 \in \mathbb{N}$ such that

$$\mathcal{D}(gx_{n_{k_0}-i_{k_0}}, gx_{n_{k_0}+1-i_{k_0}}) < \frac{\epsilon}{2} \text{ and } \mathcal{D}(gy_{n_{k_0}-i_{k_0}}, gy_{n_{k_0}+1} - i_{k_0}) < \frac{\epsilon}{2}.$$

Thus, we have

$$\begin{aligned}\epsilon &\leq \max\{\mathcal{D}(gx_{n_{k_0}}, gx_{n_{k_0}+1}), \mathcal{D}(gy_{n_{k_0}}, gy_{n_{k_0}+1})\}\\ &\leq \prod_{j=1}^{i_{k_0}} \theta(\mathcal{D}(gx_{n_{k_0}-j}, gx_{n_{k_0}+1-j}), \mathcal{D}(gy_{n_{k_0}-j}, gy_{n_{k_0}+1-j}))\\ &\quad \max\{\mathcal{D}(gx_{n_{k_0}-i_{k_0}}, gx_{n_{k_0}+1-i_{k_0}}), \mathcal{D}(gy_{n_{k_0}-i_{k_0}}, gy_{n_{k_0}+1-i_{k_0}})\}\\ &< \frac{\epsilon}{2},\end{aligned}$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty} \mathcal{D}(gx_n, gx_{n+1}) = 0 \text{ and } \lim_{n\to\infty} \mathcal{D}(gy_n, gy_{n+1}) = 0. \tag{3.4}$$

Next, we will show that $\{gx_n\}$ and $\{gy_n\}$ are $\mathcal{D}$-Cauchy sequences. By contradiction, suppose that at least one of $\{gx_n\}$ or $\{gy_n\}$ is not a $\mathcal{D}$-Cauchy sequences. Then there exists $\epsilon' > 0$ for which we can obtain subsequences $\{n_k\}$,

$\{m_k\}$ such that $n_k, m_k \geq k$ and $\epsilon' \leq \max\{\mathcal{D}g(x_{n_k}, gx_{n_k+m_k}), \mathcal{D}(gy_{n_k}, gy_{n_k+m_k})\}$. Consider

$$\begin{aligned}
&\mathcal{D}(gx_{n_k}, gx_{n_k+m_k}) \\
&= \mathcal{D}(F(x_{n_k-1}, y_{n_k-1}), F(x_{n_k+m_k-1}, y_{n_k+m_k-1})) \\
&\leq \alpha((gx_{n_k-1}, gy_{n_k-1}), (gx_{n_k+m_k-1}, gy_{n_k+m_k-1})) \\
&\quad \mathcal{D}(F(x_{n_k-1}, y_{n_k-1}), F(x_{n_k-1}, y_{n_k-1})) \\
&\leq \theta(\mathcal{D}(gx_{n_k-1}, gx_{n_k+m_k-1})), \mathcal{D}(gy_{n_k-1}, gy_{n_k+m_k-1})) \\
&\quad M((gx_{n_k-1}, gx_{n_k+m_k-1}), (gy_{n_k-1}, gy_{n_k+m_k-1}))
\end{aligned}$$

(3.5)

and

$$\begin{aligned}
&\mathcal{D}(gy_{n_k}, gy_{n_k+m_k}) \\
&= \mathcal{D}(F(y_{n_k-1}, x_{n_k-1}), F(y_{n_k+m_k-1}, x_{n_k+m_k-1})) \\
&\leq \alpha((gy_{n_k-1}, gx_{n_k-1}), (gy_{n_k+m_k-1}, gx_{n_k+m_k-1})) \\
&\quad \mathcal{D}(F(y_{n_k-1}, x_{n_k-1}), F(y_{n_k+m_k-1}, x_{n_k+m_k-1})) \\
&\leq \theta(\mathcal{D}(gx_{n_k-1}, gx_{n_k+m_k-1}), \mathcal{D}(gy_{n_k-1}, gy_{n_k+m_k-1})) \\
&\quad M((gx_{n_k-1}, gx_{n_k+m_k-1}), (gy_{n_k-1}, gy_{n_k+m_k-1})).
\end{aligned}$$

(3.6)

From (3.4),

$$\begin{aligned}
&M((gx_{n_k-1}, gx_{n_k+m_k-1}), (gy_{n_k-1}, gy_{n_k+m_k-1})) \\
&= M((gy_{n_k-1}, gy_{n_k+m_k-1}), (gx_{n_k-1}, gx_{n_k+m_k-1})) \\
&= \max\{\mathcal{D}(gx_{n_k-1}, gx_{n_k+m_k-1}), \mathcal{D}(gy_{n_k-1}, gy_{n_k+m_k-1})\}.
\end{aligned}$$

