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# EXISTENCE AND STABILITY RESULTS FOR PARTIAL IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS WITH NOT INSTANTANEOUS IMPULSES

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**Abstract.** In this paper, we investigate some existence, uniqueness and stability results for a class of partial differential equations with not instantaneous impulses in Banach spaces. We give an Ulam type stability result and present an illustrative example.

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## 1. Introduction

The fractional calculus represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas et al. [2, 3], Kilbas *et al.* [10], Miller and Ross [11], Zhou [19], the papers of Abbas et al. [5], Diethelm [6], Kilbas and Marzan [8], Podlubny [13], and Vityuk and Golushkov [14], and the references therein.

Implicit differential equations involving the regularized fractional derivative was analyzed by many authors, in the last years; see for instance [5, 15] and the references therein. In [16], Wang et al. introduced some new concepts about Ulam stability of impulsive fractional differential equations. Recently, in [4],

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Abbas et al. studied the existence, the uniqueness and the Ulam stability of solutions for the Darboux problem of partial impulsive differential equations.

In pharmacotherapy, the above instantaneous impulses can not describe the certain dynamics of evolution processes. For example, one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. In [7, 12] the authors initially offered to study some new classes of abstract semilinear impulsive differential equations with not instantaneous impulses.

Motivated by recent works [9, 17], we investigate the uniqueness and Ulam-Hyers-Rassias stability of the following partial fractional implicit differential equations with not instantaneous impulses

(1.1)

$$\begin{cases} \overline{D}^r_{\theta_k} u(t,x) = f(t,x,u(t,x),\overline{D}^r_{\theta_k} u(t,x)); \text{ if } (t,x) \in I_k,\ k=0,\dots,m, \\ u(t,x) = g_k(t,x,u(t,x)); \text{ if } (t,x) \in J_k,\ k=1,\dots,m, \\ u(t,0) = \varphi(t);\ t \in [0,a], \\ u(0,x) = \psi(x);\ x \in [0,b], \\ \varphi(0) = \psi(0), \end{cases}$$

where $I_k := (s_k, t_{k+1}] \times [0,b],\ J_k := (t_k, s_k] \times [0,b],\ a,b > 0,\ \theta_k = (s_k, 0);\ k = 0,\dots,m,\ \overline{D}^r_{\theta_k}$ is the mixed regularized derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1],\ 0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < s_{m-1} \leq t_m \leq s_m \leq t_{m+1} = a,\ f : I_k \times E \to E;\ k = 0,\dots,m$ is a given continuous function, $g_k : J_k \times E \to E;\ k = 1,\dots,m$ are given continuous functions, $\varphi : [0,a] \to E$ and $\psi : [0,b] \to E$ are given absolutely continuous functions and $E$ is a Banach space.

The paper is organized as follows: In Section 2 we present some fundamental results of fractional calculus, stability theory and a version of Gronwall's lemma for partial differential equations. In Section 3 we present our main results. Finally, an example is included to illustrate one of the main results.

## 2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $J = [0,a] \times [0,b];\ a,b > 0$, denote by $L^1(J)$ the space of Bochner-integrable functions $u : J \to E$ with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(t,x)\|_E dxdt,$$

where $\|\cdot\|_E$ denotes a suitable complete norm on $E$.

As usual, by $AC(J)$ we denote the space of absolutely continuous functions from $J$ into $E$, and $\mathcal{C} := C(J)$ is the Banach space of all continuous functions from $J$ into $E$ with the norm $\|\cdot\|_\infty$ defined by

$$\|u\|_\infty = \sup_{(t,x)\in J} \|u(t,x)\|_E.$$

Consider the Banach space

$$\begin{aligned}PC = \big\{ & u : J \to E : u \in C((t_k, t_{k+1}] \times [0,b]);\ k = 0,1,\dots,m, \text{ and there}\\ & \text{exist } u(t_k^-, x) \text{ and } u(t_k^+, x);\ k = 1,\dots,m,\\ & \text{with } u(t_k^-, x) = u(t_k, x) \text{ for each } x \in [0,b]\big\},\end{aligned}$$

with the norm

$$\|u\|_{PC} = \sup_{(t,x)\in J} \|u(t,x)\|_E.$$

Let $\theta = (0,0)$, $r_1$, $r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J)$, the expression

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-\tau)^{r_1-1}(x-\xi)^{r_2-1} u(\tau,\xi)\, d\xi d\tau,$$

is called the left-sided mixed Riemann-Liouville integral of order $r$, where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\varsigma) = \int_0^\infty t^{\varsigma-1} e^{-t} dt;\ \varsigma > 0$.

In particular,

$$(I_\theta^\theta u)(t,x) = u(t,x),\ (I_\theta^\sigma u)(t,x) = \int_0^t \int_0^x f(\tau,\xi)\, d\xi d\tau;\ \text{for almost all } (t,x) \in J,$$

where $\sigma = (1,1)$.

