Novi Sad J. Math.
Vol. 47, No. 2, 2017, 107-116

STRONG PROXIMINALITY IN METRIC SPACES

Sahil Gupta[1,2] **and T.D. Narang**[3]

Abstract. In this paper, we extend the notions of strong proximinality and strong Chebyshevity available in Banach spaces to metric spaces and prove that an approximatively compact subset W of a metric space X is strongly proximinal. Moreover, the converse holds if the set of best approximants in W to each point of the space X is compact. It is proved that strongly Chebyshev sets are precisely the sets which are strongly proximinal and Chebyshev. Further, by extending the notion of local uniform convexity from Banach spaces to metric spaces, it is proved that a proximinal convex subset of a locally uniformly convex metric space is approximatively compact. As a consequence, it is observed that closed balls in a locally uniformly convex metric space are strongly Chebyshev. The results proved in the paper generalize and extend several known results on the subject.

AMS Mathematics Subject Classification (2010): 41A50; 41A65; 46B20

Key words and phrases: Strongly proximinal set; approximatively compact set; strongly Chebyshev set; locally uniformly convex space; metric projection

1. Introduction

Let W be a non-empty closed subset of a metric space (X, d) and $x \in X$. An element $w_0 \in W$ is said to be a *best approximation* to x from W if

$$d(x, w_0) = \inf_{w \in W} d(x, w) \equiv d(x, W).$$

The set of all best approximants to x from W is denoted by $P_W(x)$, i.e., $P_W(x) = \{w_0 \in W : d(x, w_0) = d(x, W)\}$. The set W is called *proximinal* if $P_W(x) \neq \emptyset$ for every $x \in X$. If for each $x \in X$, $P_W(x)$ is a singleton then the set W is called *Chebyshev*.

A proximinal subset W of a metric space (X, d) is said to be *strongly proximinal* at $x \in X$ if for any minimizing sequence $\{y_n\} \subseteq W$ for x, i.e., $\lim_{n\to\infty} d(x, y_n) = d(x, W)$, there is a subsequence $\{y_{n_k}\}$ and a sequence $\{z_k\} \subseteq P_W(x)$ such that $d(y_{n_k}, z_k) \to 0$. Equivalently, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $y \in P_W(x, \delta)$, we can find $y' \in P_W(x)$ satisfying $d(y, y') < \varepsilon$, where $P_W(x, \delta) = \{w_0 \in W : d(x, w_0) < d(x, W) + \delta\}$.

[1]Department of Mathematics, Guru Nanak Dev University, Amritsar, India, e-mail: sahilmath@yahoo.in
[2]Corresponding author
[3]Department of Mathematics, Guru Nanak Dev University, Amritsar, India, e-mail: tdnarang1948@yahoo.co.in

The set W is said to be strongly proximinal in X if it is strongly proximinal at every point of X.

For Banach spaces, these two equivalent forms of strong proximinality have been introduced in [15] and [6] respectively.

The set W is said to be *strongly Chebyshev* (see [3]) if for any $x \in X$, every minimizing sequence $\{y_n\} \subseteq W$ for x is convergent in W.

The set W is said to be *approximatively compact* if for any $x \in X$, every minimizing sequence $\{y_n\} \subseteq W$ for x has a convergent subsequence in W.

For $\varepsilon > 0$, we define $\varepsilon-$neighborhood $V_\varepsilon(A)$ of a subset A of a metric space (X, d) by $V_\varepsilon(A) = \{x \in X : d(x, A) < \varepsilon\}$. In particular, $V_\varepsilon(A)$ is an open set containing A.

A mapping $P_W : X \to 2^W$ defined by $P_W(x) = \{w_0 \in W : d(x, w_0) = d(x, W)\}$ is called a *metric projection*.

The metric projection P_W is said to be *upper Hausdorff semi-continuous (u.H.s.c.)*(see [5]) at $x \in X$ if for every $\varepsilon > 0$, there exists an open neighborhood U of x such that $P_W(y) \subseteq V_\varepsilon(P_W(x))$ for every $y \in U$.

The metric projection P_W is said to be *upper semi-continuous (u.s.c.)* (see [5]) at $x \in X$ if for every open set $V \supseteq W$ such that $P_W(x) \subseteq V$, there exists an open neighborhood U of x such that $P_W(z) \subseteq V$ for all $z \in U$.

