Novi Sad J. Math.
Vol. 47, No. 2, 2017, 149-156

SOME GEOMETRIC PROPERTIES OF AN INTEGRAL OPERATOR INVOLVING BESSEL FUNCTIONS

Nanjundan Magesh[1], Saurabh Porwal[23] and Surya Pratap Singh[4]

Abstract. The purpose of the present paper is to obtain some sufficient conditions for an integral operator involving Bessel functions of the first kind to be in the classes $S^*(\alpha)$, $C(\alpha)$ and UCV.

AMS Mathematics Subject Classification (2010): 30C45

Key words and phrases: Analytic functions; convex functions; Bessel functions; integral operator

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ and normalized by the condition $f(0) = f'(0) - 1 = 0$. A function $f(z) \in A$ is said to starlike of order $\alpha(0 \leq \alpha < 1)$, if it satisfies the following condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}).$$

A function $f(z) \in A$ is said to be convex of order $\alpha(0 \leq \alpha < 1)$, if it satisfies

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathbb{U})$$

the classes of starlike and convex functions of order α are denoted by $S^*(\alpha)$ and $K(\alpha)$. Further we denote by $S^*(0) = S^*$ and $K(0) = K$. The classes $S^*(\alpha)$, $K(\alpha)$, S^* and K were studied by Robertson [16] and Silverman [17].

A function $f(z) \in A$ is said to be close-to-convex of order $\alpha(0 \leq \alpha < 1)$, if it satisfies the condition

$$\Re\{f'(z)\} > \alpha \qquad (z \in \mathbb{U}).$$

[1]Post-Graduate and Research Department of Mathematics, Government Arts College for Men, Krishnagiri 635001, Tamilnadu, India. e-mail: nmagi_2000@yahoo.co.in

[2]Lecturer Mathematics, Sri Radhey Lal Arya Inter College, Ahen, Hathras, U.P., India, e-mail: saurabhjcb@rediffmail.com

[3]Corresponding author

[4]Department of Mathematics, U.I.E.T., C.S.J.M. University, Kanpur-208024, (U.P.), India, e-mail: surya.neverdies@gmal.com

The class of all close-to-convex functions of order α are denoted by $C(\alpha)$.

A function $f(z)$ is uniformly convex in $\mathbb{U}$ if $f \in K$ and has the property that for every circular arc γ contained in $\mathbb{U}$, with center ξ also in $\mathbb{U}$, the arc $f(\gamma)$ is convex with respect to $f(\xi)$. The class of uniformly convex functions denoted by UCV. It is well known that

$$f \in UCV \Leftrightarrow \left|\frac{zf''(z)}{f'(z)}\right| \leq \Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} \qquad (z \in \mathbb{U}).$$

The class UCV was studied by Goodman [5] and further studied and generalized by many researchers see e.g. ([1], [3], [6], [7], [15] and [19]).

The Bessel function of the first kind of order ν is defined by the infinite series

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{n!\Gamma(n+\nu+1)},$$

where Γ stands for the Euler gamma function, $z \in \mathbb{C}$ and $v \in \mathbb{R}$.

In 1960, Brown [2] studied the univalence of Bessel functions. He introduced some criteria to determine the radius of univalence of Bessel functions. Recently Szasz and Kupan [20] investigated the univalence of the normalized Bessel function of the first kind $g_v : \mathbb{U} \to \mathbb{C}$ defined by

$$\begin{aligned} g_\nu(z) &= 2^\nu\Gamma(\nu+1)z^{1-\nu/2}J_\nu(z^{1/2}) \\ &= z + \sum_{n=1}^{\infty} \frac{(-1)^n z^{n+1}}{4^n n!(\nu+1)(\nu+2)\dots(\nu+n)}. \end{aligned} \tag{1.2}$$

Recently, Frasin [4] introduced the following integral operator which involves the normalized Bessel function of the first kind.

$$F_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{g_{v_i}(t)}{t}\right)^{\alpha_i} dt \tag{1.3}$$

and obtained several sufficient conditions for this operator to be convex and strongly convex of given order in the open disc $\mathbb{U}$. Recently, analogous to these results Porwal and Breaz [14] studied the sufficient condition for the operator defined by (1.3) for certain classes of univalent functions. Further, these results were generalized by Porwal and Kumar [12], (see also [11], [13]). In 2012 Mohammed and Darus [9] obtained some sufficient conditions for an integral transform to be in the classes $S^*(\alpha)$, $C(\alpha)$, UCV and $N(\beta)$. In the present paper we obtain some sufficient conditions for the operator defined by (1.3) to be in the classes $S^*(\alpha)$, $C(\alpha)$ and UCV.

