

# HYPERCYCLIC AND TOPOLOGICALLY MIXING PROPERTIES OF ABSTRACT DEGENERATE TIME-FRACTIONAL INCLUSIONS

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**Abstract.** In this paper, we analyze hypercyclic and topologically mixing properties of abstract degenerate (multi-term) time-fractional inclusions in separable infinite-dimensional Fréchet spaces. We use the multivalued linear operator approach.

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## 1. Introduction and preliminaries

The theory of topological dynamics of linear operators is very popular nowadays ([5, 16]). A linear operator  $T$  on a Fréchet space  $E$  is said to be hypercyclic iff there exists an element  $x \in D_\infty(T)$  whose orbit  $\{T^n x : n \in \mathbb{N}_0\}$  is dense in  $E$ ;  $T$  is said to be topologically transitive, resp. topologically mixing, iff for every pair of open non-empty subsets  $U, V$  of  $E$ , there exists  $n \in \mathbb{N}$  such that  $T^n(U) \cap V \neq \emptyset$ , resp. iff for every pair of open non-empty subsets  $U, V$  of  $E$ , there exists  $n_0 \in \mathbb{N}$  such that, for every  $n \in \mathbb{N}$  with  $n \geq n_0$ , one has  $T^n(U) \cap V \neq \emptyset$ .

In [23, 24], we have analyzed hypercyclic and topologically mixing properties of various classes of abstract degenerate Volterra integro-differential equations (cf. [19, Chapter 3] for a survey of results about non-degenerate case). The main aim of this paper, which is written in an expository manner, is to reconsider the results from the above-mentioned papers from the point of view of the theory of multivalued linear operators.

Throughout the paper, we use the standard notation. By  $E$  we denote a separable infinite-dimensional Fréchet space over the field of complex numbers. By  $L(E)$  we denote the space which consists of all continuous linear mappings from  $E$  into  $E$ . We assume that the topology of  $E$  is induced by the fundamental system  $(p_n)_{n \in \mathbb{N}}$  of increasing seminorms. The translation invariant metric  $d : E \times E \rightarrow [0, \infty)$ , defined by

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x - y)}{1 + p_n(x - y)}, \quad x, y \in E,$$

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satisfies the following properties:  $d(x+u, y+v) \leq d(x, y) + d(u, v)$ ,  $x, y, u, v \in E$ ,  $d(cx, cy) \leq (|c|+1)d(x, y)$ ,  $c \in \mathbb{C}$ ,  $x, y \in E$ , and  $d(\alpha x, \beta x) \geq \frac{|\alpha-\beta|}{1+|\alpha-\beta|}d(0, x)$ ,  $x \in E$ ,  $\alpha, \beta \in \mathbb{C}$ . The domain, kernel and range of a linear operator  $A$  on  $E$  are denoted by  $D(A)$ ,  $N(A)$  and  $R(A)$ , respectively. For any  $p \in \mathbb{N}$  and  $r \in \mathbb{N}_p \equiv \{1, 2, \dots, p\}$ , we define  $\text{Proj}_{r,p} : E^p \rightarrow E$  by  $\text{Proj}_{r,p}(x_1, \dots, x_p) := x_r$ ,  $\vec{x} = (x_1, \dots, x_p) \in E^p$ . Given  $s \in \mathbb{R}$  in advance, set  $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$ . The Gamma function is denoted by  $\Gamma(\cdot)$  and the principal branch is always used to take the powers. Set  $0^\alpha := 0$ ,  $\mathbb{N}_p^0 := \mathbb{N}_p \cup \{0\}$  and  $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$  ( $\alpha > 0, t > 0$ ).

During the past three decades, considerable interest in fractional calculus and fractional differential equations has been stimulated due to their numerous applications in various fields of science, technology and engineering. Basic information about fractional calculus and non-degenerate fractional differential equations can be obtained by consulting [6, 13, 18, 19, 30] and the references cited therein. We refer the reader to [31] and [19] for further information about abstract non-degenerate Volterra equations. Concerning abstract degenerate Volterra equations and abstract degenerate fractional differential equations, we would like to recommend for the reader the forthcoming monograph [20] (cf. [15] and [32] for the basic source of information on abstract degenerate differential equations with integer order derivatives).

In this paper, we will use the Caputo fractional derivatives. Let  $\zeta > 0$ . Then the Caputo fractional derivative  $\mathbf{D}_t^\zeta u$  ([6, 19]) is defined for those functions  $u \in C^{\lceil \zeta \rceil - 1}([0, \infty) : X)$  for which  $g_{\lceil \zeta \rceil - \zeta} * (u - \sum_{j=0}^{\lceil \zeta \rceil - 1} u^{(j)}(0)g_{j+1}) \in C^{\lceil \zeta \rceil}([0, \infty) : X)$ , by

$$\mathbf{D}_t^\zeta u(t) := \frac{d^{\lceil \zeta \rceil}}{dt^{\lceil \zeta \rceil}} \left[ g_{\lceil \zeta \rceil - \zeta} * \left( u - \sum_{j=0}^{\lceil \zeta \rceil - 1} u^{(j)}(0)g_{j+1} \right) \right].$$

If the Caputo fractional derivative  $\mathbf{D}_t^\zeta u(t)$  exists, then for each number  $\nu \in (0, \zeta)$  the Caputo fractional derivative  $\mathbf{D}_t^\nu u(t)$  exists, as well.