A metric space (X, d) is said to be *convex* if for any two distinct points x and y of X and every $t \in [0, 1]$, there exists at least one $z \in X$ such that

$$d(x, z) = (1 - t)d(x, y) \text{ and } d(z, y) = td(x, y). \tag{1.1}$$

A point z satisfying (1.1) is called a *between point* of x and y, and the set of all between points of x and y is denoted by $[x, y]$.

A subset W of a convex metric space (X, d) is said to be *convex* if for every $x, y \in W$, any point between x and y also lies in W.

A convex metric space (X, d) is said to be *strongly convex* (see [4]) or an $M - space$ (see [9]) if for any two distinct points x and y of X and every $t \in [0, 1]$, there exist exactly one $z \in X$ such that $d(x, z) = (1 - t)d(x, y)$ and $d(z, y) = td(x, y)$.

An M-space (X, d) is said to be *strictly convex* (see [12]) if for every pair x, y of X and $r > 0$ satisfying $d(x, p) \leq r$, $d(y, p) \leq r$ imply $d(z, p) < r$ unless $x = y$, where p is an arbitrary but fixed point of X and z is any point between x and y.

An M-space (X, d) is said to be *locally uniformly convex* if given $\varepsilon > 0$ and an element x with $d(x, p) \leq r$, there exists $\delta(\varepsilon, x) > 0$ such that $d(z, p) < r$ for all $y \in X$ with $d(x, y) \geq \varepsilon$ and $d(y, p) < r + \delta$, where z is any point between x and y, and p is arbitrary but fixed point of X.

It is easy to see that a locally uniformly convex metric space is strictly convex, but the converse is not true even in Banach spaces (see [11]).

Several researchers have discussed strongly proximinal and strongly Chebyshev sets in Banach spaces (see e.g. [3], [6], [15], [16] and references cited therein). By extending these notions and the notion of local uniform convexity from Banach spaces (see [11]) to metric spaces, we discuss relationships between proximinality, strong proximinality, approximative compactness and strong Chebyshevity in metric spaces and in locally uniformly convex metric spaces. We also prove that for a strongly proximinal subset W of a metric space, the metric projection P_W is u.H.s.c. Further, we show that if a metric space X is locally uniformly convex, then every proximinal convex subset of X is strongly Chebyshev, but the converse is not true. Moreover, we prove that the closed balls in a locally uniformly convex metric space are strongly Chebyshev. We also show that in an M-space (X, d), if a Chebyshev set W is strongly proximinal at $x \in X$ then W is strongly proximinal at every point between x and $P_W(x)$.

The results proved in this paper generalize and extend some results of [3], [6], [7], [9], [12] and [15].

2. Strong proximinality and approximative compactness

In this section, we discuss relationships between strong proximinality, approximative compactness and strong Chebyshevity in metric spaces and also prove that for a strongly proximinal set W of a metric space X, the metric projection P_W is u.H.s.c. We start with the following theorem.

Theorem 2.1. *A non-empty subset W of a metric space (X, d) is approximatively compact if and only if W is strongly proximinal and $P_W(x)$ is compact for every $x \in X$.*

Proof. Suppose W is approximatively compact, $x \in X$ and $\{y_n\} \subseteq W$ is any minimizing sequence for x, i.e.,

$$\lim_{n\to\infty} d(x, y_n) = d(x, W). \tag{2.1}$$

Then there exist a subsequence $\{y_{n_k}\}$ such that $\{y_{n_k}\} \to y \in W$. It follows from (2.1) that $y \in P_W(x)$ and so W is proximinal. The constant sequence $\{y\}$ satisfies the requirements of strong proximinality. Now, we show that $P_W(x)$ is compact. Let $\{y_n\}$ be any sequence in $P_W(x)$ then $d(x, y_n) = d(x, W)$ for all n and so

$$\lim_{n\to\infty} d(x, y_n) = d(x, W)$$

i.e., $\{y_n\} \subseteq W$ is a minimizing sequence for x. By the hypothesis, $\{y_n\}$ has a convergent subsequence converging to some point $y \in W$. Also $y \in P_W(x)$ and so $P_W(x)$ is compact.