(3.7)

From (3.5), (3.6) and (3.7),

$$\begin{aligned}
&\max\{\mathcal{D}(gx_{n_k}, gx_{n_k+m_k}), \mathcal{D}(gy_{n_k}, gy_{n_k+m_k})\} \\
&\leq \theta(\mathcal{D}(gx_{n_k-1}, gx_{n_k+m_k-1}), \mathcal{D}(gy_{n_k-1}, gy_{n_k+m_k-1})) \\
&\quad \max\{\mathcal{D}(gx_{n_k-1}, gx_{n_k+m_k-1}), \mathcal{D}(gy_{n_k-1}, gy_{n_k+m_k-1})\}.
\end{aligned}$$

Continuing this process, we get that

$$\begin{aligned}
&\max\{\mathcal{D}(gx_{n_k}, gx_{n_k+m_k}), \mathcal{D}(gy_{n_k}, gy_{n_k+m_k})\} \\
&\qquad \leq \prod_{i=1}^{n_k} \theta(\mathcal{D}(gx_{n_k-i}, gx_{n_k+m_k-i}), \mathcal{D}(gy_{n_k-i}, gy_{n_k+m_k-i})) \\
&\qquad\quad \max\{\mathcal{D}(gx_0, gx_{m_k}), \mathcal{D}(gy_0, gy_{m_k})\}.
\end{aligned}$$

We choose i_k such that

$$\begin{aligned}
&\theta(\mathcal{D}(gx_{n_k-i_k}, gx_{n_k+m_k-i_k}), \mathcal{D}(gy_{n_k-i_k}, gy_{n_k+m_k-i_k})) \\
&\qquad = \max_{1\leq i\leq n_k} \{\theta(\mathcal{D}(gx_{n_k-i}, gx_{n_k+m_k-i}), \mathcal{D}(gy_{n_k-i}, gy_{n_k+m_k-i}))\}.
\end{aligned}$$

Define $\eta := \limsup_{k\to\infty}\{\theta(\mathcal{D}(gx_{n_k-i_k}, gx_{n_k+m_k-i_k}), \mathcal{D}(gy_{n_k-i_k}, gy_{n_k+m_k-i_k}))\}$.
If $\eta < 1$, then $\lim_{k\to\infty} \max\{\mathcal{D}(gx_{n_k}, gx_{n_k+m_k}), \mathcal{D}(gy_{n_k}, gy_{n_k+m_k})\} = 0$, which contradicts the assumption.
If $\eta = 1$, by passing through a subsequence, then we may assume that

$$\lim_{k\to\infty} \theta(\mathcal{D}(gx_{n_k-i_k}, gx_{n_k+m_k-i_k}), \mathcal{D}(gy_{n_k-i_k}, gy_{n_k+m_k-i_k})) = 1.$$

Since $\theta \in \Theta'$ and (3.4), we have

$$\lim_{k\to\infty} \mathcal{D}(gx_{n_k-i_k}, gx_{n_k+m_k-i_k}) = 0 \text{ and } \lim_{k\to\infty} \mathcal{D}(gy_{n_k-i_k}, gy_{n_k+m_k-i_k}) = 0.$$

Then there exists $k_0 \in \mathbb{N}$ such that

$$\mathcal{D}(gx_{n_{k_0}-i_{k_0}}, gx_{n_{k_0}+m_{k_0}-i_{k_0}}) < \frac{\epsilon'}{2} \text{ and } \mathcal{D}(gy_{n_{k_0}-i_{k_0}}, gy_{n_{k_0}+m_{k_0}-i_{k_0}}) < \frac{\epsilon'}{2}.$$

Thus, we have

$$\begin{aligned}\epsilon' &\leq \max\{\mathcal{D}(gx_{n_{k_0}}, gx_{n_{k_0}+m_{k_0}}), \mathcal{D}(gy_{n_{k_0}}, gy_{n_{k_0}+m_{k_0}})\}\\ &\leq \prod_{j=1}^{i_{k_0}} \theta(\mathcal{D}(gx_{n_{k_0}-j}, gx_{n_{k_0}+m_{k_0}-j}), \mathcal{D}(gy_{n_{k_0}-j}, gy_{n_{k_0}+m_{k_0}-j}))\\ &\quad \max\{\mathcal{D}(gx_{n_{k_0}-i_{k_0}}, gx_{n_{k_0}+m_{k_0}-i_{k_0}}), \mathcal{D}g(y_{n_{k_0}-i_{k_0}}, gy_{n_{k_0}+m_{k_0}-i_{k_0}})\}\\ &< \frac{\epsilon'}{2},\end{aligned}$$