The Mittag-Leffler function  $E_{\beta, \gamma}(z)$  ( $\beta > 0, \gamma \in \mathbb{R}$ ) is defined by

$$E_{\beta, \gamma}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \gamma)}, \quad z \in \mathbb{C}.$$

In this place, we assume that  $1/\Gamma(\beta k + \gamma) = 0$  if  $\beta k + \gamma \in -\mathbb{N}_0$ . Set, for short,  $E_\beta(z) := E_{\beta, 1}(z)$ ,  $z \in \mathbb{C}$ . For further information about the Mittag-Leffler functions, cf. [6, 19] and the references cited there.

Let  $0 < \tau \leq \infty$  and  $\mathcal{F} : [0, \tau) \rightarrow P(X)$ . A single-valued function  $f : [0, \tau) \rightarrow X$  is called a section of  $\mathcal{F}$  iff  $f(t) \in \mathcal{F}(t)$  for all  $t \in [0, \tau)$ . We denote the set consisting of all continuous sections of  $\mathcal{F}$  by  $\text{sec}_c(\mathcal{F})$ .

## 2. Multivalued linear operators in locally convex spaces

In this section, we will present some necessary definitions from the theory of multivalued linear operators. For more details about this topic, we refer the reader to the monographs by R. Cross [11] and A. Favini-A. Yagi [15].

Let  $X$  and  $Y$  be two sequentially complete locally convex spaces (SCLCSs) over the field of complex numbers. A multivalued map  $\mathcal{A} : X \rightarrow P(Y)$  is said to be a multivalued linear operator (MLO) iff the following holds:

- (i)  $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$  is a linear subspace of  $X$ ;
- (ii)  $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x + y)$ ,  $x, y \in D(\mathcal{A})$  and  $\lambda \mathcal{A}x \subseteq \mathcal{A}(\lambda x)$ ,  $\lambda \in \mathbb{C}$ ,  $x \in D(\mathcal{A})$ .

If  $X = Y$ , then we say that  $\mathcal{A}$  is an MLO in  $X$ . As an almost immediate consequence of definition, we have that the equality  $\lambda \mathcal{A}x + \eta \mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$  holds for every  $x, y \in D(\mathcal{A})$  and for every  $\lambda, \eta \in \mathbb{C}$  with  $|\lambda| + |\eta| \neq 0$ . If  $\mathcal{A}$  is an MLO, then  $\mathcal{A}0$  is a linear manifold in  $Y$  and  $\mathcal{A}x = f + \mathcal{A}0$  for any  $x \in D(\mathcal{A})$  and  $f \in \mathcal{A}x$ . Define  $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$ . Then the set  $N(\mathcal{A}) := \mathcal{A}^{-1}0 = \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$  is called the kernel of  $\mathcal{A}$ . The inverse  $\mathcal{A}^{-1}$  of an MLO is defined by  $D(\mathcal{A}^{-1}) := R(\mathcal{A})$  and  $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$ . It can be easily seen that  $\mathcal{A}^{-1}$  is an MLO in  $X$ , as well as that  $N(\mathcal{A}^{-1}) = \mathcal{A}0$  and  $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$ . If  $N(\mathcal{A}) = \{0\}$ , i.e., if  $\mathcal{A}^{-1}$  is single-valued, then  $\mathcal{A}$  is said to be injective. If  $\mathcal{A}, \mathcal{B} : X \rightarrow P(Y)$  are two MLOs, then we define its sum  $\mathcal{A} + \mathcal{B}$  by  $D(\mathcal{A} + \mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$  and  $(\mathcal{A} + \mathcal{B})x := \mathcal{A}x + \mathcal{B}x$ ,  $x \in D(\mathcal{A} + \mathcal{B})$ . It is clear that  $\mathcal{A} + \mathcal{B}$  is likewise an MLO. We write  $\mathcal{A} \subseteq \mathcal{B}$  iff  $D(\mathcal{A}) \subseteq D(\mathcal{B})$  and  $\mathcal{A}x \subseteq \mathcal{B}x$  for all  $x \in D(\mathcal{A})$ .

Let  $\mathcal{A} : X \rightarrow P(Y)$  and  $\mathcal{B} : Y \rightarrow P(Z)$  be two MLOs, where  $Z$  is an SCLCS. The product of  $\mathcal{A}$  and  $\mathcal{B}$  is defined by  $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$  and  $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$ . Then  $\mathcal{B}\mathcal{A} : X \rightarrow P(Z)$  is an MLO and  $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$ . The scalar multiplication of an MLO  $\mathcal{A} : X \rightarrow P(Y)$  with the number  $z \in \mathbb{C}$ ,  $z\mathcal{A}$  for short, is defined by  $D(z\mathcal{A}) := D(\mathcal{A})$  and  $(z\mathcal{A})(x) := z\mathcal{A}x$ ,  $x \in D(\mathcal{A})$ . It is clear that  $z\mathcal{A} : X \rightarrow P(Y)$  is an MLO and  $(\omega z)\mathcal{A} = \omega(z\mathcal{A}) = z(\omega\mathcal{A})$ ,  $z, \omega \in \mathbb{C}$ .

The point spectrum of  $\mathcal{A}$ ,  $\sigma_p(\mathcal{A})$  for short, is consisted of those complex numbers  $\lambda \in \mathbb{C}$  for which there exists a non-zero vector  $x \in D(\mathcal{A})$  such that  $\lambda x \in \mathcal{A}x$ .