Conversely, suppose that W is strongly proximinal and $P_W(x)$ is compact for every $x \in X$. Let $x \in X$ be arbitrary and $\{y_n\} \subseteq W$ be any minimizing sequence for x, i.e., $\lim_{n\to\infty} d(x, y_n) = d(x, W)$. Since W is strongly proximinal, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and a sequence $\{z_k\} \subseteq P_W(x)$ such that $d(y_{n_k}, z_k) \to 0$. Since $P_W(x)$ is compact, we may assume without loss of generality that $z_k \to z \in P_W(x)$. So, $\{y_{n_k}\} \to z \in W$. Hence W is approximatively compact. □

A closed subset W of a metric space (X, d) is said to be *quasi-Chebyshev* (see [8]) if $P_W(x)$ is non-empty and compact for all $x \in X$.

From Theorem 2.1, we obtain

Corollary 2.2. *A non-empty subset W of a metric space (X,d) is approximatively compact if and only if W is strongly proximinal and quasi-Chebyshev in X.*

The following theorem gives relationships between strong proximinality, approximative compactness and strong Chebyshevity.

Theorem 2.3. *Let W be a non-empty subset of a metric space (X,d). Then the following statements are equivalent:*
(i) W is strongly Chebyshev.
(ii) W is strongly proximinal and Chebyshev.
(iii) W is approximatively compact and Chebyshev.

Proof. (i) $\Rightarrow$ (ii). Since W is strongly Chebyshev, it is approximatively compact and so by Theorem 2.1, W is strongly proximinal. Now, we show that W is Chebyshev. Suppose that for some $x \in X$ there exist $w_1, w_2 \in P_W(x)$, $w_1 \neq w_2$. Then

$$d(x,w_1) = d(x,W) = d(x,w_2).$$

Consider the sequence $\{y_n\}$ in W such that $y_{2n} = w_1$ and $y_{2n+1} = w_2$. Then $\{y_n\}$ is a minimizing sequence for x in W. Since $w_1 \neq w_2$, $\{y_n\}$ is not convergent, a contradiction to strong Chebyshevity of W. Thus $w_1 = w_2$ and hence W is Chebyshev.
(ii) $\Rightarrow$ (iii). follows from Theorem 2.1.
(iii) $\Rightarrow$ (i). Let $x \in X$ be arbitrary and $\{y_n\} \subseteq W$ a minimizing sequence for x, i.e., $\lim_{n\to\infty} d(x,y_n) = d(x,W)$. Since W is approximatively compact, $\{y_n\}$ has a subsequence $\{y_{n_k}\} \to y_0 \in W$. Then $d(x,y_0) = d(x,W)$, i.e., $y_0 \in P_W(x)$. We claim that every subsequence of $\{y_n\}$ also converges to y_0. Suppose $\{y_n\}$ has a subsequence $\{y_{n_i}\}$ such that $\{y_{n_i}\} \to z_0$, $z_0 \neq y_0$. Then $d(x,z_0) = d(x,W)$, i.e., z_0 is also a best approximation to x in W. But W is Chebyshev and so $y_0 = z_0$, a contradiction. Therefore, every subsequence of $\{y_n\}$ converges to y_0 and hence $\{y_n\} \to y_0 \in W$. □

Remarks:

1. Whereas a strongly Chebyshev subset of a metric space is approximatively compact, an approximatively compact subset of a metric space need not be strongly Chebyshev.

   Let $X = \mathbb{R}^2$ with the usual metric and $W = \{(x,y) \in X : x^2 + y^2 = 1\}$. Then W, being a compact subset of the metric space X, is approximatively compact. But $P_W((0,0)) = W$, i.e., W is not Chebyshev. Therefore, it follows from Theorem 2.3 that W is not strongly Chebyshev.

2. Whereas an approximatively compact subset of a metric space is strongly proximinal, a strongly proximinal subset of a metric space (even of a Banach space) need not be approximatively compact.

   Let $X = l_\infty$, $W = c_0$. Then W is strongly proximinal in X but for $x = (1,1,1,...) \in l_\infty$, the sequence $\{y_n\}$ such that $y_n = (1,1,...,1(nth\ place),$

$0, 0, \ldots) \in W$ is a minimizing sequence for x having no convergent subsequence (see [3]).

3. A proximinal subset of a metric space need not be strongly proximinal. Even a proximinal convex subset of a Banach space space need not be strongly proximinal.