which is a contradiction. Therefore, $\{gx_n\}$ and $\{gy_n\}$ are $\mathcal{D}$-Cauchy sequences. By completeness of $(X, \mathcal{D})$, there exist some $\omega, \omega' \in X$ such that

$$\lim_{n\to\infty} \mathcal{D}(F(x_n, y_n), \omega) = \lim_{n\to\infty} \mathcal{D}(gx_n, \omega) = 0 \qquad \text{and}$$

$$\lim_{n\to\infty} \mathcal{D}(F(y_n, x_n), \omega') = \lim_{n\to\infty} \mathcal{D}(gy_n, \omega') = 0.$$

By the continuity of g,

$$\lim_{n\to\infty} \mathcal{D}(g(F(x_n, y_n)), g\omega) = 0 \text{ and } \lim_{n\to\infty} \mathcal{D}(g(F(y_n, x_n)), g\omega') = 0.$$

By the continuity of F,

$$\lim_{n\to\infty} \mathcal{D}(F(gx_n, gy_n), F(\omega, \omega')) = 0 \text{ and } \lim_{n\to\infty} \mathcal{D}(F(gy_n, gx_n), F(\omega', \omega)) = 0.$$

Since g and F commute and by the uniqueness of the limit, $g\omega = F(\omega, \omega')$ and $g(\omega') = F(\omega', \omega)$. Therefore, $(\omega, \omega') \in X \times X$ is a coupled coincidence point of g and F. □

In our second main result, we obtain a coupled coincidence result for α-Geraghty contraction type in JS-metric spaces. Let $\mathcal{F}'$ be the family of all functions $\beta : [0, +\infty] \to [0, 1)$ satisfying the condition

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0 \qquad \text{for all} \quad t_n \in [0, +\infty].$$

Theorem 3.2. *Let $(X, \mathcal{D}, \preceq)$ be a complete partially ordered JS-metric space, and let $F : X \times X \to X$, $g : X \to X$ and $\alpha : X^2 \times X^2 \to [0, +\infty]$. Suppose that the following conditions hold:*

(i) $F(X^2) \subseteq g(X)$,

(ii) F *is* $\preceq$*-g-monotone,*

(iii) g *is* $\mathcal{D}$*-continuous, and commutes with* F,

(iv) g *and* F *are triangular* α*-admissible,*

(v) *there exists* $\beta \in \mathcal{F}'$ *such that for all* $x, y, u, v \in X$ *satisfying* $(gx, gu) \in E_{\preceq}$ *and* $(gy, gv) \in E_{\preceq}$,

$$\begin{aligned}&\alpha((gx, gy), (gu, gv))\mathcal{D}(F(x, y), F(u, v))\\&\quad \leq \beta(M((gx, gu), (gy, gv)))M((gx, gu), (gy, gv)),\end{aligned}$$

where

$$\begin{aligned}M((gx, gu), (gy, gv)) = \max\{&\mathcal{D}(gx, gu), \mathcal{D}(gy, gv), \mathcal{D}(gx, F(x, y)),\\&\mathcal{D}(gy, F(y, x)), \mathcal{D}(gu, F(u, v)), \mathcal{D}(gv, F(v, u))\},\end{aligned}$$

(vi) *there exist* $x_0, y_0 \in X$ *such that*

$$\begin{aligned}&(gx_0, F(x_0, y_0)), (gy_0, F(y_0, x_0)) \in E_{\preceq},\\&\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1 \text{ and}\\&\alpha((gy_0, gx_0), (F(y_0, x_0), F(x_0, y_0))) \geq 1,\end{aligned}$$

(vii) *if* $\{x_n\}, \{y_n\}$ *are sequences such that*

$$\lim_{n\to\infty} \mathcal{D}(gx_n, gx_{n+1}) = 0 \text{ and } \lim_{n\to\infty} \mathcal{D}(gy_n, gy_{n+1}) = 0,$$

then $\sup\{\mathcal{D}(gx_0, gx_n), \mathcal{D}(gy_0, gy_n) : n \in \mathbb{N}\} < \infty$,