We say that an MLO  $\mathcal{A} : X \rightarrow P(Y)$  is closed if for any nets  $(x_\tau)$  in  $D(\mathcal{A})$  and  $(y_\tau)$  in  $Y$  such that  $y_\tau \in \mathcal{A}x_\tau$  for all  $\tau \in I$ , we have that the suppositions  $\lim_{\tau \rightarrow \infty} x_\tau = x$  and  $\lim_{\tau \rightarrow \infty} y_\tau = y$  imply  $x \in D(\mathcal{A})$  and  $y \in \mathcal{A}x$ .

We need the following auxiliary lemma from [21].

**Lemma 2.1.** *Let  $\Omega$  be a locally compact, separable metric space, and let  $\mu$  be a locally finite Borel measure defined on  $\Omega$ . Suppose that  $\mathcal{A} : X \rightarrow P(Y)$  is a closed MLO. Let  $f : \Omega \rightarrow X$  and  $g : \Omega \rightarrow Y$  be  $\mu$ -integrable, and let  $g(x) \in \mathcal{A}f(x)$ ,  $x \in \Omega$ . Then  $\int_\Omega f d\mu \in D(\mathcal{A})$  and  $\int_\Omega g d\mu \in \mathcal{A} \int_\Omega f d\mu$ .*

In the remaining part of paper,  $\Omega$  will always be an appropriate subspace of  $\mathbb{R}$ ,  $\mu$  will always be the Lebesgue measure defined on  $\Omega$ ;  $E$  will be a separable infinite-dimensional Fréchet space over the field of complex numbers, and  $\mathcal{A}$  will be a multivalued linear operator in  $E$ .

### 3. Hypercyclic and topologically mixing properties of problem $(DFP)_\alpha$

In this section, we will consider hypercyclic and topologically mixing properties of the following abstract degenerate time-fractional inclusion:

$$(DFP)_\alpha : \begin{cases} \mathbf{D}_t^\alpha u(t) \in \mathcal{A}u(t), & t \geq 0, \\ u(0) = x; u^{(j)}(0) = 0, & 0 \leq j \leq \lceil \alpha \rceil - 1. \end{cases}$$

By a (strong) solution of  $(DFP)_\alpha$ , we mean any continuous  $E$ -valued function  $t \mapsto u(t)$ ,  $t \geq 0$  such that the term  $t \mapsto \mathbf{D}_t^\alpha u(t)$ ,  $t \geq 0$  is well defined and continuous, as well as that  $(DFP)_\alpha$  holds. Assuming that the operator  $\mathcal{A}$  is closed, we can integrate problem  $(DFP)_\alpha$   $\alpha$ -times in order to see (cf. also Theorem 2.1) that any solution  $u(t)$  of the problem  $(DFP)_\alpha$  satisfies the following integral equation:

$$u(t) - x = (g_\alpha * u_{\mathcal{A}})(t) \in \mathcal{A}(g_\alpha * u)(t), \quad t \geq 0.$$

In this place, it is worth noting that we do not require a priori the closedness of the operator  $\mathcal{A}$  henceforth. Denote by  $Z_\alpha(\mathcal{A})$  the set which consists of those vectors  $x \in E$  for which there exists a solution of the problem  $(DFP)_\alpha$ . Then  $Z_\alpha(\mathcal{A})$  is a linear subspace of  $E$ .

The following is an extension of [22, Lemma 2.1] to multivalued linear operator case. The proof is almost straightforward after pointing out that  $\mathbf{D}_t^\alpha E_\alpha(\lambda t^\alpha) = \lambda E_\alpha(\lambda t^\alpha)$ ,  $t \geq 0$ ,  $\lambda \in \mathbb{C}$ ,  $\alpha > 0$ .

**Lemma 3.1.** *Suppose  $\alpha > 0$ ,  $\lambda \in \mathbb{C}$ ,  $x \in E$  and  $\lambda x \in \mathcal{A}x$ . Then  $x \in Z_\alpha(\mathcal{A})$  and one solution of  $(DFP)_\alpha$  is given by  $u(t) \equiv u(t; x) = E_\alpha(\lambda t^\alpha)x$ ,  $t \geq 0$ .*

The notion of a (subspace-)hypercyclicity, (subspace-)topological transitivity and (subspace-)topologically mixing property of the problem  $(DFP)_\alpha$  are introduced in the following definition.

**Definition 3.2.** Let  $\alpha > 0$ , and let  $\tilde{E}$  be a closed linear subspace of  $E$ . Then it is said that:

- (i) an element  $x \in Z_\alpha(\mathcal{A}) \cap \tilde{E}$  is a  $\tilde{E}$ -hypercyclic vector for  $(DFP)_\alpha$  iff there exists a strong solution  $t \mapsto u(t; x)$ ,  $t \geq 0$  of the problem  $(DFP)_\alpha$  with the property that the set  $\{u(t; x) : t \geq 0\}$  is a dense subset of  $\tilde{E}$ .