   Let $X = (l_1, \|.\|_H)$ Then the unit ball $B(X_H)$ is proximinal in X, but is not strongly proximinal in X (see [16]).

In view of the above results and remarks, we obtain that strong Chebyshevity $\Rightarrow$ approximative compactness $\Rightarrow$ strong proximinality $\Rightarrow$ proximinality, but none of the implications can be reversed.

Concerning strong Chebyshevity, we also have the following result.

Theorem 2.4. *Let W be a closed subset of a metric space (X, d) and $x \in X$. Then W is strongly Chebyshev for x if and only if diam $P_W(x, \delta) < \varepsilon$.*

Proof. If W is strongly Chebyshev for x then $P_W(x) = \{y_0\}$. Therefore, every minimizing sequence $\{y_n\} \subseteq W$ for x converges to y_0. Suppose that the given condition does not hold. Then there exists an $\varepsilon > 0$ and $z_n \in P_W(x, \frac{1}{n})$ such that $d(z_n, y_0) \geq \varepsilon$. This implies that $\{z_n\}$ is a minimizing sequence for x that does not converge to y_0, a contradiction. Hence *diam* $P_W(x, \delta) < \varepsilon$.

Conversely, suppose that $\{y_n\} \subseteq W$ is any minimizing sequence for x, i.e., $\lim_{n\to\infty} d(x, y_n) = d(x, W)$. Then for any $\delta > 0$, $y_n \in P_W(x, \delta)$ after some stage. This implies that for any $\delta > 0$, $d(y_n, y_{n+p}) \leq$ *diam* $P_W(x, \delta) < \varepsilon$ after some stage. This implies that the sequence $\{y_n\}$ is Cauchy. Since W is a closed subset of a complete metric space, W is complete. Therefore, $\{y_n\} \to y_0 \in W$, i.e., every minimizing sequence for x is convergent. Hence W is strongly Chebyshev.

Remarks. Theorems 2.1, 2.3 and 2.4 extend the corresponding Theorems 2.2, 2.3 and Proposition 2.7 of [3] (see also Proposition 3 of [15]) from Banach spaces to metric spaces, respectively. □

We require the following lemma in the proof of our next theorem showing that for strongly proximinal sets in metric spaces, the associated metric projection is u.H.s.c.

Lemma 2.5. *[5] Let W be a closed subset of a metric space (X, d) then the following statements are equivalent:*
(i) P_W is u.H.s.c. at x.
(ii) The relations $x_n \to x$ and $y_n \in P_W(x_n)$ imply $d(y_n, P_W(x)) \to 0$.

Theorem 2.6. *If W is a strongly proximinal subset of a metric space (X, d) then P_W is u.H.s.c..*

Proof. Let $x \in X$ be arbitrary, $x_n \to x$ and $y_n \in P_W(x_n)$. Then $d(x_n, y_n) = d(x_n, W)$ and so $\lim_{n\to\infty} d(x, y_n) = d(x, W)$. This implies that $y_n \in P_W(x, \delta)$ for any $\delta > 0$ after some stage. Since W is strongly proximinal at x, for any $\varepsilon > 0$, we can choose $\delta > 0$ such that for any $y_n \in P_W(x, \delta)$ we can find $y \in P_W(x)$ satisfying $d(y_n, y) < \varepsilon$. As $\varepsilon > 0$ is arbitrary, $d(y_n, P_W(x)) \to 0$. □

It is well-known (see [5]) that the mapping P_W is u.s.c. at x if and only if it is *u.H.s.c.* at x and $P_W(x)$ is compact. Therefore using Theorem 2.1 and Theorem 2.6, we obtain the following well-known result.

Corollary 2.7. *[14] If W is an approximatively compact subset of a metric space (X, d) then the metric projection P_W is u.s.c..*

Corollary 2.8. *If W is a strongly proximinal, quasi-Chebyshev subset of a metric space (X, d) then the metric projection P_W is u.s.c..*

Using Theorem 2.3, we obtain

Corollary 2.9. *If W is a strongly Chebyshev subset of a metric space (X, d) then the metric projection P_W is continuous.*

The converse of Theorem 2.6 is not true even if W is a Chebyshev subset of X.

