

FACTORIZATION OF OPERATORS WITH $G_\alpha^\alpha(\mathbb{R}_+^d)$ AND $g_\alpha^\alpha(\mathbb{R}_+^d)$ KERNELS

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Abstract. The aim of this paper is to prove that any linear operator with kernel in the spaces $G_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$ and $g_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha > 1$ is a composition of two operators in the same space.

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1. Introduction

The test space $\mathcal{S}(\mathbb{R}_+)$ for the space of tempered distributions supported by $[0, \infty)$ is studied in [3], [10] and [13]; recently, the space $\mathcal{S}(\mathbb{R}_+^d)$ is examined in [5]. In [6] G -type spaces, $G_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$ and their dual spaces, i.e. the spaces of ultradistributions of Roumier type over $[0, \infty)^d$, are characterized in terms of their Fourier-Laguerre coefficients; cf. Duran [4] for the one-dimensional case. Actually, the result of [4] is extended and the full topological characterization is given in all dimensions, as well as applications to pseudo-differential operators with radial symbols.

In this paper we introduce g -type spaces, $g_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha > 1$ and their dual spaces i.e. the spaces of ultradistributions of Beurling type over $[0, \infty)^d$ (cf. [7]). We give the kernel theorem for $g_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha > 1$.

For any topological vector space \mathcal{B} , the set of continuous linear functionals on \mathcal{B} , denoted by $\mathcal{M} = \mathcal{L}(\mathcal{B})$, is a factorization algebra (also the term decomposition algebra can be used). This means that any operator T in \mathcal{M} is a composition of two operators T_1, T_2 in \mathcal{M} since we can choose T_1 as the identity operator and $T_2 = T$. If \mathcal{B} is a Hilbert space, then it follows from spectral decomposition that the set of compact operators on \mathcal{B} is a factorization algebra, where the factorization properties are obtained by straightforward applications of the spectral theorem.

An interesting subclass of linear and continuous operators on an L^2 space concerns the set of all linear operators whose kernels belong to the Schwartz space (see e.g. [1], [8],[12]). Similar facts hold true for the set of operators with kernels in Gelfand-Shilov spaces (cf. [11]) and Pilipović spaces (cf. [2]).

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In this paper we consider the case when \mathcal{M} is the set of all linear operators with kernels in $G_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$ and $g_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha > 1$ spaces. We prove that any such \mathcal{M} is a factorization algebra. Note that the identity operator does not belong to these operator classes.

2. Preliminaries

We denote by \mathbb{N} , \mathbb{Z} , \mathbb{R} and \mathbb{C} the sets of positive integers, integers, real and complex numbers, respectively; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}_+^d = (0, \infty)^d$ and $\overline{\mathbb{R}_+^d} = [0, \infty)^d$. We use the standard multi-index notation. Let $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $k = (k_1, \dots, k_d) \in \mathbb{N}_0^d$. Then $|x| = \sqrt{x_1^2 + \dots + x_d^2}$, $|k| = k_1 + \dots + k_d$, $k! = k_1! \cdots k_d!$, $x^k = \prod_{i=1}^d x_i^{k_i}$, $D^k = \prod_{i=1}^d \partial^{k_i} / \partial x_i^{k_i}$. Furthermore, if $x, \gamma \in \overline{\mathbb{R}_+^d}$ we also use $x^\gamma = \prod_{j=1}^d x_j^{\gamma_j}$. In this case, if $x_j = 0$ and $\gamma_j = 0$, we use the convention $0^0 = 1$.

For $j \in \mathbb{N}_0$ and $\gamma > -1$, the j -th Laguerre polynomial of order γ is defined by

$$L_j^\gamma(x) = \frac{x^{-\gamma} e^x}{j!} \frac{d^j}{dx^j} (e^{-x} x^{\gamma+j}), \quad x \geq 0.$$

For $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{R}^d$ such that $\gamma_j > -1$, $j = 1, \dots, d$ and $n \in \mathbb{N}_0^d$, the d -dimensional n -th Laguerre polynomial of order γ is defined by $L_n^\gamma(x) = L_{n_1}^{\gamma_1}(x_1) \cdots L_{n_d}^{\gamma_d}(x_d)$. For $\gamma = 0$, we write $L_n(x)$ instead of $L_n^0(x)$.