For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I_\theta^r u) \in C(J)$, moreover

$$(I_\theta^r u)(t,0) = (I_\theta^r u)(0,x) = 0;\ t \in [0,a],\ x \in [0,b].$$

**Example 2.1.** Let $\lambda, \omega \in (-1,0) \cup (0,\infty)$, $r = (r_1, r_2)$, $r_1, r_2 \in (0,\infty)$ and $h(t,x) = t^\lambda x^\omega$; $(t,x) \in J$. We have $h \in L^1(J)$, and we get

$$(I_\theta^r h)(t,x) = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2},\ \text{for almost all } (t,x) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D_{tx}^2 := \frac{\partial^2}{\partial t \partial x}$, the mixed second order partial derivative.

**Definition 2.2.** [14] Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J)$. The mixed fractional Riemann-Liouville derivative of order $r$ of $u$ is defined by the expression

$$\begin{aligned} D_\theta^r u(t,x) &= (D_{tx}^2 I_\theta^{1-r} u)(t,x)\\ &= \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)} D_{tx}^2 \int_0^t \int_0^x \frac{u(\tau,\xi)}{(t-\tau)^{r_1}(x-\xi)^{r_2}} d\xi d\tau.\end{aligned}$$

and the Caputo fractional-order derivative of order $r$ of $u$ is defined by the expression

$$\begin{aligned} {}^cD_\theta^r u(t,x) &= (I_\theta^{1-r} D_{tx}^2 u)(t,x)\\ &= \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)} \int_0^t \int_0^x \frac{D_{\tau\xi}^2 u(\tau,\xi)}{(t-\tau)^{r_1}(x-\xi)^{r_2}} d\xi d\tau.\end{aligned}$$

The case $\sigma = (1,1)$ is included and we have

$$D^{\sigma}_{\theta}u)(t,x) = (^cD^{\sigma}_{\theta}u)(t,x) = (D^2_{tx}u)(t,x); \text{ for almost all } (t,x) \in J.$$

**Example 2.3.** Let $\lambda, \omega \in (-1,0)\cup(0,\infty)$ and $r = (r_1, r_2) \in (0,1]\times(0,1]$, then

$$D^r_{\theta} t^{\lambda} x^{\omega} =^c D^r_{\theta} t^{\lambda} x^{\omega} = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1} x^{\omega-r_2};$$

for almost all $(t,x) \in J$.

**Definition 2.4.** [15] For a function $u : J \to E$, we set

$$q(t,x) = u(t,x) - u(t,0) - u(0,x) + u(0,0).$$

By the mixed regularized derivative of order $r = (r_1, r_2) \in (0,1]\times(0,1]$ of a function $u(t,x)$, we name the function

$$\overline{D}^r_{\theta} u(t,x) = D^r_{\theta} q(t,x).$$

Let $a_1 \in [0,a]$, $z^+ = (a_1, 0) \in J$, $J_z = (a_1, a]\times[0,b]$, $r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J_z)$, the expression

$$(I^r_{z^+}u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^{t}\int_0^x (t-\tau)^{r_1-1}(x-\xi)^{r_2-1}u(\tau,\xi)d\xi d\tau,$$

is called the left-sided mixed Riemann-Liouville integral of order $r$ of $u$.

**Definition 2.5.** [15] For $u \in L^1(J_z)$, the mixed regularized derivative of order $r$ of $u$ is defined by the expression

$$\overline{D}^r_{z^+} u(t,x) = D^r_{z^+} q_k(t,x),$$

where

$$q_k(t,x) = u(t,x) - u(t,0) - u(a_1,x) + u(a_1,0).$$

As a consequence of Lemma 3.2 in [1], we have the following Lemma

**Lemma 2.6.** *Let $r_1, r_2 \in (0,1]$, $\mu(t,x) = \varphi(t) + \psi(x) - \varphi(0)$. A function $u \in PC$ is solution of the problem (1.1), if and only if $u$ satisfies*

$$(2.1) \qquad \begin{cases} u(t,x) = \mu(t,x) + (I^r_{\theta}h)(t,x); \quad \textit{if} \ \ (t,x) \in [0,t_1]\times[0,b], \\[2ex] u(t,x) = \varphi(t) + g_k(s_k, x, u(s_k,x)) - g_k(s_k, 0, u(s_k,0)) \\ +(I^r_{\theta_k}h)(t,x); \quad \textit{if} \ \ (t,x) \in I_k, \ k = 1,\dots,m, \\[2ex] u(t,x) = g_k(t,x,u(t,x)); \ \textit{if} \ (t,x) \in J_k, \ k = 1,\dots,m, \end{cases}$$

*where $h \in \mathcal{C}(I_k)$; $k = 0,\dots,m$, such that*

$$h(t,x) = f(t,x,u(t,x),h(t,x))); \quad \textit{for} \ \ (t,x) \in I_k, \ k = 0,\dots,m.$$

