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COUPLED FIXED POINT THEOREM IN $b-$FUZZY METRIC SPACES[1]

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Abstract. The aim of this paper is to prove a coupled coincidence fixed point theorem in complete $b-$fuzzy metric space. The results presented in this paper are generalizations of some well known, up to date research.

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1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [19] in 1965. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [7] , Kramosil and Michalek [10] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics, particularly in connections with both string and E-infinity theory which were given and studied by El Naschie [3, 4, 5, 6, 18]. For more information about the fuzzy metric and probabilistic metric spaces and fixed point theory in these spaces, we recommend [1, 8, 12, 13, 14, 17].

In this paper we dealt with a b-fuzzy metric spaces, and we proved a coupled coincidence point theorem in that spaces.

2. Preliminaries

This section we will start with the basic definitions and notations.

Definition 2.1. [9] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

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1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0, 1]$,
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Two typical examples of a continuous t-norm are $a * b = a \cdot b$ and $a * b = \min(a, b)$.

Further on, by a fuzzy set A on some universal set X we shall consider its membership function (see [19]). For the sake of simplicity, the membership function will be also denoted by A, i.e., $A : X \to [0, 1]$.

Definition 2.2. [9]A 3-tuple $(X, M, *)$ is called a *fuzzy metric space* if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1$ if and only if $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
5. $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

The function M is called a fuzzy metric.

Definition 2.3. [15] A 3-tuple $(X, M, *)$ is called a $b-$fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$, $t, s > 0$ and a given real number $b \geq 1$,

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1$ if and only if $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, \frac{t}{b}) * M(y, z, \frac{s}{b}) \leq M(x, z, t + s)$,
5. $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

The function M is called a b-fuzzy metric.

It should be noted that the class of $b-$fuzzy metric spaces is effectively larger than that of fuzzy metric spaces, since a $b-$fuzzy metric space is a fuzzy metric space when $b = 1$.

We present an example that shows that a $b-$fuzzy metric on X need not be a fuzzy metric on X.

Example 2.4. Let $M(x,y,t) = e^{\frac{-|x-y|^p}{t}}$, where $p > 1$ is a real number, and $a * b = a \cdot b$. We will show that $(X, M, *)$ is a $b-$fuzzy metric space with $b = 2^{p-1}$.

Obviously conditions (1), (2),(3) and (5) of Definition 2.3 are satisfied.

If $1 < p < \infty$, then the convexity of the function $f(x) = x^p$ $(x > 0)$ implies

$$\left(\frac{a+c}{2}\right)^p \leq \frac{1}{2}\left(a^p + c^p\right),$$

and hence, $(a + c)^p \leq 2^{p-1}(a^p + c^p)$ holds. Therefore,

$$\begin{aligned}
\frac{|x-y|^p}{t+s} &\leq 2^{p-1}\frac{|x-z|^p}{t+s} + 2^{p-1}\frac{|z-y|^p}{t+s} \\
&\leq 2^{p-1}\frac{|x-z|^p}{t} + 2^{p-1}\frac{|z-y|^p}{s} \\
&= \frac{|x-z|^p}{t/2^{p-1}} + \frac{|z-y|^p}{s/2^{p-1}}
\end{aligned}$$

Thus for each $x, y, z \in X$ we obtain

$$\begin{aligned}
M(x,y,t+s) &= e^{\frac{-|x-y|^p}{t+s}} \\
&\geq M(x,z,\frac{t}{2^{p-1}}) * M(z,y,\frac{s}{2^{p-1}}).
\end{aligned}$$

So condition (4) of Definition 2.3 holds and $(X, M, *)$ is a $b-$ fuzzy metric space.

It should be noted that in preceding example, for $p = 2$ $(X, M, *)$ is not a fuzzy metric space.

Example 2.5. Let $M(x,y,t) = e^{\frac{-d(x,y)}{t}}$ or $M(x,y,t) = \frac{t}{t+d(x,y)}$, where d is a b-metric on X and $a * c = a \cdot c$ for all $a, c \in [0,1]$. Then it is easy to show that $(X, M, *)$ is a $b-$fuzzy metric space.