Example 2.10. *(see [13]) Let X be the dual space of the normed linear space constructed by Klee [10] by suitable renorming of l^2. Lambert (unpublished) (see [13]) has shown that in the space X the metric projection P_W supported by any Chebyshev subspace W of X is continuous. However, X does not satisfy the Effimov-Steckin property and hence contains a closed hyperplane K which is not approximatively compact (see [14], Theorem 3). Since X is strictly convex and reflexive, K is Chebyshev and thus supports a continuous metric projection. Since K is Chebyshev, approximative compactness and strong proximinality are equivalent and so K is not strongly proximinal.*

Remarks. *It was erroneously stated in [6] (Lemma 4.1) that for a strongly proximinal subset W of a Banach space X, the associated metric projection P_W is u.s.c.. In fact, it is only u.H.s.c.. Theorem 2.6 extends this result to metric spaces.*

3. Strong proximinality in convex spaces

In this section, we prove that proximinality and approximative compactness are equivalent for convex sets in locally uniformly convex metric spaces. As a consequence, we prove that closed balls in a locally uniformly convex metric space are strongly Chebyshev. We also prove that in an M-space (X, d), if a Chebyshev set W is strongly proximinal at x, then W is strongly proximinal at every point between x and $P_W(x)$. The results proved in this section are motivated by the corresponding results proved for best approximation in Banach spaces given in [7] and in metric spaces given in [9] and [12]. Using the fact that a proximinal convex subset of a strictly convex metric space is Chebyshev (see [9]), we prove the following theorem:

Theorem 3.1. *A closed convex subset W of a complete locally uniformly convex metric space (X, d) is proximinal if and only if W is approximatively compact.*

Proof. Suppose W is proximinal and $x \in X$. Since a locally uniformly convex metric space is strictly convex, proximinal convex subsets of locally uniformly convex metric spaces are Chebyshev. Thus W is Chebyshev. Suppose, $P_W(x) = \{y'\}$. Let $\{y_n\} \subseteq W$ be any minimizing sequence for x, i.e.,

$$\lim_{n\to\infty} d(x, y_n) = d(x, W). \tag{3.1}$$

If $x \in W$ then $\lim_{n\to\infty} d(x, y_n) = 0$, i.e., $\{y_n\} \to x$ and so W is approximatively compact.

Now, suppose $x \in X\backslash W$ then $\lim_{n\to\infty} d(x, y_n) = d(x, W) \equiv r > 0$. Let $\varepsilon > 0$ be arbitrary and $\delta(\varepsilon, y')$ be taken as in the definition of local uniform convexity. Using (3.1), we can find $m \in \mathbb{N}$ such that $d(x, y_n) < r + \delta$ for all $n \geq m$. Let $p \geq m$ then $d(x, y_p) < r + \delta$.

Let z_p be a between point of y_p and y'. Then, by the convexity of the set W, we have $z_p \in W$, $d(x, z_p) \geq r$. Therefore, by the local uniform convexity of the space X, we obtain

$$d(y_p, y') < \varepsilon.$$

Similarly, we can choose y_l for $l > m$ such that

$$d(y_l, y') < \varepsilon.$$

Then $d(y_p, y_l) < 2\varepsilon$ for all $l, p \geq m$, i.e., $\{y_n\}$ is a Cauchy sequence in W. Now, W being a closed subset of complete metric space is complete, $\{y_n\}$ converges to some element of W and hence W is approximatively compact.

The converse is obvious. □

Since a proximinal convex subset of a locally uniformly convex metric space is Chebyshev, using Theorem 2.3 and Corollary 2.9, we obtain

Corollary 3.2. *Let W be a closed convex subset of a locally uniformly convex metric space (X, d) then the following are equivalent:*
(i) W is proximnal.
(ii) W is approximatively compact.
(iii) W is strongly Chebyshev.
(iv) W is Chebyshev and P_W is continuous.