The j -th Laguerre function (of order 0) is defined by $l_j(x) = L_j(x)e^{-x/2}$, $x \geq 0$, $j \in \mathbb{N}_0$ and in a d -dimensional case we have $l_n(x) = l_{n_1}(x_1) \cdots l_{n_d}(x_d)$, $x \in \overline{\mathbb{R}_+^d}$, $n \in \mathbb{N}_0^d$. The Laguerre functions form an orthonormal basis for $L^2(\mathbb{R}_+^d)$. Also, they have a special role for the characterisation of the spaces $G_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$ and $g_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha > 1$ considered below.

2.1. Basic spaces

We denote by \mathbb{R}_+^d the set $(0, \infty)^d$ and by $\overline{\mathbb{R}_+^d}$ its closure, i.e. $[0, \infty)^d$. The space $\mathcal{S}(\mathbb{R}_+^d)$ consists of all $f \in C^\infty(\overline{\mathbb{R}_+^d})$ such that all derivatives $D^p f$, $p \in \mathbb{N}_0^d$, extend to continuous functions on $\overline{\mathbb{R}_+^d}$ and

$$\sup_{x \in \mathbb{R}_+^d} x^k |D^p f(x)| < \infty, \quad \forall k, p \in \mathbb{N}_0^d.$$

Let $A > 0$. We denote by $G_{\alpha, A}^{\alpha, A}(\mathbb{R}_+^d)$ the space of all $f \in \mathcal{S}(\mathbb{R}_+^d)$ for which

$$\sup_{p, k \in \mathbb{N}_0^d} \frac{\|t^{(p+k)/2} D^p f(t)\|_2}{A^{|p+k|} k^{(\alpha/2)k} p^{(\alpha/2)p}} < \infty.$$

With the following seminorms

$$\sigma_{A, j}(f) = \sup_{p, k \in \mathbb{N}_0^d} \frac{\|t^{(p+k)/2} D^p f(t)\|_{L^2(\mathbb{R}_+^d)}}{A^{|p+k|} k^{(\alpha/2)k} p^{(\alpha/2)p}} + \sup_{\substack{|p| \leq j \\ |k| \leq j}} \sup_{t \in \mathbb{R}_+^d} |t^k D^p f(t)|, \quad j \in \mathbb{N}_0,$$

one easily verifies that it becomes an (F) -space.

Define $G_\alpha^\alpha(\mathbb{R}_+^d) = \varinjlim_{A \rightarrow \infty} G_{\alpha,A}^{\alpha,A}(\mathbb{R}_+^d)$ and $g_\alpha^\alpha(\mathbb{R}_+^d) = \varprojlim_{A \rightarrow 0} G_{\alpha,A}^{\alpha,A}(\mathbb{R}_+^d)$. Clearly, $G_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$ and $g_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha > 1$ are continuously injected into $\mathcal{S}(\mathbb{R}_+^d)$ i.e.

$$g_\alpha^\alpha(\mathbb{R}_+^d) \hookrightarrow G_\alpha^\alpha(\mathbb{R}_+^d) \hookrightarrow \mathcal{S}(\mathbb{R}_+^d).$$

We immediately have

$$(2.1) \quad (G_\alpha^\alpha(\mathbb{R}_+^d))' = \bigcup_{A \in \mathbb{N}} (G_{\alpha,A}^{\alpha,A}(\mathbb{R}_+^d))'$$

in the set theoretical sense (see [9, (1.2), p.34]). Since the sequence $G_{\alpha,A}^{\alpha,A}(\mathbb{R}_+^d)$, when $A \rightarrow \infty$, is reduced, we have

$$(2.2) \quad (g_\alpha^\alpha(\mathbb{R}_+^d))' = \bigcap_{A \in \mathbb{N}} (G_{\alpha,1/A}^{\alpha,1/A}(\mathbb{R}_+^d))'$$

in the set theoretical sense (see [9, (1.1), p.33]).