Obviously conditions (1), (2),(3) and (5) of Definition 2.3 are satisfied. For each $x, y, z \in X$ we obtain

$$\begin{aligned}
M(x,y,t+s) &= e^{\frac{-d(x,y)}{t+s}} \\
&\geq e^{-b\frac{d(x,z)+d(z,y)}{t+s}} \\
&= e^{-b\frac{d(x,z)}{t+s}}.e^{-b\frac{d(z,y)}{t+s}} \\
&\geq e^{\frac{-d(x,z)}{t/b}}.e^{\frac{-d(z,y)}{s/b}} \\
&= M(x,z,\frac{t}{b}) * M(z,y,\frac{s}{b}).
\end{aligned}$$

So condition (4) of Definition 2.3 holds and $(X, M, *)$ is a $b-$ fuzzy metric space. Similarly, it is easy to see that for $M(x,y,t) = \frac{t}{t+d(x,y)}$, $(X, M, *)$ is a $b-$ fuzzy metric space.

Before stating and proving our results, we present a definition and a proposition in $b-$metric space.

Definition 2.6. [15] A function $f : \mathbb{R} \to \mathbb{R}$ is called $b-$nondecreasing, if $x > by$ implies $f(x) \geq f(y)$ for each $x, y \in \mathbb{R}$.

Lemma 2.7. *[15] Let $(X, M, *)$ be a $b-$fuzzy metric space. Then $M(x, y, t)$ is $b-$nondecreasing with respect to t, for all x, y in X. Also,*

$$M(x, y, b^n t) \geq M(x, y, t), \forall n \in \mathbb{N}.$$

Let $(X, M, *)$ be a $b-$fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

We recall the notions of convergence and completeness in a $b-$fuzzy metric space [15].

Let $(X, M, *)$ be a $b-$fuzzy metric space. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the $b-$fuzzy metric M). A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. The $b-$fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Lemma 2.8. *[15]In a $b-$fuzzy metric space $(X, M, *)$ the following assertions hold:*

(i) If the sequence $\{x_n\}$ in X converges to x, then x is unique,

(ii) If the sequence $\{x_n\}$ in X is converges to x, then $\{x_n\}$ is a Cauchy sequence.

In a $b-$fuzzy metric space we have the following Proposition.

Proposition 2.9. *[16, Prop. 1.10] Let $(X, M, *)$ be a $b-$fuzzy metric space and suppose that $\{x_n\}$ is b-convergent to x then we have*

$$M(x, y, \frac{t}{b}) \leq \limsup_{n\to\infty} M(x_n, y, t) \leq M(x, y, bt),$$

$$M(x, y, \frac{t}{b}) \leq \liminf_{n\to\infty} M(x_n, y, t) \leq M(x, y, bt).$$

Remark 2.10. In general, a $b-$fuzzy metric is not continuous.

Definition 2.11. [2] An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.12. [11] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 2.13. [11] Let X be a nonempty set. Then we say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if $gF(x, y) = F(gx, gy)$.

3. The Main Results

Let Φ denote the class of all functions $\phi : [0,1] \to [0,1]$ such that ϕ is increasing, continuous and let $\phi(t) > t$ for all $t \in (0,1)$.

Note that if $\phi(0) = 0$ and $\phi(1) = 1$ additionally hold, then $\phi(t) \geq t$, $t \in [0,1]$, for all functions from Φ.

We start our work by proving the following crucial lemma.

Lemma 3.1. *Let $(X, M, *)$ be a b-fuzzy metric space with $b \geq 1$ and let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that*

$$M(F(x,y), F(u,v), t) \geq \phi(\min\{M(gx, gu, t), M(gy, gv, t)\}), \tag{3.1}$$

for some $\phi \in \Phi$ and for all $x, y, u, v \in X$ and $t > 0$. Assume that (x, y) is a coupled coincidence point of the mappings F and g. Then $F(x,y) = gx = gy = F(y,x)$.