To prove our main results we shall require the following lemmas:

Lemma 1.1. *([20]) Let $v > (-5+\sqrt{5})/4$ and consider the normalized Bessel function of the first kind $g_v : \mathbb{U} \to \mathbb{C}$, defined by*

$$g_v(z) = 2^v\Gamma(v+1)z^{1-v/2}J_v(z^{1/2}),$$

where J_v *stands for the Bessel function of the first kind, then the following inequality holds for all* $z \in \mathbb{U}$

$$\left|\frac{zg_v'(z)}{g_v(z)} - 1\right| \leq \frac{v+2}{4v^2+10v+5}. \tag{1.4}$$

Lemma 1.2. *([18]) If* $f \in A$ *satisfies*

$$\Re\left\{\frac{zf''(z)}{f'(z)} + 1\right\} < \frac{3}{2}, \qquad (z \in \mathbb{U})$$

then $f \in S^*$.

Lemma 1.3. *([8]) If* $f \in A$ *satisfies*

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| < 2, \qquad (z \in \mathbb{U})$$

then $f \in S^*$.

Lemma 1.4. *([10]) If* $f \in A$ *satisfies*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \frac{3\alpha+1}{2(\alpha+1)} \qquad (z \in \mathbb{U},\ 0 \leq \alpha < 1)$$

then

$$\Re\{f'(z)\} > \frac{\alpha+1}{2}$$

or equivalently

$$f \in C\left(\frac{\alpha+1}{2}\right), \qquad (z \in \mathbb{U}).$$

Lemma 1.5. *([15]) If* $f \in A$ *satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{2}, \qquad (z \in \mathbb{U})$$

then $f \in UCV$.

2. Main results

Theorem 2.1. *Let* n *be a natural number and let* $v_1, v_2, \ldots, v_n > \dfrac{-5+\sqrt{5}}{4}$ *consider the functions* $g_{v_i} : \mathbb{U} \to \mathbb{C}$, *defined by*

$$g_{v_i}(z) = 2^{v_i}\Gamma(v_i+1)z^{1-v_i/2}J_{v_i}(z^{1/2}). \tag{2.1}$$

Let $v = \min\{v_1, v_2, \ldots, v_n\}$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *be positive real numbers. Moreover, suppose that these numbers satisfy the following inequality*

$$\frac{v+2}{4v^2+10v+5}\sum_{i=1}^{n}\alpha_i < \frac{1}{2}.$$

Then the function $F_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n} : \mathbb{U} \to \mathbb{C}$ *defined by (1.3) is in the class* S^*.

Proof. Since for all $i \in \{1, 2, \ldots, n\}$, we have $g_{v_i} \in A$, that is

$$g_{v_i}(0) = g'_{v_i}(0) - 1 = 0$$

clearly

$$F_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n} \in A$$

that is

$$F_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(0) = F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(0) - 1 = 0.$$

On the other hand, it is easy to see that,

$$F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z) = \prod_{i=1}^{n} \left(\frac{g_{v_i}(z)}{z} \right)^{\alpha_i}$$

which implies

$$\frac{zF''_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zg'_{v_i}(z)}{g_{v_i}(z)} - 1 \right) \tag{2.2}$$

or equivalently

$$1 + \frac{zF''_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zg'_{v_i}(z)}{g_{v_i}(z)} \right) + 1 - \sum_{i=1}^{n} \alpha_i \tag{2.3}$$

$$1 + \frac{zF''_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zg'_{v_i}(z)}{g_{v_i}(z)} - 1 \right) + 1.$$

Taking absolute value on both sides

$$\begin{aligned} \left| 1 + \frac{zF''_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)} \right| &\leq \sum_{i=1}^{n} \alpha_i \left| \frac{zg'_{v_i}(z)}{g_{v_i}(z)} - 1 \right| + 1 \\ &\leq \sum_{i=1}^{n} \alpha_i \frac{(v_i + 2)}{4v_i^2 + 10v_i + 5} + 1. \end{aligned}$$

We observe that the function $\phi : (-1, \infty) \to \mathbb{R}$, defined by

$$\phi(x) = \frac{x+2}{4x^2 + 10x + 5},$$

is decreasing and consequently for all $i \in \{1, 2, ..., n\}$ we have

$$\frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \leq \frac{\nu + 2}{4\nu^2 + 10\nu + 5}.$$

$$\begin{aligned} &\leq \tfrac{v+2}{4v^2+10v+5} \textstyle\sum_{i=1}^{n} \alpha_i + 1 \\ &\leq \tfrac{1}{2} + 1 \leq \tfrac{3}{2}. \end{aligned}$$

Hence by Lemma 1.2, we get $f \in S^*$. □

Theorem 2.2. *Let n be a natural number and let $v_1, v_2, \dots, v_n > -5+\sqrt{5}/4$. Consider the functions $g_{v_i} : \mathbb{U} \to \mathbb{C}$ defined by (2.1). Let $v = \min\{v_1, v_2, \dots v_n\}$ and $\alpha_1, \alpha_2, \dots, \alpha_n$ be positive real numbers. Moreover, suppose that these numbers satisfy the following inequality*

$$\frac{v+2}{4v^2+10v+5}\sum_{i=1}^{n}\alpha_i < 1.$$

Then the function $F_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}$ defined by (1.3) is in the class S^*.