Let $\alpha \geq 1$ and $a > 1$. We define $s^{\alpha,a}$ as the space of all complex sequences $\{a_n\}_{n \in \mathbb{N}_0^d}$ for which $\|\{a_n\}_{n \in \mathbb{N}_0^d}\|_{s^{\alpha,a}} = \sup_{n \in \mathbb{N}_0^d} |a_n| a^{|n|^{1/\alpha}} < \infty$. With this norm $s^{\alpha,a}$ becomes a (B) -space. We define $s^\alpha = \varinjlim_{a \rightarrow 1^+} s^{\alpha,a}$ (resp. $\sigma^\alpha = \varprojlim_{a \rightarrow \infty} \sigma^{\alpha,a}$).

In particular, s^α is a (DFN) -space (resp. σ^α is a (FN) -space). The strong dual $(s^\alpha)'$ of s^α is an (FN) -space of all complex valued sequences $\{b_n\}_{n \in \mathbb{N}_0^d}$ such that, for each $a > 1$, $\|\{b_n\}_{n \in \mathbb{N}_0^d}\|_{(s^{\alpha,a})'} = \sum_{n \in \mathbb{N}_0^d} |b_n| a^{-|n|^{1/\alpha}} < \infty$ (resp. the strong dual $(\sigma^\alpha)'$ of σ^α is a (DFN) -space of all complex valued sequences $\{b_n\}_{n \in \mathbb{N}_0^d}$ such that there exists $a > 1$ such that $\|\{b_n\}_{n \in \mathbb{N}_0^d}\|_{(\sigma^{\alpha,a})'} < \infty$).

Theorem 2.1. ([6, Theorem 5.7.]) *Let $\alpha \geq 1$. For $f \in L^2(\mathbb{R}_+^d)$ let $a_n = \int_{\mathbb{R}_+^d} f(t) l_n(t) dt$, $n \in \mathbb{N}_0^d$. Then $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$ if and only if there exist $c > 0$ and $a > 1$ such that $|a_n| \leq ca^{-|n|^{1/\alpha}}$.*

With small modifications of the arguments (using the closed graph theorem for an F -space), we have the next theorem

Theorem 2.2. *Let $\alpha > 1$. For $f \in L^2(\mathbb{R}_+^d)$ let $a_n = \int_{\mathbb{R}_+^d} f(t) l_n(t) dt$, $n \in \mathbb{N}_0^d$. Then $f \in g_\alpha^\alpha(\mathbb{R}_+^d)$ if and only if for every $a > 1$ there exists $c > 0$ such that $|a_n| \leq ca^{-|n|^{1/\alpha}}$.*

Theorem 2.3. ([6, Theorem 6.1]) *Let $\alpha \geq 1$. The mapping $\iota : G_\alpha^\alpha(\mathbb{R}_+^d) \rightarrow s^\alpha$, $\iota(f) = \{\langle f, l_n \rangle\}_{n \in \mathbb{N}_0^d}$, is a topological isomorphism between $G_\alpha^\alpha(\mathbb{R}_+^d)$ and s^α .*

For each $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$, $\sum_{n \in \mathbb{N}_0^d} \langle f, l_n \rangle l_n$ is summable to f in $G_\alpha^\alpha(\mathbb{R}_+^d)$.

In a similar way, we prove the next theorem.

Theorem 2.4. *Let $\alpha > 1$. The mapping $\iota : g_\alpha^\alpha(\mathbb{R}_+^d) \rightarrow \sigma^\alpha$, $\iota(f) = \{\langle f, l_n \rangle\}_{n \in \mathbb{N}_0^d}$, is a topological isomorphism between $g_\alpha^\alpha(\mathbb{R}_+^d)$ and σ^α .*

For each $f \in g_\alpha^\alpha(\mathbb{R}_+^d)$, $\sum_{n \in \mathbb{N}_0^d} \langle f, l_n \rangle l_n$ is summable to f in $g_\alpha^\alpha(\mathbb{R}_+^d)$.

The last four results are crucial and we will often tacitly apply them throughout the rest of this article.