Proof. Since (x, y) is a coupled coincidence point of the mappings F and g, we have $gx = F(x,y)$ and $gy = F(y,x)$. Assume $gx \neq gy$. Then by (3.1), we get

$$\begin{aligned} M(gx, gy, t) &= M(F(x,y), F(y,x), t) \geq \phi(\min\{M(gx, gy, t), M(gy, gx, t)\}) \\ &= \phi(M(gx, gy, t)) \\ &> M(gx, gy, t), \end{aligned}$$

which is a contradiction, since the values of M can not be either 0 or 1. So $gx = gy$, and hence $F(x,y) = gx = gy = F(y,x)$. □

The following is the main result of this section.

Theorem 3.2. *Let $(X, M, *)$ be a complete b-fuzzy metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that*

$$M(F(x,y), F(u,v), t) \geq \phi(\min\{M(gx, gu, b^2 t), M(gy, gv, b^2 t)\}) \tag{3.2}$$

for all $x, y, u, v \in X$ and $t > 0$. Assume that F and g satisfy the following conditions:

1. *$F(X \times X) \subseteq g(X)$,*
2. *$g(X)$ is complete, and*
3. *g is continuous and commutes with F.*

If $\phi \in \Phi$, then there is a unique x in X such that $gx = F(x,x) = x$.

Proof. Let $x_0, y_0 \in X$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X

such that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$. For $n \in \mathbb{N} \cup \{\mathbf{0}\}$, by (3.2) we have

$$\begin{aligned} M(gx_{n-1}, gx_n, t) &= M(F(x_{n-2}, y_{n-2}), F(x_{n-1}, y_{n-1}), t) \\ &\geq \phi(\min\{M(gx_{n-2}, gx_{n-1}, b^2 t), M(gy_{n-2}, gy_{n-1}, b^2 t)\}). \end{aligned}$$

Similarly by (3.2) we have

$$\begin{aligned} M(gy_{n-1}, gy_n, t) &= M(F(y_{n-2}, x_{n-2}), F(y_{n-1}, x_{n-1}), t) \\ &\geq \phi(\min\{M(gy_{n-2}, gy_{n-1}, b^2 t), M(gx_{n-2}, gx_{n-1}, b^2 t)\}). \end{aligned}$$

Hence, we have

$$\begin{aligned} a_n(t) &= \min\{M(gx_{n-1}, gx_n, t), M(gy_{n-1}, gy_n, t)\} \\ &\geq \phi(\min\{M(gx_{n-2}, gx_{n-1}, b^2 t), M(gy_{n-2}, gy_{n-1}, b^2 t)\}) \\ &= \phi(a_{n-1}(b^2 t)) \end{aligned}$$

holds for all $n \in \mathbb{N}$. Thus, we get that

$$a_n(t) \geq \phi(a_{n-1}(b^2 t)) > a_{n-1}(b^2 t) \geq a_{n-1}(t).$$

Thus $\{a_n(t)\}$ is an increasing sequence in $[0, 1]$ for every $t > 0$. Therefore, $\{a_n(t)\}$ tends to a limit $a(t) \leq 1$. We claim that $a(t) = 1$. For if $a(t) < 1$, letting $n \to \infty$ in the above inequality we get $a(t) \geq \phi(a(b^2 t)) > a(b^2 t) \geq a(t)$, a contradiction. Hence $a(t) = 1$, i.e.,

$$\lim_{n\to\infty} M(gx_n, gx_{n+1}, t) = 1, \ \lim_{n\to\infty} M(gy_n, gy_{n+1}, t) = 1.$$

Now, we prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequence in $g(X)$ for $n = 1, 2, 3, \ldots$.