Now, we consider the Ulam stability for the problem (1.1). Let $\epsilon > 0$, $\Psi \geq 0$ and $\Phi : J \to [0,\infty)$ be a continuous function. We consider the following inequalities

$$(2.2)\quad \begin{cases} \|\overline{D}^r_{\theta_k} u(t,x) - f(t,x,u(t,x),\overline{D}^r_{\theta_k} u(t,x))\|_E \leq \epsilon; & \text{if } (t,x) \in I_k,\ k = 0,\dots,m, \\ \|u(t,x) - g_k(t,x,u(t,x))\|_E \leq \epsilon; & \text{if } (t,x) \in J_k,\ k = 1,\dots,m. \end{cases}$$

$$(2.3)\quad \begin{cases} \|\overline{D}^r_{\theta_k} u(t,x) - f(t,x,u(t,x),\overline{D}^r_{\theta_k} u(t,x))\|_E \leq \Phi(t,x); & \text{if } (t,x) \in I_k,\ k = 0,\dots,m, \\ \|u(t,x) - g_k(t,x,u(t,x))\|_E \leq \Psi; & \text{if } (t,x) \in J_k,\ k = 1,\dots,m. \end{cases}$$

$$(2.4)\quad \begin{cases} \|\overline{D}^r_{\theta_k} u(t,x) - f(t,x,u(t,x),\overline{D}^r_{\theta_k} u(t,x))\|_E \leq \epsilon\Phi(t,x); & \text{if } (t,x) \in I_k,\ k = 0,\dots,m, \\ \|u(t,x) - g_k(t,x,u(t,x))\|_E \leq \epsilon\Psi; & \text{if } (t,x) \in J_k,\ k = 1,\dots,m. \end{cases}$$

**Definition 2.7.** [16] Problem (1.1) is Ulam-Hyers stable if there exists a real number $c_{f,g_k} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC$ of the inequality (2.2) there exists a solution $v \in PC$ of problem (1.1) with

$$\|u(t,x) - v(t,x)\|_E \leq \epsilon c_{f,g_k};\ (t,x) \in J.$$

**Definition 2.8.** [16] Problem (1.1) is generalized Ulam-Hyers stable if there exists $c_{f,g_k} : C([0,\infty),[0,\infty))$ with $c_{f,g_k}(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC$ of the inequality (2.2) there exists a solution $v \in PC$ of problem (1.1) with

$$\|u(t,x) - v(t,x)\|_E \leq c_{f,g_k}(\epsilon);\ (t,x) \in J.$$

**Definition 2.9.** [16] Problem (1.1) is Ulam-Hyers-Rassias stable with respect to $(\Phi,\Psi)$ if there exists a real number $c_{f,g_k,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC$ of the inequality (2.4) there exists a solution $v \in PC$ of problem (1.1) with

$$\|u(t,x) - v(t,x)\|_E \leq \epsilon c_{f,g_k,\Phi}(\Psi + \Phi(t,x));\ (t,x) \in J.$$

**Definition 2.10.** [16] Problem (1.1) is generalized Ulam-Hyers-Rassias stable with respect to $(\Phi,\Psi)$ if there exists a real number $c_{f,g_k,\Phi} > 0$ such that for each solution $u \in PC$ of the inequality (2.3 ) there exists a solution $v \in PC$ of problem (1.1) with $\|u(t,mx) - v(t,x)\|_E \leq c_{f,g_k,\Phi}(\Psi + \Phi(t,x))$; $(t,x) \in J$.

*Remark* 2.11. It is clear that: (i) Definition 2.7 $\Rightarrow$ Definition 2.8, (ii) Definition 2.9 $\Rightarrow$ Definition 2.10, (iii) Definition 2.9 for $\Phi(\cdot,\cdot) = \Psi = 1 \Rightarrow$ Definition 2.7.

*Remark* 2.12. A function $u \in PC$ is a solution of the inequality (2.2) if and only if there exist a function $G \in PC$ and a sequence $G_k;\ k = 1,\ldots,m$ in E (which depend on u) such that

(*i*) $\|G(t,x)\|_E \le \epsilon$ and $\|G_k\|_E \le \epsilon;\ k = 1,\ldots,m,$

(*ii*) $\overline{D}_{\theta_k}^r u(t,x) = f(t,x,u(t,x),\overline{D}_{\theta_k}^r u(t,x)) + G(t,x);$
if $(t,x) \in I_k,\ k = 0,\ldots,m,$

(*iii*) $u(t,x) = g_k(t,x,u(t,x)) + G_k;$ if $(t,x) \in J_k,\ k = 1,\ldots,m,$

One can have similar remarks for the inequalities (2.3) and (2.4). So, the Ulam stabilities of the impulsive fractional differential equations are some special types of data dependence of the solutions of impulsive fractional differential equations.

We recall now an integral inequality which based on an iteration argument.

**Lemma 2.13.** *[18] Suppose $\beta > 0,\ a(t)$ is a nonnegative function locally integrable on $0 \le t < T$ (some $T \le +\infty$) and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \le t < T,\ g(t) \le M$ (constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \le t < T$ with*

$$u(t) \le a(t) + g(t)\int_0^t (t-s)^{\beta-1}u(s)ds$$

*on this interval. Then*

$$u(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\right]ds,\ 0 \le t < T.$$

From the above lemma, we concluded the following lemma.