Proof. From equation (2.2) we have

$$\begin{aligned}
\frac{zF''_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}(z)}{F'_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}(z)} &= \sum_{i=1}^{n}\alpha_i\left(\frac{zg'_{v_i}(z)}{g_{v_i}(z)}-1\right)\\
1+\frac{zF''_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}(z)}{F'_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}(z)} &= \sum_{i=1}^{n}\alpha_i\left(\frac{zg'_{v_i}(z)}{g_{v_i}(z)}-1\right)+1\\
\left|1+\frac{zF''_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}(z)}{F'_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}(z)}\right| &\leq \sum_{i=1}^{n}\alpha_i\left|\frac{zg'_{v_i}(z)}{g_{v_i}(z)}-1\right|+1\\
&\leq \sum_{i=1}^{n}\alpha_i\left(\frac{v_i+2}{4v_i^2+10v_i+5}\right)+1\\
\left|1+\frac{zF''_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}(z)}{F'_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}(z)}\right| &\leq \frac{v+2}{4v^2+10v+5}\sum_{i=1}^{n}\alpha_i+1.
\end{aligned}$$

But

$$\frac{v+2}{4v^2+10v+5}\sum_{i=1}^{n}\alpha_i \leq 1.$$

Therefore,

$$\left|1+\frac{zF''_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}(z)}{F'_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}(z)}\right| \leq 1+1 \leq 2.$$

Hence by Lemma 1.3, we get $f \in S^*$. □

Theorem 2.3. *Let n be a natural number and let $v_1, v_2, \dots, v_n > \frac{-5+\sqrt{5}}{4}$. Consider the function $g_{v_i} : \mathbb{U} \to \mathbb{C}$ defined by (2.1). Let $v = \min\{v_1, v_2, \dots, v_n\}$ and $\alpha_1, \alpha_2, \dots, \alpha_n$ be positive real numbers. Moreover, suppose that these numbers satisfy the following inequality*

$$\frac{v+2}{4v^2+10v+5}\sum_{i=1}^{n}\alpha_i < \frac{1-\alpha}{2(1+\alpha)}.$$

Then the function $F_{v_1,\dots,v_n,\alpha_1,\dots,\alpha_n}(z)$ defined by (1.3) is in the class $C\left(\frac{1-\alpha}{2(\alpha+1)}\right)$.

Proof. From equation (2.2), we have

$$\begin{aligned} \frac{zF''_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)} &= \sum_{i=1}^{n} \alpha_i \left(\frac{zg'_{v_i}(z)}{g_{v_i}(z)} - 1 \right) \\ 1 + \frac{zF''_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)} &= \sum_{i=1}^{n} \alpha_i \left(\frac{zg'_{v_i}(z)}{g_{v_i}(z)} - 1 \right) + 1. \end{aligned}$$

We have

$$\begin{aligned} &\Re\left\{1 + \frac{zF''_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}\right\} \\ &= 1 + \sum_{i=1}^{n} \alpha_i \Re\left\{ \frac{zg'_{v_i}(z)}{g_{v_i}(z)} - 1 \right\} \\ &= 1 - \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \alpha_i \Re\left\{ \frac{zg'_{v_i}(z)}{g_{v_i}(z)} \right\} \\ &\geq 1 - \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \alpha_i \left\{ 1 - \frac{v_i + 2}{4v_i^2 + 10v_i + 5} \right\} + 1 - \sum_{i=1}^{n} \alpha_i \\ &\geq 1 - \left(\frac{v+2}{4v^2 + 10v + 5} \right) \sum_{i=1}^{n} \alpha_i \\ &> \frac{3\alpha + 1}{2(\alpha + 1)}, \end{aligned}$$

by the given hypothesis.