First, we prove that for every $\varepsilon \in (0, 1)$, there exist two numbers $n, m \in \mathbb{N}$ such that

$$\min\{M(gx_n, gx_m, t), M(gy_n, gy_m, t)\} > 1 - \varepsilon.$$

Suppose that this is not true. Then there is an $\varepsilon \in (0, 1)$ such that for each integer k, there exist integers $m(k)$ and $n(k)$ with $m(k) > n(k) \geq k$ such that

(3.3)

$$\min\{M(gx_{n(k)}, gx_{m(k)}, t), M(gy_{n(k)}, gy_{m(k)}, t)\} \leq 1-\varepsilon \quad \text{for} \quad k = 1, 2, \cdots.$$

We may assume that

$$\min\{M(gx_{n(k)}, gx_{m(k)-1}, t), M(gy_{n(k)}, gy_{m(k)-1}, t)\} > 1 - \varepsilon, \tag{3.4}$$

by choosing $m(k)$ be the smallest number exceeding $n(k)$ for which (3.3) holds.

Let

$$d_k(t) = \min\{M(gx_{n(k)}, gx_{m(k)}, t), M(gy_{n(k)}, gy_{m(k)}, t)\}.$$

Using (3.3), and the fact that $a * b \geq \min\{a, c\} * \min\{b, d\}$ we have

$$
\begin{aligned}
&1-\varepsilon \\
&\geq \quad d_k(t) \geq \min\{M(gx_{n(k)}, gx_{m(k)-1}, \frac{t}{2b}) * M(gx_{m(k)-1}, gx_{m(k)}, \frac{t}{2b}), \\
&\qquad M(gy_{n(k)}, gy_{m(k)-1}, \frac{t}{2b}) * M(gy_{m(k)-1}, gy_{m(k)}, \frac{t}{2b})\} \\
&\geq \quad \min\{\min\{M(gx_{n(k)}, gx_{m(k)-1}, \frac{t}{2b}), M(gy_{n(k)}, gy_{m(k)-1}, \frac{t}{2b})\} \\
&* \quad \min\{M(gx_{m(k)-1}, gx_{m(k)}, \frac{t}{2b}), M(gy_{m(k)-1}, gy_{m(k)}, \frac{t}{2b})\}, \\
&\qquad \min\{M(gx_{n(k)}, gx_{m(k)-1}, \frac{t}{2b}), M(gy_{n(k)}, gy_{m(k)-1}, \frac{t}{2b})\} \\
&* \quad \min\{M(gx_{m(k)-1}, gx_{m(k)}, \frac{t}{2b}), M(gy_{m(k)-1}, gy_{m(k)}, \frac{t}{2b})\} \\
&\geq \quad \min\{M(gx_{m(k)-1}, gx_{m(k)}, \frac{t}{2b}), M(gy_{m(k)-1}, gy_{m(k)}, \frac{t}{2b})\} * a_k(\frac{t}{2b}),
\end{aligned}
$$

Thus, as $k \to \infty$ in the above inequality we have

$$1-\varepsilon \geq \lim_{k\to\infty} d_k(t) \geq (1-\varepsilon) * \lim_{k\to\infty} a_k(\frac{t}{2b}) = 1-\varepsilon,$$

that is

$$\lim_{k\to\infty} d_k(t) = 1-\varepsilon,$$

for every $t > 0$.