**Lemma 2.14.** *Suppose $r_1, r_2 > 0,\ a(t,x)$ is a nonnegative function locally integrable on $J$ and $g(t,x)$ is a nonnegative, nondecreasing continuous function on $J,\ g(t,x) \le M$ (constant), and suppose $u(t,x)$ is nonnegative and locally integrable on $J$ with*

$$u(t,x) \le a(t,x) + g(t,x)\int_0^t\int_0^x (t-\tau)^{r_1-1}(x-\xi)^{r_2-1}u(\tau,\xi)d\xi d\tau$$

*on $J$. Then*

$$\begin{aligned} u(t,x) & \\ \le\ & a(t,x) \\ +\ & \int_0^t\int_0^x \left[\sum_{n=1}^{\infty} \frac{(g(t,x)\Gamma(r_1)\Gamma(r_2))^n}{\Gamma(nr_1)\Gamma(nr_2)}(t-\tau)^{nr_1-1}(x-\xi)^{nr_2-1}a(\tau,\xi)\right]d\xi d\tau \end{aligned}$$

*on $J$.*

## 3. Uniqueness and Ulam stabilities results

In this section, we present conditions for the Ulam stability of problem (1.1).

**Lemma 3.1.** *If $u \in PC$ is a solution of the inequality (2.2) then $u$ is a solution of the following integral inequality*

$$
(3.1) \qquad \begin{cases}
\|u(t,x) - \mu(t,x) - \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} h(\tau,\xi) d\xi d\tau\|_E \\
\leq \frac{\epsilon a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}; \quad \text{if} \ \ (t,x) \in [0,t_1] \times [0,b], \\
\|u(t,x) - \varphi(t) - g_k(s_k, x, u(s_k,x)) + g_k(s_k, 0, u(s_k,0)) \\
- \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} h(\tau,\xi) d\xi d\tau\|_E \\
\leq \frac{\epsilon a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}; \quad \text{if} \ \ (t,x) \in I_k, \ k = 1, \dots, m, \\
\|u(t,x) - g_k(t,x,u(t,x))\|_E \leq \epsilon; \ \text{if} \ (t,x) \in J_k, \ k = 1, \dots, m,
\end{cases}
$$

*where $h \in \mathcal{C}(I_k);\ k = 0, \dots, m,$ such that*

$$
h(t,x) = f(t,x,u(t,x),h(t,x))); \quad \text{for} \ \ (t,x) \in I_k, \ k = 0, \dots, m.
$$

*Proof.* By Remark 2.12 we have that

$$
\begin{cases}
\overline{D}_{\theta_k}^r u(t,x) = f(t,x,u(t,x), \overline{D}_{\theta_k}^r u(t,x)) + G(t,x); & \text{if } (t,x) \in I_k, \ k = 0, \dots, m, \\
u(t,x) = g_k(t,x,u(t,x)) + G_k; & \text{if } (t,x) \in J_k, \ k = 1, \dots, m.
\end{cases}
$$

Then

$$
\begin{cases}
u(t,x) = \mu(t,x) \\
+ \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} (h(\tau,\xi) + G(\tau,\xi)) d\xi d\tau; \quad \text{if} \ \ (t,x) \in [0,t_1] \times [0,b], \\
u(t,x) = \varphi(t) + g_k(s_k, x, u(s_k,x)) - g_k(s_k, 0, u(s_k,0)) \\
+ \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} (h(\tau,\xi) + G(\tau,\xi)) d\xi d\tau; \quad \text{if} \ \ (t,x) \in I_k, \ k = 1, \dots, m, \\
u(t,x) = g_k(t,x,u(t,x)) + G_k; \ \text{if} \ (t,x) \in J_k, \ k = 1, \dots, m,
\end{cases}
$$

where $h \in \mathcal{C}(I_k);\ k = 0, \dots, m,$ such that

$$
h(t,x) = f(t,x,u(t,x),h(t,x))); \quad \text{for} \ \ (t,x) \in I_k, \ k = 0, \dots, m.
$$

Thus, it follows that

$$\begin{cases} \|u(t,x)-\mu(t,x)-\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}h(\tau,\xi)d\xi d\tau\|_E \\ = \|\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}G(\tau,\xi)d\xi d\tau\|_E; \ \text{ if } \ (t,x)\in[0,t_1]\times[0,b], \\ \|u(t,x)-\varphi(t)-g_k(s_k,x,u(s_k,x))+g_k(s_k,0,u(s_k,0)) \\ -\int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}h(\tau,\xi)d\xi d\tau\|_E \\ = \|\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}G(\tau,\xi)d\xi d\tau\|_E; \ \text{ if } \ (t,x)\in I_k, \ k=1,\dots,m, \\ \|u(t,x)-g_k(t,x,u(t,x))\|_E=\|G_k\|_E; \text{ if } (t,x)\in J_k, \ k=1,\dots,m. \end{cases}$$

Hence, we obtain (3.1). □

*Remark* 3.2. We have similar results for the solutions of the inequalities (2.3) and (2.4).