3. G - and g - kernels

Firstly, we state the kernel theorems:

Theorem 3.1. (*[6, Theorem 6.4.]*) *Let $\alpha \geq 1$. We have the following canonical isomorphism:*

$$(3.1) \quad (G_\alpha^\alpha(\mathbb{R}_+^{d_1}))' \hat{\otimes} (G_\alpha^\alpha(\mathbb{R}_+^{d_2}))' \cong (G_\alpha^\alpha(\mathbb{R}_+^{d_1+d_2}))' \cong \mathcal{L}(G_\alpha^\alpha(\mathbb{R}_+^{d_1}), (G_\alpha^\alpha(\mathbb{R}_+^{d_2}))').$$

Theorem 3.2. *Let $\alpha > 1$. We have the following canonical isomorphism:*

$$(3.2) \quad (g_\alpha^\alpha(\mathbb{R}_+^{d_1}))' \hat{\otimes} (g_\alpha^\alpha(\mathbb{R}_+^{d_2}))' \cong (g_\alpha^\alpha(\mathbb{R}_+^{d_1+d_2}))' \cong \mathcal{L}(g_\alpha^\alpha(\mathbb{R}_+^{d_1}), (g_\alpha^\alpha(\mathbb{R}_+^{d_2}))').$$

Proof. The proof for $(g_\alpha^\alpha(\mathbb{R}_+^d))'$ can be obtained in the same way as for $(G_\alpha^\alpha(\mathbb{R}_+^d))'$ in Theorem 3.1. \square

The isomorphisms (3.1) and (3.2) require some comment. In what follows we use the convention that if T is a linear and continuous operator from $G_1^1(\mathbb{R}_+^{d_1})$ to $(G_1^1(\mathbb{R}_+^{d_2}))'$, and $g \in (G_1^1(\mathbb{R}_+^{d_0}))'$, then $T \otimes g$ is the linear and continuous operator from $G_1^1(\mathbb{R}_+^{d_1})$ to $(G_1^1(\mathbb{R}_+^{d_2+d_0}))'$, given by

$$(T \otimes g) : f \mapsto (Tf) \otimes g.$$

The following theorem is the main result of this paper.

Theorem 3.3. *Let T be a linear and continuous operator from $G_1^1(\mathbb{R}_+^{d_1})$ to $(G_1^1(\mathbb{R}_+^{d_2}))'$ with the kernel K , and let $d_0 \geq \min(d_1, d_2)$. Then the following is true:*

(R) *If $\alpha \geq 1$ and $K \in G_\alpha^\alpha(\mathbb{R}_+^{d_2+d_1})$, then there are operators T_1 and T_2 with kernels $K_1 \in G_\alpha^\alpha(\mathbb{R}_+^{d_0+d_1})$ and $K_2 \in G_\alpha^\alpha(\mathbb{R}_+^{d_2+d_0})$, respectively, such that $T = T_2 \circ T_1$.*

(B) *If $\alpha > 1$ and $K \in g_\alpha^\alpha(\mathbb{R}_+^{d_2+d_1})$, then there are operators T_1 and T_2 with kernels $K_1 \in g_\alpha^\alpha(\mathbb{R}_+^{d_0+d_1})$ and $K_2 \in g_\alpha^\alpha(\mathbb{R}_+^{d_2+d_0})$, respectively, such that $T = T_2 \circ T_1$.*

Remark 3.4. Let $\alpha \geq 1$. An operator with kernel in $G_\alpha^\alpha(\mathbb{R}_+^{2d})$ is sometimes called a regularizing operator with respect to $G_\alpha^\alpha(\mathbb{R}_+^d)$, because it extends uniquely to a continuous mapping from $(G_\alpha^\alpha(\mathbb{R}_+^d))'$, into $G_\alpha^\alpha(\mathbb{R}_+^d)$.

Let $\alpha > 1$. An operator with kernel in $g_\alpha^\alpha(\mathbb{R}_+^{2d})$ is sometimes called a regularizing operator with respect to $g_\alpha^\alpha(\mathbb{R}_+^d)$.

Proof. First we assume that $d_0 = d_1$, and start to prove (R). Let $l_{d,n}(x)$ be the Laguerre function on \mathbb{R}_+^d of order $n \in \mathbb{N}^d$. Then K possesses the expansion

$$(3.3) \quad K(x, y) = \sum_{n \in \mathbb{N}^{d_2}} \sum_{k \in \mathbb{N}^{d_1}} a_{n,k} l_{d_2,n}(x) l_{d_1,k}(y),$$

where the coefficients $a_{n,k}$ satisfy

$$(3.4) \quad \sup_{n,k} |a_{n,k} e^{A(|n|^{1/\alpha} + |k|^{1/\alpha})}| < \infty,$$