Remark. *Theorem 3.1 extends Proposition 3 of [7] from Banach spaces to metric spaces.*

As a consequence of Theorem 3.1, we have

Theorem 3.3. *Let (X, d) be an M-space then for the statements*
(i) X is locally uniformly convex,
(ii) every proximinal convex subset of X is strongly Chebyshev,
(iii) closed balls in X are strongly Chebyshev,
we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. (i) $\Rightarrow$(ii). follows from Corollary 3.2.
(ii)$\Rightarrow$ (iii). Since every poximinal convex subset of X is strongly Chebyshev, it follows from Theorems 2.1 and 2.3 of [9] that the closed balls $B[z,r]$ are Chebyshev and convex for every $z \in X$ and $r > 0$. Therefore, the result follows. □

Now, it is natural to ask whether all the three statements of Theorem 3.3 are equivalent? We proceed to prove that $(ii) \not\Rightarrow (i)$ even in Banach spaces.

Let $S_X(S_{X^*})$ be the unit sphere and $B_X(B_{X^*})$ the closed unit ball in $X(X^*)$. Recall that a Banach space $(X, \|.\|)$ is said to be
(i) *strongly rotund* (see [7]) if given $x \in S_X, x^* \in S_{X^*}$ such that $x^*(x) = 1$ and $\{x_n\} \subseteq B_X$ a sequence in B_X such that $x^*(x_n) \to x^*(x) = 1$ then $\{x_n\}_{n\in\mathbb{N}}$ is convergent.
(ii) *almost locally uniform rotund* (ALUR) [1] if for every $x \in S_X, \{x_n\} \subseteq B_X$ and $\{x_n^*\} \subseteq B_{X^*}$, the condition $\lim_m \lim_n x_m^*(\frac{x_n+x}{2}) = 1$ imply $x_n \to x$.

An $x^* \in S_{X^*}$ is said to be *strongly exposing functional* (see [3]) if $x^* \in NA(X)$ and every sequence $\{x_n\} \subseteq B_X$ with $\lim_{n\to\infty} x^*(x_n) = 1$ is convergent where $NA(X)$ is the set of all norm attaining functionals in X^*.

Using the above definitions, we prove the following result:

Theorem 3.4. *Let X be Banach space. Then the following are equivalent:*
(i) X is strongly rotund.
(ii) Every $x^* \in NA(X)$ is strongly exposing functional.
(iii) Every proximinal convex subset of X is strongly Chebyshev.
(iv) X is almost locally uniform rotund.

Proof. (i) $\Rightarrow$(ii) Suppose that x^* attains its norm at $x \in S_X$, i.e., $x^*(x) = \|x^*\| = 1$. Let $\{x_n\} \subseteq B_X$ be such that $x^*(x_n) \to 1$. Since X is strongly rotund, $\{x_n : n \in \mathbb{N}\}$ is convergent.
(ii) $\Rightarrow$ (i) Let $x \in S_X, x^* \in S_{X^*}$ be such that $x^*(x) = 1$ and $\{x_n\} \subseteq B_X$ a sequence in B_X such that $x^*(x_n) \to x^*(x) = 1$. Since x^* is norm attaining, x^* is strongly exposing functional and so $\{x_n\}$ is convergent. Hence X is strongly rotund.
(i)$\Leftrightarrow$ (iii) The proof runs on similar lines as that of Theorem 5 in Guiro and Montesinos[7].
(ii) $\Leftrightarrow$ (iv) follows from Corollary 4.6 of [2]. □

Therefore, to prove $(ii) \not\Rightarrow (i)$ in Theorem 3.3, it is sufficient to show that almost locally uniformly rotund Banach space need not be locally uniformly convex. Since this is well-known (see [1]- Proposition 11 and Corollary 12), we obtain the desired result.

Remark. The authors do not know whether (iii)$\Rightarrow$ (ii) in Theorem 3.3.

We require the following lemma proved in [12] for our next theorem.

Lemma 3.5. *Let W be a Chebyshev subset of an M-space (X, d) and $x \in X$. If $P_W(x) = \{w_0\}$, then $P_W(y) = \{w_0\}$ for every $y \in [w_0, x)$.*

Theorem 3.6. *Let W be a Chebyshev subset of an M-space (X, d) and $x \in X$. If W is strongly proximinal at $x \in X$, then W is strongly proximinal at every point between x and $P_W(x)$.*

Proof. Let $\varepsilon > 0$ and $x \in X$. Then there exists some $y' \in W$ such that $P_W(x) = \{y'\}$. Since W is strongly proximinal at x, there exists a $\delta > 0$ such that for every $y \in P_W(x, \delta)$ there is $y' \in P_W(x)$ satisfying $d(y, y') < \varepsilon$.