**Theorem 3.3.** *Assume that the following hypotheses hold:*

$(H_1)$ *There exist constants $l_f>0$ and $0<l'_f<1$ such that*

$$\|f(t,x,u,v)-f(t,x,\overline{u},\overline{v})\|_E\le l_f\|u-\overline{u}\|_E+l'_f\|v-\overline{v}\|_E;$$

*for each $(t,x)\in I_k;\ k=0,\dots,m$, and each $u,v,\overline{u},\overline{v}\in E$,*

$(H_2)$ *There exist constants $l_{g_k}>0;\ k=1,\dots,m$, such that*

$$\|g_k(t,x,u)-g_k(t,x,\overline{u})\|_E\le l_{g_k}\|u-\overline{u}\|_E,$$

*for each $(t,x)\in J_k$, and each $u,\overline{u}\in E,\ k=1,\dots,m$.*

*If*

$$\ell:=2l_g+\frac{l_f a^{r_1}b^{r_2}}{(1-l'_f)\Gamma(1+r_1)\Gamma(1+r_2)}<1, \tag{3.2}$$

*where $l_g=\max_{k=1,\dots,m} l_{g_k}$, then the problem (1.1) has a unique solution on $J$.*

*Furthermore, if the following hypothesis*

$(H_3)$ *There exists $\lambda_\Phi>0$ such that for each $(t,x)\in J$, we have*

$$\int_{s_k}^t\int_0^x\left[\sum_{n=1}^\infty\frac{(l_f)^n(t-\tau)^{nr_1-1}(x-\xi)^{nr_2-1}}{(1-l'_f)^n(1-2l_g)^n\Gamma(nr_1)\Gamma(nr_2)}\Phi(\tau,\xi)\right]d\xi d\tau\le\lambda_\Phi\Phi(t,x);$$

*for $k=0,\dots,m$, holds, then the problem (1.1) is generalized Ulam-Hyers-Rassias stable.*

*Proof.* Consider the operator $N : PC \to PC$ defined by

$$\begin{cases} (Nu)(t,x) = \mu(t,x) \\ + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} h(\tau,\xi)d\xi d\tau; \ \text{ if } \ (t,x) \in [0,t_1] \times [0,b], \\ (Nu)(t,x) = \varphi(t) + g_k(s_k,x,u(s_k,x)) - g_k(s_k,0,u(s_k,0)) \\ + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} h(\tau,\xi)d\xi d\tau; \ \text{ if } \ (t,x) \in I_k, \ k = 1,\dots,m, \\ (Nu)(t,x) = g_k(t,x,u(t,x)); \text{ if } (t,x) \in J_k, \ k = 1,\dots,m. \end{cases}$$

where $h \in \mathcal{C}(I_k); \ k = 0,\dots,m$, such that

$$h(t,x) = f(t,x,u(t,x),h(t,x))); \ \text{ for } \ (t,x) \in I_k, \ k = 0,\dots,m.$$

Clearly, the fixed points of the operator $N$ are solutions of the problem (1.1). We shall use the Banach contraction principle to prove that $N$ has a fixed point. *$N$ is a contraction.* Let $u, v \in PC$, then, for each $(t,x) \in J$, we have

$$\begin{cases} \|(Nu)(t,x) - (Nv)(t,x)\|_E \\ \leq \| \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} [h(\tau,\xi) - h_v(\tau,\xi)]d\xi d\tau\|_E; \\ \text{ if } \ (t,x) \in [0,t_1] \times [0,b], \\ \|(Nu)(t,x) - (Nv)(t,x)\|_E \leq \|g_k(s_k,x,u(s_k,x)) - g_k(s_k,x,v(s_k,x))\|_E \\ +\|g_k(s_k,0,u(s_k,0)) - g_k(s_k,0,v(s_k,0))\|_E \\ +\| \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} [h(\tau,\xi) - h_v(\tau,\xi)]d\xi d\tau\|_E; \\ \text{ if } \ (t,x) \in I_k, \ k = 1,\dots,m, \\ \|(Nu)(t,x) - (Nv)(t,x)\|_E = \|g_k(t,x,u(t,x)) - g_k(t,x,v(t,x))\|_E; \\ \text{if } (t,x) \in J_k, \ k = 1,\dots,m. \end{cases}$$

where $h_v \in \mathcal{C}(I_k); \ k = 0,\dots,m$, such that

$$h_v(t,x) = f(t,x,v(t,x),h_v(t,x))); \ \text{ for } \ (t,x) \in I_k, \ k = 0,\dots,m.$$

However, $(H_1)$ gives

$$\|h(t,x) - h_v(t,x)\|_E \leq l_f\|u(t,x) - v(t,x)\|_E + l'_f\|h(t,x) - h_v(t,x)\|_E.$$

Then

$$\begin{aligned} \|h(t,x) - h_v(t,x)\|_E &\leq \frac{l_f}{1-l'_f}\|u(t,x) - v(t,x)\|_E \\ &\leq \frac{l_f}{1-l'_f}\|u - v\|_{PC}. \end{aligned}$$