for some $A > 0$. Let $z \in \mathbb{R}_+^{d_1}$ and

$$(3.5) \quad \begin{aligned} K_{0,1}(z, y) &= \sum_{n,k \in \mathbb{N}^{d_1}} c_{n,k} l_{d_1,n}(z) l_{d_1,k}(y) \\ K_{0,2}(x, z) &= \sum_{n \in \mathbb{N}^{d_2}} \sum_{k \in \mathbb{N}^{d_1}} b_{n,k} l_{d_2,n}(x) l_{d_1,k}(z), \end{aligned}$$

where

$$c_{n,k} = \chi_{n,k} e^{-\frac{A}{2}|n|^{1/\alpha}} \quad \text{and} \quad b_{n,k} = a_{n,k} e^{\frac{A}{2}|k|^{1/\alpha}}$$

and $\chi_{n,k}$ is the Kronecker delta. Then we have

$$\int K_{0,2}(x, z) K_{0,1}(z, y) dz = \sum_{n \in \mathbb{N}^{d_2}} \sum_{k \in \mathbb{N}^{d_1}} a_{n,k} l_{d_2,n}(x) l_{d_1,k}(y) = K(x, y).$$

Hence, if T_j is the operator with kernel $K_{0,j}$, $j = 1, 2$, then $T = T_2 \circ T_1$. Furthermore,

$$\sup_{n,k} |b_{n,k} e^{\frac{A}{2}(|n|^{1/\alpha} + |k|^{1/\alpha})}| \leq \sup_{n,k} |a_{n,k} e^{A(|n|^{1/\alpha} + |k|^{1/\alpha})}| < \infty$$

and

$$\sup_{n,k} |c_{n,k} e^{\frac{A}{4}(|n|^{1/\alpha} + |k|^{1/\alpha})}| = \sup_n |e^{-\frac{A}{2}|n|^{1/\alpha}} e^{\frac{A}{2}|n|^{1/\alpha}}| < \infty.$$

This implies that $K_{0,1} \in G_\alpha^\alpha(\mathbb{R}_+^{d_1+d_1})$ and $K_{0,2} \in G_\alpha^\alpha(\mathbb{R}_+^{d_2+d_1})$ (see Theorem 2.1). If we put $K_1 = K_{0,1}$ and $K_2 = K_{0,2}$, we proved (R) in the case when $d_0 = d_1$.

In order to prove (B) assume that $K \in g_\alpha^\alpha(\mathbb{R}_+^{d_2+d_1})$ and let $a_{n,k}$ be the same as the above. Then (3.4) holds for any $A > 0$, which implies that if $N \geq 0$ is an integer, then

$$(3.6) \quad \Sigma_N = \sup\{|k| : |a_{n,k}| \geq e^{-2(N+1)(|n|^{1/\alpha} + |k|^{1/\alpha})} \text{ for some } n \in \mathbb{N}^{d_2}\}$$

is finite. Let $I_1 = \{k \in \mathbb{N}^{d_1} : |k| \leq \Sigma_1 + 1\}$ and define inductively

$$I_j = \{k \in \mathbb{N}^{d_1} \setminus I_{j-1} : |k| \leq \Sigma_j + j\}, \quad j \geq 2.$$

Then

$$I_j \cap I_k = \emptyset \quad \text{when } j \neq k, \quad \text{and} \quad \mathbb{N}^{d_1} = \bigcup_{j \geq 1} I_j.$$

Let $K_{0,1}$ and $K_{0,2}$ be as in (3.5) where

$$c_{n_1,k} = \chi_{n_1,k} e^{-j|k|^{1/\alpha}} \quad \text{and} \quad b_{n_2,k} = a_{n_2,k} e^{j|k|^{1/\alpha}},$$

$n_1 \in \mathbb{N}^{d_1}$, $n_2 \in \mathbb{N}^{d_2}$ and $k \in I_j$. If T_j is the operator with kernel $K_{0,j}$, $j = 1, 2$, then it follows that $T = T_2 \circ T_1$. Moreover, if $A > 0$ we have

$$\sup_{n,k} |b_{n,k} e^{A(|n|^{1/\alpha} + |k|^{1/\alpha})}| \leq J_1 + J_2,$$

where

$$(3.7) \quad J_1 = \sup_{j \leq A+1} \sup_n \sup_{k \in I_j} |b_{n,k} e^{A(|n|^{1/\alpha} + |k|^{1/\alpha})}|$$

and

$$(3.8) \quad J_2 = \sup_{j > A+1} \sup_n \sup_{k \in I_j} |b_{n,k} e^{A(|n|^{1/\alpha} + |k|^{1/\alpha})}|.$$

We will prove that J_1 and J_2 are finite. Since in (3.7) we have the finite numbers of k , from (3.4) and the definition of $b_{n,k}$ we obtain that J_1 is finite.