Furthermore, we say that the abstract Cauchy problem  $(DFP)_\alpha$  is:

- (ii)  $\tilde{E}$ -topologically transitive iff for every  $y, z \in \tilde{E}$  and for every  $\varepsilon > 0$ , there exist  $x \in Z_\alpha(\mathcal{A}) \cap \tilde{E}$ , a strong solution  $t \mapsto u(t; x)$ ,  $t \geq 0$  of the problem  $(DFP)_\alpha$  and  $t \geq 0$  such that  $d(x, y) < \varepsilon$  and  $d(u(t; x), z) < \varepsilon$ ;
- (iii)  $\tilde{E}$ -topologically mixing iff for every  $y, z \in \tilde{E}$  and for every  $\varepsilon > 0$ , there exists  $t_0 \geq 0$  such that, for every  $t \geq t_0$ , there exist  $x_t \in Z_\alpha(\mathcal{A}) \cap \tilde{E}$  and a strong solution  $t \mapsto u(t; x_t)$ ,  $t \geq 0$  of the problem  $(DFP)_\alpha$  such that  $d(x_t, y) < \varepsilon$  and  $d(u(t; x_t), z) < \varepsilon$ .

In the case  $\tilde{E} = E$ , it is also said that a  $\tilde{E}$ -hypercyclic vector for  $(DFP)_\alpha$  is a hypercyclic vector for  $(DFP)_\alpha$ , and that  $(DFP)_\alpha$  is topologically transitive, resp. topologically mixing.

In a series of our recent research studies of hypercyclicity and topologically mixing properties of abstract integro-differential equations, we have reconsidered several times the Desch-Schappacher-Webb and Banasiak-Moszyński criteria for chaos of strongly continuous semigroups ([12, 3]). It is very important to observe that this useful criterion continues to hold in our framework (cf. [22, Theorem 2.3] and its proof):

**Theorem 3.3.** *Assume  $\alpha \in (0, 2)$  and there exists an open connected subset  $\Omega$  of  $\mathbb{C}$  which satisfies  $\Omega \cap (-\infty, 0] = \emptyset$ ,  $\Omega^\alpha := \{\lambda^\alpha : \lambda \in \Omega\} \subseteq \sigma_p(\mathcal{A})$  and  $\Omega \cap i\mathbb{R} \neq \emptyset$ . Let  $f : \Omega^\alpha \rightarrow E$  be an analytic mapping such that  $f(\lambda^\alpha) \in N(\mathcal{A} - \lambda^\alpha) \setminus \{0\}$ ,  $\lambda \in \Omega$  and let  $\tilde{E} := \text{span}\{f(\lambda^\alpha) : \lambda \in \Omega\}$ . Then the abstract degenerate inclusion  $(DFP)_\alpha$  is  $\tilde{E}$ -topologically mixing.*

The assertion of [23, Theorem 9] can be reformulated for multivalued linear operators, as well:

**Theorem 3.4.** *Suppose that  $\alpha > 0$  and  $(t_n)_{n \in \mathbb{N}}$  is a sequence of positive reals tending to  $+\infty$ . If the set  $E_{0,\alpha}$ , which consists of those elements  $y \in Z_\alpha(\mathcal{A}) \cap \tilde{E}$  for which there exists a strong solution  $t \mapsto u(t; y)$ ,  $t \geq 0$  of the problem  $(DFP)_\alpha$  such that  $\lim_{n \rightarrow \infty} u(t_n; y) = 0$ , is dense in  $\tilde{E}$ , and if the set  $E_{\infty,\alpha}$ , which consists of those elements  $z \in Z_\alpha(\mathcal{A}) \cap \tilde{E}$  for which there exist a null sequence  $(\omega_n)_{n \in \mathbb{N}} \in Z_\alpha(\mathcal{A}) \cap \tilde{E}$  and a sequence  $(u_n(\cdot; \omega_n))_{n \in \mathbb{N}}$  of strong solutions of the problem  $(DFP)_\alpha$  such that  $\lim_{n \rightarrow \infty} u(t_n; \omega_n) = z$ , is also dense in  $\tilde{E}$ , then the problem  $(DFP)_\alpha$  is  $\tilde{E}$ -topologically transitive.*

#### 4. Hypercyclic and topologically mixing properties of abstract degenerate Cauchy problems of first and second order

Concerning linear dynamical properties, the abstract degenerate Cauchy problems of first and second order have a numerous peculiarities compared with the abstract degenerate fractional Cauchy problems. The main aim of this section is to investigate some of these peculiarities in more detail.

We start by stating the following simple proposition.

**Proposition 4.1.** *(cf. [23, Proposition 13])*

- (i) *Suppose that  $\alpha = 1$ ,  $x \in Z_1(\mathcal{A})$  and the function  $t \mapsto u(t; x)$ ,  $t \geq 0$  is a solution of the problem  $(DFP)_1$ . Then, for every  $s \geq 0$ ,  $u(s; x) \in Z_1(\mathcal{A})$  and a solution of  $(DFP)_1$ , with initial condition  $x$  replaced by  $u(s; x)$ , is given by  $u(t; u(s; x)) := u(t + s; x)$ ,  $t \geq 0$ .*
- (ii) *Suppose that  $\alpha = 2$ ,  $x \in Z_2(\mathcal{A})$  and the function  $t \mapsto u(t; x)$ ,  $t \geq 0$  is a solution of  $(DFP)_2$ . Then, for every  $s \geq 0$ ,  $u(s; x) \in Z_2(\mathcal{A})$  and a solution of  $(DFP)_2$ , with initial condition  $x$  replaced by  $u(s; x)$ , is given by  $u(t; u(s; x)) := \frac{1}{2}[u(t + s; x) + u(|t - s|; x)]$ ,  $t \geq 0$ .*

In [14, Theorem 2.1], S. El Mourchid has investigated the connection between the imaginary point spectrum and hypercyclicity of strongly continuous semigroups. It is well known that this result cannot be transferred to fractional equations. In [23, Theorem 18], we have seen that the assertion of [14, Theorem 2.1] can be extended to linear Sobolev equations of first order. Now we would like to formulate a similar result for multivalued linear operators.