Thus, we get

$$\begin{cases} \|(Nu)(t,x)-(Nv)(t,x)\|_E \leq \int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \frac{l_f}{1-l'_f}\|u-v\|_{PC}d\xi d\tau; \\ \leq \frac{l_f a^{r_1}b^{r_2}}{(1-l'_f)\Gamma(1+r_1)\Gamma(1+r_2)}\|u-v\|_{PC};\ \text{if}\ \ (t,x)\in[0,t_1]\times[0,b], \\ \|(Nu)(t,x)-(Nv)(t,x)\|_E \leq 2l_g\|u-v\|_{PC} \\ +\int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \frac{l_f}{1-l'_f}\|u-v\|_{PC}d\xi d\tau \\ \leq \left(2l_g+\frac{l_f a^{r_1}b^{r_2}}{(1-l'_f)\Gamma(1+r_1)\Gamma(1+r_2)}\right)\|u-v\|_{PC};\ \text{if}\ \ (t,x)\in I_k,\ k=1,\dots,m, \\ \|(Nu)(t,x)-(Nv)(t,x)\|_E \leq l_g\|u-v\|_{PC}; \text{if}\ (t,x)\in J_k,\ k=1,\dots,m. \end{cases}$$

Hence

$$\|N(u)-N(v)\|_{PC} \leq \ell\|u-v\|_{PC}.$$

By the condition (3.2), we conclude that $N$ is a contraction. As a consequence of the Banach fixed point theorem, we deduce that $N$ has a unique fixed point $v$ which is a solution of the problem (1.1). Then we have

$$\begin{cases} v(t,x)=\mu(t,x) \\ +\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}h_v(\tau,\xi)d\xi d\tau;\ \ \text{if}\ \ (t,x)\in[0,t_1]\times[0,b], \\ v(t,x)=\varphi(t)+g_k(s_k,x,v(s_k,x))-g_k(s_k,0,v(s_k,0)) \\ +\int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}h_v(\tau,\xi)d\xi d\tau;\ \ \text{if}\ \ (t,x)\in I_k,\ k=1,\dots,m, \\ v(t,x)=g_k(t,x,v(t,x));\ \text{if}\ (t,x)\in J_k,\ k=1,\dots,m, \end{cases}$$

where $h_v\in\mathcal{C}(I_k);\ k=0,\dots,m$, such that

$$h_v(t,x)=f(t,x,v(t,x),h_v(t,x)));\quad\text{for}\ \ (t,x)\in I_k,\ k=0,\dots,m.$$

Let $u\in PC$ be a solution of the inequality (2.3). By integrating this inequality, for each $(t,x)\in J$, we have

$$\begin{cases} \|u(t,x)-\mu(t,x)-\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}h(\tau,\xi)d\xi d\tau\|_E \\ \leq \|\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\Phi(\tau,\xi)d\xi d\tau\|_E;\ \ \text{if}\ \ (t,x)\in[0,t_1]\times[0,b], \\ \|u(t,x)-\varphi(t)-g_k(s_k,x,u(s_k,x))+g_k(s_k,0,u(s_k,0)) \\ -\int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}h(\tau,\xi)d\xi d\tau\|_E \\ \leq \|\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\Phi(\tau,\xi)d\xi d\tau\|_E;\ \ \text{if}\ \ (t,x)\in I_k,\ k=1,\dots,m, \\ \|u(t,x)-g_k(t,x,u(t,x))\|_E \leq \Psi;\ \text{if}\ (t,x)\in J_k,\ k=1,\dots,m, \end{cases}$$

where $h \in \mathcal{C}(I_k);\ k = 0, \dots, m$, such that

$$h(t,x) = f(t,x,v(t,x),h(t,x))); \quad \text{for} \quad (t,x) \in I_k,\ k = 0, \dots, m.$$

Thus, by $(H_3)$ for each $(t,x) \in J$, we get

$$\begin{cases} \|u(t,x) - \mu(t,x) - \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} h(\tau,\xi) d\xi d\tau\|_E \\ \leq \frac{\lambda_\Phi}{l_f}\Phi(t,x); \quad \text{if} \quad (t,x) \in [0,t_1] \times [0,b], \\ \|u(t,x) - \varphi(t) - g_k(s_k,x,u(s_k,x)) + g_k(s_k,0,u(s_k,0)) \\ - \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} h(\tau,\xi) d\xi d\tau\|_E \\ \leq \frac{\lambda_\Phi}{l_f}\Phi(t,x); \quad \text{if} \quad (t,x) \in I_k,\ k = 1, \dots, m, \\ \|u(t,x) - g_k(t,x,u(t,x))\|_E \leq \Psi; \text{ if } (t,x) \in J_k,\ k = 1, \dots, m. \end{cases}$$

Hence

$$\begin{cases} \|u(t,x) - v(t,x)\|_E \leq \frac{\lambda_\Phi}{l_f}\Phi(t,x) \\ + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|h(\tau,\xi) - h_v(\tau,\xi)\|_E d\xi d\tau; \\ \text{ if } \ (t,x) \in [0,t_1] \times [0,b], \\ \|u(t,x) - v(t,x)\|_E \leq \frac{\lambda_\Phi}{l_f}\Phi(t,x) + 2l_g\|u(t,x) - v(t,x)\|_E \\ + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|(h\tau,\xi) - h_v(\tau,\xi)\|_E d\xi d\tau; \\ \text{ if } \ (t,x) \in I_k,\ k = 1, \dots, m, \\ \|u(t,x) - v(t,x)\|_E \leq \Psi + \|g_k(t,x,u(t,x)) - g_k(t,x,v(t,x))\|_E \\ \leq \Psi + l_g\|u(t,x) - v(t,x)\|_E; \text{ if } (t,x) \in J_k,\ k = 1, \dots, m. \end{cases}$$