Hence by Lemma 1.4, we have $f \in C\left(\frac{1-\alpha}{2(\alpha+1)}\right)$. □

Theorem 2.4. *Let n be a natural number and let $v_1, v_2, \ldots, v_n > -5 + \sqrt{5}/4$. Consider the function $g_{v_i} : \mathbb{U} \to \mathbb{C}$ defined by (2.1). Let $v = \min\{v_1, v_2, \ldots, v_n\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive real numbers. Moreover, suppose that these numbers satisfy the following inequality*

$$\frac{v+2}{4v^2 + 10v + 5} \sum_{i=1}^{n} \alpha_i < \frac{1}{2}.$$

Then the function $F_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}$ defined by (1.3) is in UCV.

Proof. From equation (2.2) we have

$$\frac{zF''_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zg'_{v_i}(z)}{g_{v_i}(z)} - 1 \right).$$

Taking absolute value on both sides we have,

$$\begin{aligned}\left|\frac{zF''_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}\right| &\leq \sum_{i=1}^{n}\alpha_i\left|\frac{zg'_{v_i}(z)}{g_{v_i}(z)}-1\right| \leq \sum_{i=1}^{n}\alpha_i\frac{v_i+2}{4v_i^2+10v_i+5} \\ &\leq \frac{v+2}{4v^2+10v+5}\sum_{i=1}^{n}\alpha_i < \frac{1}{2}.\end{aligned}$$

That is,

$$\left|\frac{zF''_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}{F'_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n}(z)}\right| < \frac{1}{2}.$$

Hence by Lemma 1.5, we get $F_{v_1,\ldots,v_n,\alpha_1,\ldots,\alpha_n} \in UCV$. □

Acknowledgement

The authors are thankful to the referee for his/her valuable comments and observations which helped in improving the paper.

References

[1] Bharti, R., Parvatham, R., Swaminathan, A., On subclass of uniformly convex functions and corressponding class of starlike functions. Tamkang J. Math. 28 (1997), 17-32.

[2] Brown, R.K., Univalence of Bessel functions. Proc. Amer. Math. Soc., 11(1960), 278-283.

[3] Dixit, K.K., Porwal, S., On a certain class of k-uniformly convex functions with negative coefficients. Bull. Cal. Math. Soc. 100(6) (2008), 639-652.

[4] Frasin, B.A., Sufficient conditions for integral operator defined by Bessel functions. J. Math. Inequal. 4(2) (2010), 301–306.

[5] Goodman, A.W., On Uniformly convex functions. Ann. Polon. Math. 56 (1991), 87-92.

[6] Kanas, S., Wishiwoska, A., Conic regions and k-uniform convexity, J. Comput. Appl. Math. Pure Appl. 105 (1999), 327-336.

[7] Kanas, S., Wishiowska, A., Conic region and k-starlike functions, Rev. Roum. Math., 45 (2000), 647-657.

[8] Miller, S.S., Mocanu, P.T., On some classes of first-order differential subordinations. Michigan Math. J. 32(2) (1985), 185–195.

[9] Mohammed, A., Darus, M., Sufficient conditions for new integral transformation. Malaysian J Fundamental Applied Sci., 8(5)(2012), 181–186.

[10] Owa, S., Nunokawa, M., Saitoh, H., Srivastava, H.M., Close-to-convexity, starlikeness, and convexity of certain analytic functions, Appl. Math. Lett. 15(1) (2002), 63–69.

[11] Porwal, S., Mapping properties of an integral operator. Acta Univ. Apulensis Math. Inform. 27 (2011), 151–155.

[12] Porwal, S. and Kumar, M., Mapping properties of an integral operator involving Bessel functions. Afrika Matematika 28(1)(2017), 165-170.

[13] Porwal, S., Singh, M.K., Mapping properties a general integral operator defined by the Hadamard product. Le Matematiche 69(1) (2014), 179-184.

[14] Porwal, S., Breaz, D., Mapping properties of an integral operator involving Bessel functions. In: Analytic number theory, Approximation theory, and Special functions 2014, pp. 821–826.

[15] Ravichandran, V., On uniformly convex functions. Ganita 53(2) (2002), 117–124.

[16] Robertson, M.S., On the theory of univalent functions. Ann. of Math. 37(2) (1936), 374–408.

[17] Silverman, H., Univalent functions with negative coefficients. Proc. Amer. Math. Soc. 51 (1975), 109–116.

[18] Singh, R.and Singh, S., Some sufficient conditions for univalence and starlikeness. Colloq. Math. 47(2) (1982), 309–314.

[19] Subramanian, K.G., Murugsundramoorthy, G., Balasubramaniyam, P., Silverman, H., Subclass of uniformly convex and uniformly starlike functions. Math. Japon., 42 (1995), 512-522.

[20] Szász , R., Kupán, P.A., About the univalence of the Bessel functions. Studia Univ. Babeş-Bolyai Math. 54(1) (2009), 127–132.

Received by the editors July 3, 2017
First published online July 21, 2017