Let $z \in X$ be such that

$$d(x, z) + d(z, y') = d(x, y'). \tag{3.2}$$

Then using Lemma 3.5, $y' \in P_W(z)$. Suppose $z' \in P_W(z, \delta)$. Then

$$d(z, z') < d(z, w) + \delta \ \textit{for all} \ w \in W. \tag{3.3}$$

We claim that $z' \in P_W(x, \delta)$. Consider

$$d(x, z') \leq d(x, z) + d(z, z') < d(x, z) + d(z, w) + \delta \ \textit{for all} \ w \in W \ \textit{using} \ (3.3)$$

$i.e.,\ d(x, z') < d(x, z) + d(z, y') + \delta,\ as\ y' \in W$. Using (3.2), we obtain

$$d(x, z') < d(x, y') + \delta$$

i.e., $d(x, z') < d(x, w) + \delta$ for all $w \in W$, as $y' \in P_W(x)$. Therefore, $z' \in P_W(x, \delta)$. Hence the claim holds. Since W is Chebyshev and strongly proximinal at x, for $y' \in P_W(x)$, we have $d(z', y') < \varepsilon$. Also $y' \in P_W(z)$. Therefore, for any $z' \in P_W(z, \delta)$ there exist $y' \in P_W(z)$ satisfying $d(z', y') < \varepsilon$. Hence W is strongly proximinal at z. □

Acknowledgement

The authors are thankful to the learned referee for valuable comments and suggestions leading to an improvement of the paper. The research work of the first author has been supported by University Grants Commission, India under Senior Research Fellowship.

References

[1] Bandyopadhyay, P., Huang, D., Lin, Bor-Luh, Troyanski, S.L., Some generalizations of locally uniform rotundity. J. Math. Anal. Appl. 252 (2000), 906-916.

[2] Bandyopadhyay, P., Huang, D., and Lin, Bor-Luh, Rotund points, nested sequences of balls and smoothness in Banach spaces. Comment. Math. Prace Mat. 44 (2004), 163-186.

[3] Bandyopadhyay P., Li, Y., Lin, Bor-Luh and Narayana, D., Proximinality in Banach Spaces. J. Math. Anal. Appl. 341 (2008), 309-317.

[4] Borsuk, K., On a metrization of polytopes. Fundamenta Mathematicae, 47 (1959), 325-341.

[5] Deutsch, F., Pollul, W. and Singer, I., On set-valued metric projections, Hahn-Banch extension maps, and spherical maps. Duke Math. J. 40 (1973), 355-370.

[6] Godefroy, G., Indumathi, V., Strong proximinality and polyhedral spaces. Rev. Mat. Complut., 14 (2001), 105-125.

[7] Guirao, A.J. and Montesinos, V., A note in approximative compactness and continuity of metric projections in Banach spaces. J. Convex Anal., 18 (2011), 397-401.

[8] Mohebi, H., On quasi-Chebyshev subspaces of Banach spaces. J. Approx. Theory, 107 (2000), 87-95.

[9] Khalil, R., Best approximation in metric spaces. Proc. Amer. Math. Soc. 103 (1988), 579-586.

[10] Klee, V., Two renorming constructions related to a question of Anselone. Studia Math. 33 (1969), 231-242.

[11] Lovaglia, A.R., Locally uniformly convex Banach spaces. Trans. Amer. Math. Soc. 78 (1955), 225-238.

[12] Narang, T.D., Best approximation in metric spaces. Publ. Sec. Mat. Univ. Autonoma Barcelona 27 (1983), 71-80.

[13] Panda, B.B., Kapoor, O.P., Approximative compatness and continuity of metric projection. Bull. Austral. Math. Soc., 11 (1974), 47-55.

[14] Singer, I., Some remarks on approximative compactness. Rev. Roumaine Math. Pures Appl. 9 (1964), 167-177.

[15] Vlasov, L.P., The concept of approximative compactness and its variants. Mat. Zametki 16 (1974), 337-348. English transl. in Math. Notes 16(2) (1974), 786-792.

[16] Zhang, Z.H., Liu, C.Y., Zhou, Z., Some examples concerning proximinality in Banach spaces. J. Approx. Theory 200 (2015), 136-143.

Received by the editors February 24, 2017
First published online November 7, 2017