On the other hand, we have

$$
\begin{aligned}
&d_k(t) \\
&\geq \quad \min\{M(gx_{n(k)}, gx_{n(k)+1}, \frac{t}{3b}) * M(gx_{n(k)+1}, gx_{m(k)+1}, \frac{t}{3b}) \\
&* \quad M(gx_{m(k)+1}, gx_{m(k)}, \frac{t}{3b})\}, \{M(gy_{n(k)}, gy_{n(k)+1}, \frac{t}{3b}) \\
&* \quad M(gy_{n(k)+1}, gy_{m(k)+1}, \frac{t}{3b}) * M(gy_{m(k)+1}, gy_{m(k)}, \frac{t}{3b})\} \\
&\geq \quad \min\{\min\{M(gx_{n(k)}, gx_{n(k)+1}, \frac{t}{3b}), M(gy_{n(k)}, gy_{n(k)+1}, \frac{t}{3b})\} \\
&* \quad \min\{M(gx_{n(k)+1}, gx_{m(k)+1}, \frac{t}{3b}), M(gy_{n(k)+1}, gy_{m(k)+1}, \frac{t}{3b})\} \\
&* \quad \min\{M(gx_{m(k)+1}, gx_{m(k)}, \frac{t}{3b}), M(gy_{m(k)+1}, gy_{m(k)}, \frac{t}{3b})\}, \\
&\qquad \min\{M(gx_{n(k)}, gx_{n(k)+1}, \frac{t}{3b}), M(gy_{n(k)}, gy_{n(k)+1}, \frac{t}{3b})\} \\
&* \quad \min\{M(gx_{n(k)+1}, gx_{m(k)+1}, \frac{t}{3b}), M(gy_{n(k)+1}, gy_{m(k)+1}, \frac{t}{3b})\}
\end{aligned}
$$

$$
\begin{aligned}
&* \quad \min\{M(gx_{m(k)+1}, gx_{m(k)}, \frac{t}{3b}), M(gy_{m(k)+1}, gy_{m(k)}, \frac{t}{3b})\}\\
&= \quad \min\{M(gx_{n(k)}, gx_{n(k)+1}, \frac{t}{3b}), M(gy_{n(k)}, gy_{n(k)+1}, \frac{t}{3b})\}\\
&* \quad \min\{M(gx_{n(k)+1}, gx_{m(k)+1}, \frac{t}{3b}), M(gy_{n(k)+1}, gy_{m(k)+1}, \frac{t}{3b})\}\\
&* \quad \min\{M(gx_{m(k)+1}, gx_{m(k)}, \frac{t}{3b}), M(gy_{m(k)+1}, gy_{m(k)}, \frac{t}{3b})\}\\
&\geq \quad a_k(\frac{t}{3b}) * \min\{M(gx_{n(k)+1}, gx_{m(k)+1}, \frac{t}{3b}),\\
&\qquad M(gy_{n(k)+1}, gy_{m(k)+1}, \frac{t}{3b})\} * a_k(\frac{t}{3b})\\
&= \quad a_k(\frac{t}{3b}) * \min\left\{\begin{array}{l} M(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)}), \frac{t}{3b}),\\ M(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}), \frac{t}{3b})\end{array}\right\} * a_k(\frac{t}{3b})\\
&\geq \quad a_k(\frac{t}{3b}) * \min\{\phi(M(gx_{n(k)}, gx_{m(k)}, \frac{tb}{3}), M(gy_{n(k)}, gy_{m(k)}, \frac{tb}{3}))\} * a_k(\frac{t}{3b})\\
&= \quad a_k(\frac{t}{3b}) * \phi(d_k(\frac{tb}{3})) * a_k(\frac{t}{3b}).
\end{aligned}
$$

Thus, as $k \to \infty$ in the above inequality we have

$$1 - \varepsilon \geq 1 * \phi(1-\varepsilon) * 1 = \phi(1-\varepsilon) > 1 - \varepsilon,$$

which is a contradiction.

Thus $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, we obtain $\{gx_n\}$ and (gy_n) are convergent to some $x \in X$ and $y \in X$, respectively. Since g is continuous, we have $\{ggx_n\}$ is convergent to gx and $\{ggy_n\}$ is convergent to gy. Also, since g and F do commute, we have

$$ggx_{n+1} = g(F(x_n, y_n)) = F(gx_n, gy_n),$$

and

$$ggy_{n+1} = g(F(y_n, x_n)) = F(gy_n, gx_n).$$

Thus

$$
\begin{aligned}
M(ggx_{n+1}, F(x,y), t) &= M(F(gx_n, gy_n), F(x,y), t)\\
&\geq \phi(\min\{M(ggx_n, gx, b^2t), M(ggy_n, gy, b^2t)\}).
\end{aligned}
$$