For J_2 we have

$$\begin{aligned} J_2 &= \sup_{j > A+1} \sup_n \sup_{k \in I_j} |a_{n,k} e^{A|n|^{1/\alpha} + (A+j)|k|^{1/\alpha}}| \\ &\leq \sup_{j > A+1} \sup_n \sup_{k \in I_j} |e^{-2j(|n|^{1/\alpha} + |k|^{1/\alpha})} e^{A|n|^{1/\alpha} + (A+j)|k|^{1/\alpha}}| < \infty, \end{aligned}$$

where the first inequality follows from (3.6). Hence,

$$\sup_{n,k} |b_{n,k} e^{A(|n|^{1/\alpha} + |k|^{1/\alpha})}| < \infty,$$

which implies that $K_{0,2} \in g_\alpha^\alpha(\mathbb{R}_+^{d_2+d_1})$.

In a similar way if we replace $b_{n,k}$ with $c_{n,k}$ in the definition of J_1 and J_2) we obtain

$$\sup_{n,k} |c_{n,k} e^{A(|n|^{1/\alpha} + |k|^{1/\alpha})}| < \infty,$$

which implies $K_{0,1} \in g_\alpha^\alpha(\mathbb{R}_+^{d_1+d_1})$ and (B) follows in the case $d_0 = d_1$.

Next, assume that $d_0 > d_1$ and let $d = d_0 - d_1 \geq 1$. Then we set

$$K_1(z, y) = K_{0,1}(z_1, y) l_{d,0}(z_2) \quad \text{and} \quad K_2(x, z) = K_{0,2}(x, z_1) l_{d,0}(z_2),$$

where $z_1 \in \mathbb{R}_+^{d_1}$, $z_2 \in \mathbb{R}_+^d$ and hence, $z = (z_1, z_2) \in \mathbb{R}_+^{d_0}$. Next, we obtain

$$\int_{\mathbb{R}_+^{d_0}} K_2(x, z) K_1(z, y) dz = \int_{\mathbb{R}_+^{d_1}} K_{0,2}(x, z_1) K_{0,1}(z_1, y) dz_1 = K(x, y).$$

In this case the assertion (R) and (B) follows from the equivalences

$$K_1 \in G_\alpha^\alpha(\mathbb{R}_+^{d_0+d_1}) (g_\alpha^\alpha(\mathbb{R}_+^{d_0+d_1})) \iff K_{0,1} \in G_\alpha^\alpha(\mathbb{R}_+^{d_1+d_1}) (g_\alpha^\alpha(\mathbb{R}_+^{d_1+d_1}))$$

and

$$K_2 \in G_\alpha^\alpha(\mathbb{R}_+^{d_2+d_0}) (g_\alpha^\alpha(\mathbb{R}_+^{d_2+d_0})) \iff K_{0,1} \in G_\alpha^\alpha(\mathbb{R}_+^{d_2+d_1}) (g_\alpha^\alpha(\mathbb{R}_+^{d_2+d_1})).$$

It remains to prove the result in the case $d_0 \geq d_2$. The rules of d_1 and d_2 are interchanged when taking the adjoint operators. Hence, the result follows from the first part of the proof in combination with the facts that G_α^α and g_α^α are invariant under pullbacks of bijective linear transformations i.e. $(x, y) \mapsto F(x, y)$ belongs to $G_\alpha^\alpha(\mathbb{R}_+^{d_1} \times \mathbb{R}_+^{d_2})$ if and only if $(y, x) \mapsto \overline{F(x, y)}$ belongs to $G_\alpha^\alpha(\mathbb{R}_+^{d_2} \times \mathbb{R}_+^{d_1})$. The proof is complete. \square

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