**Theorem 4.2.** *Assume that  $\alpha = 1$ ,  $\omega_1, \omega_2 \in \mathbb{R} \cup \{-\infty, \infty\}$ ,  $\omega_1 < \omega_2$ ,  $t_0 > 0$  and  $k \in \mathbb{N}$ . Let  $f_j : (\omega_1, \omega_2) \rightarrow E$  be integrable (in the sense of [21, Definition 1.2(ii)]), and let for each  $j = 1, \dots, k$  we have  $f_j(s) \in \text{isAf}_j(s)$  for a.e.  $s \in (\omega_1, \omega_2)$ . Put  $\psi_{r,j} := \int_{\omega_1}^{\omega_2} e^{irs} f_j(s) ds$ ,  $r \in \mathbb{R}$ ,  $1 \leq j \leq k$ . Put  $\tilde{E} := \overline{\text{span}\{\psi_{r,j} : r \in \mathbb{R}, 1 \leq j \leq k\}}$ . If the operator  $\mathcal{A}$  is closed, then the problem  $(DFP)_1$  is  $\tilde{E}$ -topologically mixing.*

The assertion of [23, Theorem 21] can be formulated for multivalued linear operators, as well:

**Theorem 4.3.** *Let  $\tilde{E}$  be a closed linear subspace of  $E$ , and let  $\alpha = 2$ .*

- (i) *Suppose that  $(t_n)_{n \in \mathbb{N}}$  is a sequence of positive reals tending to  $+\infty$ . Denote by  $X_{1, \tilde{E}}$  the set which consists of those elements  $x \in Z_2(\mathcal{A}) \cap \tilde{E}$  for which there exists a solution  $t \mapsto u(t; x)$ ,  $t \geq 0$  of the problem  $(DFP)_2$  such that  $u(0; x) = x$  and  $\lim_{n \rightarrow \infty} u(t_n; x) = \lim_{n \rightarrow \infty} u(2t_n; x) = 0$ . If  $X_{1, \tilde{E}}$  is dense in  $\tilde{E}$ , then the problem  $(DFP)_2$  is  $\tilde{E}$ -topologically transitive.*
- (ii) *Denote by  $X'_{1, \tilde{E}}$  the set which consists of those elements  $x \in Z_2(\mathcal{A}) \cap \tilde{E}$  for which there exists a strong solution  $t \mapsto u(t; x)$ ,  $t \geq 0$  of the problem  $(DFP)_2$  such that  $u(0; x) = x$  and  $\lim_{t \rightarrow +\infty} u(t; x) = 0$ . If  $X'_{1, \tilde{E}}$  is dense in  $\tilde{E}$ , then the problem  $(DFP)_2$  is  $\tilde{E}$ -topologically mixing.*

Unfortunately, Theorem 3.3 is no longer true in the case that  $\alpha = 2$ . If so, then we can pass to the equation of first order by considering the multivalued linear operator  $\begin{bmatrix} 0 & I \\ \mathcal{A} & 0 \end{bmatrix}$  and the vector  $(f(\lambda^2) \lambda f(\lambda^2))^T$  for  $\lambda \in \Omega$ , where  $\Omega$  is an open connected subset of  $\mathbb{C}$  intersecting the imaginary axis, and apply Theorem 3.3, with  $\alpha = 1$ , after that (cf. Theorem 6.3 below and [23, Remark 6] for more details).

The Hypercyclicity Criterion for degenerate first order equations has been stated in [23, Theorem 20]. The interested reader may try to formulate a version of this result for abstract degenerate differential inclusions of first order.

## 5. Hypercyclic and topologically mixing properties of certain classes of abstract degenerate multi-term fractional differential inclusions

In this section, we assume that  $n \in \mathbb{N} \setminus \{1\}$ ,  $\mathcal{A}_1, \dots, \mathcal{A}_{n-1}$ ,  $\mathcal{A}$  and  $\mathcal{B}$  are multivalued linear operators on  $E$  (not necessarily closed),  $0 \leq \alpha_1 < \dots <$

$\alpha_{n-1} < \alpha_n$  and  $0 \leq \alpha < \alpha_n$ . By  $\mathcal{L}$  and  $\mathcal{L}^{-1}$  we denote the Laplace transform and its inverse transform, respectively (cf. [2, 19, 21] for more details). Fix a number  $i \in \mathbb{N}_{m_n-1}^0$ . Set  $\mathcal{A}_0 := \mathcal{A}$ ,  $\mathcal{A}_n := \mathcal{B}$ ,  $\alpha_0 := \alpha$ ,  $m_j := \lceil \alpha_j \rceil$  ( $j \in \mathbb{N}_0^n$ ),  $\mathcal{D}_i := \{j \in \mathbb{N}_{n-1} : m_j - 1 \geq i\}$  and  $\mathcal{D}_i := \{j \in \mathbb{N}_{n-1}^0 : m_j - 1 \geq i\}$ . We will consider henceforth hypercyclic and topologically mixing properties of the following degenerate abstract multi-term fractional inclusion:

$$(5.1) \quad \begin{aligned} 0 \in \mathcal{B} \mathbf{D}_t^{\alpha_n} u(t) + \sum_{j=1}^{n-1} \mathcal{A}_j \mathbf{D}_t^{\alpha_j} u(t) - \mathcal{A} \mathbf{D}_t^{\alpha} u(t), \quad t \geq 0; \\ u^{(k)}(0) = x_k, \quad k \in \mathbb{N}_{m_n-1}^0. \end{aligned}$$