(viii) *(a)* F *is* $\mathcal{D}$*-continuous or (b) for* $\{x_n\}$ *and* $\{y_n\}$ *are sequences in* X *such that*

$$\begin{aligned}&(gx_n, gx_{n+1}), (gy_n, gy_{n+1}) \in E_{\preceq},\\&\alpha((gx_n, gy_n), (gx_{n+1}, gy_{n+1})) \geq 1,\\&\alpha((gy_n, gx_n), (gy_{n+1}, gx_{n+1})) \geq 1 \text{ for all } \mathbb{N}\end{aligned}$$

and

$$\lim_{n\to\infty} \mathcal{D}(gx_n, \omega) = 0 \text{ and } \lim_{n\to\infty} \mathcal{D}(gy_n, \omega') = 0,$$

we have

$$\begin{aligned}&(gx_n, g\omega), (gy_n, g\omega') \in E_{\preceq},\\&\alpha((gx_n, gy_n), (g\omega, g\omega')) \geq 1,\\&\alpha((gy_n, gx_n), (g\omega', g\omega)) \geq 1 \text{ for all } \mathbb{N}.\end{aligned}$$

Then g and F have a coupled coincidence point.

Proof. Using similar idea as in the proof of Theorem 3.1, we can construct Cauchy sequences $\{gx_n\}$ and $\{gy_n\}$ in complete JS-metric space $(X, \mathcal{D})$. Then there exist $\omega, \omega' \in X$ such that

$$\lim_{n\to\infty} \mathcal{D}(F(x_n, y_n), \omega) = \lim_{n\to\infty} \mathcal{D}(gx_n, \omega) = 0 \qquad \text{and}$$
$$\lim_{n\to\infty} \mathcal{D}(F(y_n, x_n), \omega') = \lim_{n\to\infty} \mathcal{D}(gy_n, \omega') = 0.$$

By the continuity of g,

$$\lim_{n\to\infty} \mathcal{D}(gF(x_n, y_n), g\omega) = \lim_{n\to\infty} \mathcal{D}(ggx_n, g\omega) = 0 \text{ and}$$
$$\lim_{n\to\infty} \mathcal{D}(gF(y_n, x_n), g\omega') = \lim_{n\to\infty} \mathcal{D}(ggy_n, g\omega') = 0.$$

If F is $\mathcal{D}$-continuous, it is easy to show that g and F have a coupled coincidence point. Otherwise, By assumption $(v), (vii)$, we have

$$\begin{aligned} &\mathcal{D}(F(gx_n, gy_n), F(\omega, \omega')) \\ &\leq \alpha((ggx_n, ggy_n), (g\omega, g\omega'))\mathcal{D}(F(gx_n, gy_n), F(\omega, \omega')) \\ &\leq \beta(M((ggx_n, g\omega), (ggy_n, g\omega')))M((ggx_n, g\omega), (ggy_n, g\omega')) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\mathcal{D}(F(gy_n, gx_n), F(\omega', \omega)) \\ &\leq \alpha((ggy_n, ggx_n), (g\omega', g\omega))\mathcal{D}(F(gy_n, gx_n), F(\omega', \omega)) \\ &\leq \beta(M((ggy_n, g\omega'), (ggx_n, g\omega)))M((ggy_n, g\omega'), (ggx_n, g\omega)), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} &M((ggx_n, g\omega), (ggy_n, g\omega')) \\ &= M((ggy_n, g\omega'), (ggx_n, g\omega)) \\ &= \max\{\mathcal{D}(ggx_n, g\omega), \mathcal{D}(ggy_n, g\omega'), \mathcal{D}(ggx_n, F(gx_n, gy_n)), \\ &\quad \mathcal{D}(ggy_n, F(gy_n, gx_n)), \mathcal{D}(g\omega, F(\omega, \omega'), \mathcal{D}(g\omega', F(\omega', \omega)\}. \end{aligned} \tag{3.10}$$

Suppose $g\omega \neq F(\omega, \omega')$ or $g\omega' \neq F(\omega', \omega)$, that is,

$$D := \max\{\mathcal{D}(g\omega, F(\omega, \omega')), \mathcal{D}(g\omega', F(\omega', \omega))\} > 0.$$

Letting $n \to \infty$ in (3.10), we have

$$\lim_{n\to\infty} M((ggx_n, g\omega), (ggy_n, g\omega')) = D. \tag{3.11}$$

From (3.8), (3.9),

$$\frac{\max\{\mathcal{D}(F(gx_n, gy_n), F(\omega, \omega')), \mathcal{D}(F(gy_n, gx_n), F(\omega', \omega))\}}{M((ggx_n, g\omega), (ggy_n, g\omega'))}$$
$$\leq \beta(M((ggx_n, g\omega), (ggy_n, g\omega'))).$$