For each $(t,x) \in [0,t_1] \times [0,b]$, we have

$$\|u(t,x) - v(t,x)\|_E \leq \frac{\lambda_\Phi}{l_f}\Phi(t,x) \\ + \frac{l_f}{1-l'_f}\int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|u(\tau,\xi) - v(\tau,\xi)\|_E d\xi d\tau.$$

From Lemma 2.14, we obtain

$$
\begin{aligned}
\|u(t,x)-v(t,x)\|_E &\leq \frac{\lambda_\Phi}{l_f}\Phi(t,x) \\
&+ \frac{\lambda_\Phi}{l_f}\int_0^t\int_0^x \left[\sum_{n=1}^{\infty}\frac{(l_f)^n}{(1-l'_f)^n\Gamma(nr_1)\Gamma(nr_2)}(t-\tau)^{nr_1-1}(x-\xi)^{nr_2-1}\Phi(\tau,\xi)\right]d\xi d\tau \\
&\leq \frac{\lambda_\Phi}{l_f}(1+\lambda_\Phi)\Phi(t,x) \\
&:= c_{1,f,g_k,\Phi}\Phi(t,x).
\end{aligned}
$$

Thus, for each $(t,x)\in[0,t_1]\times[0,b]$, we get

$$\|u(t,x)-v(t,x)\|_E \leq c_{1,f,g_k,\Phi}(\Psi+\Phi(t,x)).$$

Now, for each $(t,x)\in I_k,\ k=1,\dots,m$, we have

$$
\begin{aligned}
\|u(t,x)-v(t,x)\|_E &\leq \frac{\lambda_\Phi}{l_f}\Phi(t,x) \\
&+ 2l_g\|u(t,x)-v(t,x)\|_E \\
&+ \frac{l_f}{1-l'_f}\int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|u(\tau,\xi)-v(\tau,\xi)\|_E d\xi d\tau.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
\|u(t,x)-v(t,x)\|_E &\leq \frac{\lambda_\Phi}{l_f(1-2l_g)}\Phi(t,x) \\
&+ \frac{l_f}{(1-l'_f)(1-2l_g)}\int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|u(\tau,\xi)-v(\tau,\xi)\|_E d\xi d\tau.
\end{aligned}
$$

Again, from Lemma 2.14, we get

$$
\begin{aligned}
\|u(t,x)-v(t,x)\|_E &\leq \frac{\lambda_\Phi}{l_f(1-2l_g)}\Phi(t,x) \\
&+ \frac{\lambda_\Phi}{l_f(1-2l_g)}\int_{s_k}^t\int_0^x \left[\sum_{n=1}^{\infty}\frac{(l_f)^n(t-\tau)^{nr_1-1}(x-\xi)^{nr_2-1}}{(1-l'_f)^n(1-2l_g)^n\Gamma(nr_1)\Gamma(nr_2)}\Phi(\tau,\xi)\right]d\xi d\tau \\
&\leq \frac{\lambda_\Phi}{l_f(1-2l_g)}(1+\lambda_\Phi)\Phi(t,x) \\
&:= c_{2,f,g_k,\Phi}\Phi(t,x).
\end{aligned}
$$

Hence, for each $(t,x)\in I_k,\ k=1,\dots,m$, we get

$$\|u(t,x)-v(t,x)\|_E \leq c_{2,f,g_k,\Phi}(\Psi+\Phi(t,x)).$$

Now, for each $(t,x)\in J_k,\ k=1,\dots,m$, we have

$$\|u(t,x)-v(t,x)\|_E \leq \Psi + l_g\|u(t,x)-v(t,x)\|_E.$$

This gives

$$\|u(t,x)-v(t,x)\|_E \leq \frac{\Psi}{1-l_g} := c_{3,f,g_k,\Phi}\Psi.$$

Thus, for each $(t,x)\in J_k,\ k=1,\dots,m$, we get

$$\|u(t,x)-v(t,x)\|_E \leq c_{3,f,g_k,\Phi}(\Psi+\Phi(t,x)).$$

Set $c_{f,g_k,\Phi} := \max\limits_{i\in\{1,2,3\}} c_{i,f,g_k,\Phi}$. Hence, for each $(t,x)\in J$, we obtain

$$\|u(t,x)-v(t,x)\|_E \leq c_{f,g_k,\Phi}(\Psi+\Phi(t,x)).$$

Consequently, problem (1.1) is generalized Ulam-Hyers-Rassias stable. □

## 4. An Example

Let $E=l^1=\{w=(w_1,w_2,\dots,w_n,\dots): \sum_{n=1}^{\infty}|w_n|<\infty\}$, be the Banach space with norm