Letting $n \to \infty$, and using the Proposition 2.9, we get that

$$
\begin{aligned}
M(gx, F(x,y), bt) &\geq \limsup_{n\to\infty} M(F(gx_n, gy_n), F(x,y), t)\\
&\geq \limsup_{n\to\infty} \phi(\min\{M(ggx_n, gx, b^2t), M(ggy_n, gy, b^2t)\})\\
&\geq \phi(\min\{M(gx, gx, bt), M(gy, gy, bt)\}) = 1.
\end{aligned}
$$

Hence $gx = F(x,y)$. Similarly, we may show that $gy = F(y,x)$. By Lemma 3.1, (x,y) is coupled fixed point of the mappings F and g. So

$$gx = F(x,y) = F(y,x) = gy.$$

Thus, using the Proposition 2.9 we have

$$\begin{aligned} M(x, gx, bt) &\geq \limsup_{n\to\infty} M(gx_{n+1}, gx, t) \\ &= \limsup_{n\to\infty} M(F(x_n, y_n), F(x, y), t) \\ &\geq \limsup_{n\to\infty} \phi(\min\{M(gx_n, gx, b^2t), M(gy_n, gy, b^2t)\}) \\ &\geq \phi(\min\{M(x, gx, bt), M(y, gy, bt)\}). \end{aligned}$$

Hence, we get

$$M(x, gx, bt) \geq \phi(\min\{M(x, gx, bt), M(y, gy, bt)\}).$$

Similarly, we may show that

$$M(y, gy, bt) \geq \phi(\min\{M(x, gx, bt), M(y, gy, bt)\}).$$

Thus

$$\begin{aligned} \min\{M(x, gx, bt), M(y, gy, bt)\} &\geq \phi(\min\{M(x, gx, bt), M(y, gy, bt)\}) \\ &> \min\{M(x, gx, bt), M(y, gy, bt)\}. \end{aligned}$$

The last inequality happened only if $M(x, gx, t) = 1$ and $M(y, gy, t) = 1$. Hence $x = gx$ and $y = gy$. Thus we get

$$gx = F(x, x) = x.$$

To prove the uniqueness, let $z \in X$ with $z \neq x$ such that

$$z = gz = F(z, z).$$

Then

$$\begin{aligned} M(x, z, t) &= M(F(x, x), F(z, z), t) \\ &\geq \phi(\min\{M(gx, gz, b^2t), M(gx, gz, b^2t)\}) \\ &= \phi(M(gx, gz, b^2t)) \\ &> M(gx, gz, b^2t) = M(x, z, b^2t) \\ &\geq M(x, z, t). \end{aligned}$$

We get $M(x, z, t) > M(x, z, t)$, which is a contradiction. Thus F and g have a unique common fixed point. □

Corollary 3.3. *Let $(X, M, *)$ be a complete b-fuzzy metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that*

$$M(F(x, y), F(u, v), t) \geq \sqrt{\min\{M(gx, gu, b^2t), M(gy, gv, b^2t)\}} \tag{3.5}$$

for all $x, y, u, v \in X$ and $t > 0$. Assume that F and g satisfy the following conditions:

1. *$F(X \times X) \subseteq g(X)$,*

2. *$g(X)$ is complete, and*

3. *g is continuous and commutes with F.*

Then there is a unique x in X such that $gx = F(x,x) = x$.

Corollary 3.4. *Let $(X, M, *)$ be a complete b-fuzzy metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that*

$$\begin{aligned} M(F(x,y), F(u,v), t) & \\ \geq & \ 2\min\{M(gx, gu, b^2t), M(gy, gv, b^2t)\} \\ & -(\min\{M(gx, gu, b^2t), M(gy, gv, b^2t)\})^2 \end{aligned} \tag{3.6}$$

for all $x, y, u, v \in X$ and $t > 0$. Assume that F and g satisfy the following conditions:

1. *$F(X \times X) \subseteq g(X)$,*

2. *$g(X)$ is complete, and*

3. *g is continuous and commutes with F.*

Then there is a unique x in X such that $gx = F(x,x) = x$.