In this section, we will consider the case in which:

$$(5.2) \quad u^{(i)}(0) = x_i = x \text{ and } u^{(k)}(0) = x_k = 0, \quad k \in \mathbb{N}_{m_n-1}^0 \setminus \{i\}.$$

Because no confusion seems likely, we will denote such a degenerate inclusion by the same symbol (5.1). By a strong solution of (5.1), we mean any continuous  $E$ -valued function  $t \mapsto u(t)$ ,  $t \geq 0$  such that the Caputo fractional derivative  $\mathbf{D}_t^{\alpha_n} u(t)$  is well-defined, as well as that the initial conditions in (5.1) hold and that there exist continuous sections  $a_j(t) \in \text{sec}_c(\mathcal{A}_j \mathbf{D}_t^{\alpha_j} u(t))$  ( $0 \leq j \leq n, t \geq 0$ ) such that

$$0 = \sum_{j=1}^n a_j(t) - a_0(t), \quad t \geq 0.$$

If (5.2) holds, then  $u(t)$  will be also denoted by  $u_i(t) \equiv u_i(t; x)$ .

We will use the following definition.

**Definition 5.1.** Let  $\tilde{E}$  be a closed linear subspace of  $E$ . Then it is said that the equation (5.1) is:

- (i)  $\tilde{E}$ -hypercyclic iff there exist an element  $x \in \tilde{E}$  and a strong solution  $t \mapsto u_i(t; x)$ ,  $t \geq 0$  of (5.1) such that  $\{u_i(t; x) : t \geq 0\}$  is a dense subset of  $\tilde{E}$ ; such an element is called a  $\tilde{E}$ -hypercyclic vector of (5.1);
- (ii)  $\tilde{E}$ -topologically transitive iff for every  $y, z \in \tilde{E}$  and for every  $\varepsilon > 0$ , there exist an element  $x \in \tilde{E}$ , a strong solution  $t \mapsto u_i(t; x)$ ,  $t \geq 0$  of (5.1) and a number  $t \geq 0$  such that  $d(x, y) < \varepsilon$  and  $d(u_i(t; x), z) < \varepsilon$ ;
- (iii)  $\tilde{E}$ -topologically mixing iff for every  $y, z \in \tilde{E}$  and for every  $\varepsilon > 0$ , there exists  $t_0 \geq 0$  such that, for every  $t \geq t_0$ , there exist an element  $x_t \in \tilde{E}$  and a strong solution  $t \mapsto u_i(t; x_t)$ ,  $t \geq 0$  of (5.1), with  $x$  replaced by  $x_t$ , such that  $d(x_t, y) < \varepsilon$  and  $d(u_i(t; x_t), z) < \varepsilon$ .

In the case  $\tilde{E} = E$ , it is also said that a  $\tilde{E}$ -hypercyclic vector of (5.1) is a hypercyclic vector of (5.1) and that (5.1) is topologically transitive, resp. topologically mixing.

The assertion of [23, Theorem 11] can be extended to multivalued linear operators as follows:

**Theorem 5.2.** *Suppose that  $\emptyset \neq \Omega$  is an open connected subset of  $\mathbb{C} \setminus \{0\}$ ,  $f : \Omega \rightarrow E \setminus \{0\}$  is an analytic function,  $f_j : \Omega \rightarrow \mathbb{C} \setminus \{0\}$  is a scalar-valued function ( $1 \leq j \leq n$ ),  $g : \Omega \rightarrow E$  satisfies  $g(\lambda) \in \mathcal{A}f(\lambda)$ ,  $\lambda \in \Omega$  and*

$$g(\lambda) \in f_n(\lambda)\mathcal{B}f(\lambda); \quad g(\lambda) \in f_j(\lambda)\mathcal{A}_j f(\lambda), \quad \lambda \in \Omega, \quad 1 \leq j \leq n-1.$$

*Suppose, further, that  $\Omega_+$  and  $\Omega_-$  are two non-empty subsets of  $\Omega$ , and each of them admits a cluster point in  $\Omega$ . Define  $\tilde{E} := \overline{\text{span}\{f(\lambda) : \lambda \in \Omega\}}$ ,*

$$H_i(\lambda, t) := \mathcal{L}^{-1} \left( \frac{z^{\alpha_n - i - 1} + \sum_{j \in D_i} \frac{f_n(\lambda)}{f_j(\lambda)} z^{\alpha_j - i - 1} - \chi_{D_i}(0) f_n(\lambda) z^{\alpha - i - 1}}{z^{\alpha_n} + \sum_{j=1}^{n-1} \frac{f_n(\lambda)}{f_j(\lambda)} z^{\alpha_j} - f_n(\lambda) z^{\alpha}} \right) (t),$$