$$\|w\|_E=\sum_{n=1}^{\infty}|w_n|.$$

Consider the following partial fractional differential equations with not instantaneous impulses

$$(4.1)\qquad \begin{cases} \overline{D}^r_{\theta_k}u(t,x)=f(t,x,u(t,x),\overline{D}^r_{\theta_k}u(t,x)); \\ \text{if } (t,x)\in([0,1]\cup(2,3])\times[0,1],\ k\in\{0,1\}, \\ u(t,x)=g(t,x,u(t,x));\ \text{if } (t,x)\in(1,2]\times[0,1], \\ u(t,0)=1+e^t;\ t\in[0,3], \\ u(0,x)=2+x^2;\ x\in[0,1], \end{cases}$$

where $r=(r_1,r_2)\in(0,1]\times(0,1],\ \theta_0=(0,0),\ \theta_1=(2,0),\ 0=s_0<t_1=1<s_1=2<t_2=3,\ u=(u_1,u_2,\dots,u_n,\dots),\ f=(f_1,f_2,\dots,f_n,\dots),\ g=(g_1,g_2,\dots,g_n,\dots)$,

$${}^cD^r_\theta u=({}^cD^r_\theta u_1,{}^c D^r_\theta u_2,\dots,{}^c D^r_\theta u_n,\dots,$$

$$f_n(t,x,u_n)=\frac{1}{(1+110e^{t+x})(1+|u_n|+|\overline{D}^r_{\theta_k}u_n|)};$$

for $(t,x)\in([0,1]\cup(2,3])\times[0,1]$ and $n\in\mathbb{N}$, and

$$g_n(t,x,u_n)=\frac{1}{110e^{t+x}}\ln(1+t^2+x^2+|u_n|);$$

for $(t,x)\in(1,2]\times[0,1]$ and $n\in\mathbb{N}$.

Clearly, the functions $f$ and $g$ are continuous. For each $n\in\mathbb{N}$, $u,\overline{u},\in E$ and $(t,x)\in([0,1]\cup(2,3])\times[0,1]$, we have

$$|f_n(t,x,u_n(t,x),\overline{D}^r_{\theta_k}u_n(t,x))-f_n(t,x,\overline{u}_n(t,x),\overline{D}^r_{\theta_k}\overline{u}_n(t,x))|$$

$$\leq \frac{1}{111}(|u_n - \overline{u}_n| + |\overline{D}^r_{\theta_k} u_n - \overline{D}^r_{\theta_k}\overline{u}_n|).$$

Thus, for each $u, \overline{u} \in E$ and $(t,x) \in ([0,1] \cup (2,3]) \times [0,1]$ we get

$$\begin{aligned}
&\|f(t,x,u(t,x),\overline{D}^r_{\theta_k} u_n(t,x)) - f(t,x,\overline{u}(t,x),\overline{D}^r_{\theta_k}\overline{u}_n(t,x))\|_E \\
&= \sum_{n=1}^{\infty} |f_n(t,x,u_n(t,x),\overline{D}^r_{\theta_k} u_n(t,x)) - f_n(t,x,\overline{u}_n(t,x),\overline{D}^r_{\theta_k}\overline{u}_n(t,x))| \\
&\leq \frac{1}{111}\sum_{n=1}^{\infty}(|u_n - \overline{u}_n| + |\overline{D}^r_{\theta_k} u_n - \overline{D}^r_{\theta_k}\overline{u}_n|) \\
&= \frac{1}{111}(\|u - \overline{u}\|_E + \|\overline{D}^r_{\theta_k} u - \overline{D}^r_{\theta_k}\overline{u}\|_E).
\end{aligned}$$

Also, for each $n \in \mathbb{N}$, $u, \overline{u}, \in E$ and $(t,x) \in (1,2] \times [0,1]$, we have

$$\|g(t,x,u(t,x)) - g(t,x,\overline{u}(t,x))\|_E \leq \frac{1}{110}\|u - \overline{u}\|_E.$$

Hence the conditions $(H_1)$ and $(H_2)$ are satisfied with $l_f = l'_f = \frac{1}{111}$, $l_g = \frac{1}{110}$. We shall show that condition (3.2) holds with $a = 3$ and $b = 1$. Indeed, for each $(r_1, r_2) \in (0,1] \times (0,1]$ we get

$$\begin{aligned}
\ell &= 2l_g + \frac{l_f a^{r_1} b^{r_2}}{(1 - l'_f)\Gamma(1+r_1)\Gamma(1+r_2)} \\
&= \frac{1}{55} + \frac{3^{r_1}}{110\Gamma(1+r_1)\Gamma(1+r_2)} \\
&< \frac{7}{55} < 1.
\end{aligned}$$

Finally, the hypothesis $(H_3)$ is satisfied with $\Phi(t,x) = x^2$ and

$$\lambda_\Phi = \sum_{n=1}^{\infty} \frac{55^n}{(54 \times 110)^n \Gamma(1+nr_1)\Gamma(1+nr_2)} 3^{nr_1}.$$

Consequently, Theorem 3.3 implies that the problem (4.1) is generalized Ulam-Hyers-Rassias stable.

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