Taking limit on both sides of above inequalities, we have

$$\lim_{n\to\infty} \beta(M((ggx_n, g\omega), (ggy_n, g\omega'))) = 1.$$

This implies $\lim_{n\to\infty} M((ggx_n, g\omega), (ggy_n, g\omega')) = 0$ which contradicts equation (3.11). Therefore, $g\omega = F(\omega, \omega')$ and $g(\omega') = F(\omega', \omega)$, that is, $(\omega, \omega') \in X \times X$ is a coupled coincidence point of g and F. □

Example 3.3. Let $X = [0, +\infty], D(x, y) = \max\{x, y\}$ for all $x, y \in X$. Define mappings $F : X \times X \to X$ and $g : X \to X$ by

$$F(x,y) = \begin{cases} \dfrac{x+y}{6}, & \text{if } x, y \in [0, +\infty), \\ +\infty, & \text{otherwise,} \end{cases} \qquad gx = \begin{cases} 2x, & \text{if } x \in [0, +\infty), \\ +\infty, & \text{otherwise.} \end{cases}$$

A mapping $\alpha : X^2 \times X^2 \to [0, +\infty]$ is given by

$$\alpha((x, y), (u, v)) = \begin{cases} 1, & \text{if } x \leq y \text{ and } u \leq v, \\ 0, & \text{otherwise.} \end{cases}$$

Let $x \leq u$ and $y \leq v$. If $x > y$ or $u > v$, then it is obvious that the assumption (v) of Theorem 3.1 holds. Otherwise,

$$\alpha((gx, gy), (gu, gv))D(F(x, y), F(u, v)) = \max\{\frac{x+y}{6}, \frac{u+v}{6}\} = \frac{u+v}{6} \leq \frac{2v}{6}$$
$$= \frac{1}{6} M((gx, gu), (gy, gv)).$$

Thus, the assumption (v) of Theorem 3.1 holds for $\theta(s, t) = \frac{1}{6}$ for all $s, t \in [0, +\infty]$. We can easily check that all conditions of Theorem 3.1 hold. Therefore, g and F have coupled coincidence point which is $(0, 0)$. However, we cannot apply Theorem 1.3 to show the existence of a coupled coincidence point for g and F.

4. Backmatter

Conflict of Interest

No conflict of interest was declared by the authors.

Authors' contributions

Both authors have contributed equally to this paper. Both authors read and approved the final manuscript.

Acknowledgements

The authors are grateful to thank the editors and referees for their valuable comments. The first author was supported by Chiang Mai University.

References

[1] Altun, I., Sadarangani, K., Generalized Geraghty type mappings on partial metric spaces and fixed point results. Arab. J. Math. 2013 (2013), 247-253.

[2] Banach, S., Sur les opérationes dans les ensembles abstraits et leur application aux équation intégrales. Fundam. Math. 3 (1922), 133-18.

[3] Bhaskar, T.G., Lakshmikantham, V., Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. 65 (2006), 1379-1393.

[4] Cho, S.H, Bae, J.S., Karapinar, E., Fixed point theorems for α-Geraghty contraction type maps in metric spaces. Fixed Point Theory Appl. 2013 (2013), Article ID 329.

[5] Dukić, D., Kadelburg, Z., Radenović, S., Fixed points of Geraghty-type mappings in various generalized metric spaces. Abstract and Applied Analysis. 2011 (2011), Article ID 561245.

[6] Geraghty, M.A., On contractive mappings, Proc. Amer. Math. Soc. 40 (1973), 604-608.

[7] Guo,D., Lakshmikantham, V., Coupled fixed points of nonlinear operators with applications. Nonlinear Anal. 11 (1987), 623-632.

[8] Jleli, M., Samet, B., A generalized metric space and related fixed point theorems. Fixed Point Theory Appl. 2015 (2015), 61 page.

[9] Kadelburg, Z., Kumam, P., Radenović, S., Sintunavarat, W., Common coupled fixed point theorems for Geraghty-type contraction mappings using monotone property. Fixed Point Theory Appl. 2015 (2015), 27 pages.

[10] Karapinar, E., A discussion on "$\alpha - \psi$-Geraghty contraction type mappings". Filomat. 28 (2014), 761-766.

[11] Radenović, S., Coupled fixed point theorems for monotone mappings in partially ordered metric spaces. Kragujevac J. Math. 38 (2014), 249-257.

Received by the editors June 27, 2017
First published online July 21, 2017