and

$$F_i(\lambda, t) := H_i(\lambda, t)f(\lambda),$$

for any  $t \geq 0$  and  $\lambda \in \Omega$ . If

$$\lim_{t \rightarrow +\infty} |H_i(\lambda, t)| = +\infty, \quad \lambda \in \Omega_+ \quad \text{and} \quad \lim_{t \rightarrow +\infty} H_i(\lambda, t) = 0, \quad \lambda \in \Omega_-,$$

then (5.1) is  $\tilde{E}$ -topologically mixing. Furthermore, there exist continuous sections  $a_{j,i}(\lambda, t) \in \text{sec}_c(\mathcal{A}_j F_i(\lambda, t))$  such that the terms  $\mathbf{D}_t^{\alpha_j} a_{j,i}(\lambda, t)$  are well-defined ( $0 \leq j \leq n$ ,  $t \geq 0$ ,  $\lambda \in \Omega$ ) and

$$0 = \sum_{j=1}^n \mathbf{D}_t^{\alpha_j} a_{j,i}(\lambda, t) - \mathbf{D}_t^{\alpha} a_{0,i}(\lambda, t), \quad t \geq 0, \quad \lambda \in \Omega.$$

## 6. $\mathcal{D}$ -Hypercyclic and $\mathcal{D}$ -topologically mixing properties of abstract degenerate multi-term fractional differential inclusions

In this section, we will briefly explain how we can, following the method proposed in [24], slightly generalize the notion introduced in the previous parts of our paper. For the sake of simplicity, we will not consider here the orbits of multilinear mappings (cf. [9, 17] for further information in this direction).

Denote by  $\mathfrak{Z}$  ( $\mathfrak{Z}_{\text{uniq}}$ ) the set of all tuples of initial values  $\vec{x} = (x_0, x_1, \dots, x_{m_n-1}) \in E^{m_n}$  for which there exists a (unique) strong solution of problem (5.1). Then  $\mathfrak{Z}$  is a linear subspace of  $E^{m_n}$  and  $\mathfrak{Z}_{\text{uniq}} \subseteq \mathfrak{Z}$ . It is clear that the equality  $\mathfrak{Z} = \mathfrak{Z}_{\text{uniq}}$  holds iff the zero function is a unique strong solution of the problem (5.1) with the initial value  $\vec{x} = \vec{0}$  (we refer the reader to [28, Proposition 3.8] and [27, Theorem 2.2] for some results on the uniqueness of solutions of (5.1) and related problems). For any  $\vec{x} \in \mathfrak{Z}$ , we denote by  $\mathfrak{S}(\vec{x})$  the set consisting of all strong solutions of problem (5.1) with the initial value  $\vec{x}$ . Assume that  $\mathfrak{P} : \mathfrak{Z} \rightarrow P(\cup_{\vec{x} \in \mathfrak{Z}} \mathfrak{S}(\vec{x}))$  is a fixed mapping satisfying  $\emptyset \neq \mathfrak{P}(\vec{x}) \subseteq \mathfrak{S}(\vec{x})$ ,  $\vec{x} \in \mathfrak{Z}$ .

Let  $\emptyset \neq W \subseteq \mathbb{N}_{m_n}$ , let  $\hat{E}_i$  be a linear subspace of  $E$  ( $i \in W$ ), and let  $\tilde{E}, \check{E}$  be linear subspaces of  $E^{m_n}$ . Suppose that the tuple  $\vec{\beta} = (\beta_0, \beta_1, \dots, \beta_{m_n-1}) \in [0, \alpha_n]^{m_n}$  is fixed. Set, with a little abuse of notation in comparison with [24],

$$\mathfrak{D} := \left( \tilde{E}, \check{E}, \{\hat{E}_i : i \in W\}, \vec{\beta} \right).$$

Denote by  $\mathcal{M}_{\mathfrak{D}}$  the set consisting of those tuples  $\vec{x} \in \mathfrak{Z}$  for which  $\text{Proj}_{i, m_n}(\vec{x}) \in \hat{E}_i$ ,  $i \in W$ .

**Definition 6.1.** (cf. also [24, Definition 3]) The abstract Cauchy problem (5.1) is said to be:

- (i)  $(\mathfrak{D}, \mathfrak{P})$ -hypercyclic iff there exist a tuple  $\vec{x} \in \mathcal{M}_{\mathfrak{D}} \cap \tilde{E}$  and a function  $u(\cdot; \vec{x}) \in \mathfrak{P}(\vec{x})$  such that

$$\mathfrak{G} := \left\{ \left( \left( \mathbf{D}_s^{\beta_0} u(s; \vec{x}) \right)_{s=t}, \left( \mathbf{D}_s^{\beta_1} u(s; \vec{x}) \right)_{s=t}, \dots, \left( \mathbf{D}_s^{\beta_{m_n-1}} u(s; \vec{x}) \right)_{s=t} \right) : t \geq 0 \right\}$$

is a dense subset of  $\check{E}$ ; such a vector is called a  $(\mathfrak{D}, \mathfrak{P})$ -hypercyclic vector of problem (5.1).