Proof. It is enough to set $\phi(t) = 2t - t^2$ in Theorem 3.2. □

Corollary 3.5. *Let $(X, M, *)$ be a complete fuzzy metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that*

$$M(F(x,y), F(u,v), t) \geq \phi(\min\{M(gx, gu, t), M(gy, gv, t)\}) \tag{3.7}$$

for all $x, y, u, v \in X$ and $t > 0$. Assume that F and g satisfy the following conditions:

1. *$F(X \times X) \subseteq g(X)$,*

2. *$g(X)$ is complete, and*

3. *g is continuous and commutes with F.*

If $\phi \in \Phi$, then there is a unique x in X such that $gx = F(x,x) = x$.

Proof. It is enough to set $b = 1$ in Theorem 3.2. □

Now we give an example to support our Theorem 3.2.

Example 3.6. Let $X = [0, 1]$ and $a * c = ac$ for all $a, c \in [0, 1]$ and let M be the $b-$fuzzy set on $X \times X \times (0, +\infty)$ defined as follows:

$$M(x, y, t) = e^{\frac{-(x-y)^2}{t}},$$

for all $t \in \mathbb{R}^+$. Then $(X, M, *)$ is a $b-$fuzzy metric space for $b = 2$. Define $g(x) = \frac{x}{4}$, $F(x, y) = \frac{2x+y}{32\sqrt{2}}$ and $\phi(t) = \sqrt{t}$, , for $t > 0$. It is evident that $F(X \times X) \subseteq g(X)$, g is continuous.

Since,

$$\begin{aligned}
(\frac{2x-2u}{32\sqrt{2}} + \frac{y-v}{32\sqrt{2}})^2 &\leq \frac{2}{32}[(\frac{x}{4} - \frac{u}{4})^2 + (\frac{y}{4} - \frac{v}{4})^2] \\
&= \frac{1}{16}[(\frac{x}{4} - \frac{u}{4})^2 + (\frac{y}{4} - \frac{v}{4})^2] \\
&\leq \frac{2}{16}\max\{(\frac{x}{4} - \frac{u}{4})^2, (\frac{y}{4} - \frac{v}{4})^2\} \\
&= \frac{1}{8}\max\{(\frac{x}{4} - \frac{u}{4})^2, (\frac{y}{4} - \frac{v}{4})^2\},
\end{aligned}$$

hence it follows that

$$\begin{aligned}
M(F(x, y), F(u, v), t) &= e^{\frac{-(\frac{2x+y}{32\sqrt{2}} - \frac{2u+v}{32\sqrt{2}})^2}{t}} \\
&= e^{\frac{-(\frac{2x-2u}{32\sqrt{2}} + \frac{y-v}{32\sqrt{2}})^2}{t}} \\
&\geq e^{\frac{-[(\frac{x}{4} - \frac{u}{4})^2 + (\frac{y}{4} - \frac{v}{4})^2]}{8t}} \\
&\geq e^{\frac{-\max\{(\frac{x}{4} - \frac{u}{4})^2, (\frac{y}{4} - \frac{v}{4})^2\}}{8t}} \\
&= \sqrt{e^{\frac{-\max\{(\frac{x}{4} - \frac{u}{4})^2, (\frac{y}{4} - \frac{v}{4})^2\}}{4t}}} \\
&= \sqrt{\min\{e^{\frac{-(\frac{x}{4} - \frac{u}{4})^2}{4t}}, e^{\frac{-(\frac{y}{4} - \frac{v}{4})^2}{4t}}\}} \\
&= \sqrt{\min\{M(gx, gu, 4t), M(gy, gv, 4t)\}}
\end{aligned}$$

for all x, y, u, v in X. Thus all the conditions of last theorem 3.2 are satisfied and 0 is a unique point in X such that $g0 = F(0, 0) = 0$.

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