- (ii)  $\mathfrak{D}$ -hypercyclic iff it is  $(\mathfrak{D}, \mathfrak{G})$ -hypercyclic; any  $(\mathfrak{D}, \mathfrak{G})$ -hypercyclic vector of problem (5.1) will be also called a  $\mathfrak{D}$ -hypercyclic vector of problem (5.1).
- (iii)  $\mathfrak{D}_{\mathfrak{P}}$ -topologically transitive iff for every pair of open non-empty subsets  $U$  and  $V$  of  $E^{m_n}$  satisfying  $U \cap \tilde{E} \neq \emptyset$  and  $V \cap \check{E} \neq \emptyset$ , there exist a tuple  $\vec{x} \in \mathcal{M}_{\mathfrak{D}}$ , a function  $u(\cdot; \vec{x}) \in \mathfrak{P}(\vec{x})$  and a number  $t \geq 0$  such that  $\vec{x} \in U \cap \tilde{E}$  and

$$\left( \left( \mathbf{D}_s^{\beta_0} u(s; \vec{x}) \right)_{s=t}, \left( \mathbf{D}_s^{\beta_1} u(s; \vec{x}) \right)_{s=t}, \dots, \left( \mathbf{D}_s^{\beta_{m_n-1}} u(s; \vec{x}) \right)_{s=t} \right) \in V \cap \check{E}.$$

- (iv)  $\mathfrak{D}$ -topologically transitive iff it is  $\mathfrak{D}_{\mathfrak{G}}$ -topologically transitive.
- (v)  $\mathfrak{D}_{\mathfrak{P}}$ -topologically mixing iff for every pair of open non-empty subsets  $U$  and  $V$  of  $E^{m_n}$  satisfying  $U \cap \tilde{E} \neq \emptyset$  and  $V \cap \check{E} \neq \emptyset$ , there exists a number  $t_0 \geq 0$  such that, for every number  $t \geq t_0$ , there exist a tuple  $\vec{x}_t \in \mathcal{M}_{\mathfrak{D}}$  and a function  $u(\cdot; \vec{x}_t) \in \mathfrak{P}(\vec{x}_t)$  such that  $\vec{x}_t \in U \cap \tilde{E}$  and

$$\left( \left( \mathbf{D}_s^{\beta_0} u(s; \vec{x}_t) \right)_{s=t}, \left( \mathbf{D}_s^{\beta_1} u(s; \vec{x}_t) \right)_{s=t}, \dots, \left( \mathbf{D}_s^{\beta_{m_n-1}} u(s; \vec{x}_t) \right)_{s=t} \right) \in V \cap \check{E}.$$

- (vi)  $\mathfrak{D}$ -topologically mixing iff it is  $\mathfrak{D}_{\mathfrak{G}}$ -topologically mixing.

*Remark 6.2.* Let  $0 \leq \beta \leq \alpha < 2$ , and let the requirements of Theorem 3.3 hold (here the notation used to denote the space  $\tilde{E}$  is slightly different from that used in the formulation of above-mentioned theorem). Then the consideration from

[24, Remark 1(ii)] shows that the problem  $(\text{DFP})_\alpha$  is  $\mathfrak{D}_{\mathfrak{P}}$ -topologically mixing, provided that  $\vec{\beta} = (\beta, \beta)$ ,  $W = \{1\}$ ,  $\hat{E}_1 = \overline{\text{span}\{f(\lambda^\alpha) : \lambda \in \Omega\}}$ ,  $\tilde{E} = \hat{E}_1 \times \{0\}$ ,  $\tilde{E} = \{(z, z) : z \in \hat{E}_1\}$  and  $\mathfrak{P}((\sum_{i=1}^m \alpha_i f(\lambda_i^\alpha), 0)) = \{\sum_{i=1}^m \alpha_i E_\alpha(\cdot^\alpha \lambda_i^\alpha) f(\lambda_i^\alpha)\}$  ( $m \in \mathbb{N}$ ,  $\alpha_i \in \mathbb{C}$ ,  $\lambda_i \in \Omega$  for  $1 \leq i \leq m$ ). By assuming some extra conditions, a similar assertion can be proved for a general problem (5.1) (cf. [24, Remark 1(iii)]).

The conjugacy lemma stated in [24, Theorem 2] admits a very simple reformulation in our context. Details are left to the interested reader.

Using the proof of [24, Theorem 3] and the usual matrix conversion of abstract higher-order differential equations with integer order derivatives into the first order matrix differential equation, we can simply verify the validity of the following theorem.

**Theorem 6.3.** *Let  $\alpha_i = i$  for all  $i \in \mathbb{N}_n$ , let  $\Omega$  be an open non-empty subset of  $\mathbb{C}$  intersecting the imaginary axis, and let  $f : \Omega \rightarrow E$  be an analytic mapping satisfying*

$$0 \in \left( \lambda^{\alpha_n} \mathcal{B} + \sum_{i=1}^{n-1} \lambda^{\alpha_i} \mathcal{A}_i - \mathcal{A} \right) f(\lambda), \quad \lambda \in \Omega.$$

Set  $\vec{x}_\lambda := [f(\lambda) \ \lambda f(\lambda) \ \cdots \ \lambda^{n-1} f(\lambda)]^T$  ( $\lambda \in \Omega$ ),  $E_0 := \text{span}\{\vec{x}_\lambda : \lambda \in \Omega\}$ ,  $\tilde{E} := \tilde{E} := \overline{E_0}$ ,  $\vec{\beta} := (0, 1, \dots, n-1)$ ,  $W := \mathbb{N}_n$  and  $\hat{E}_i := \text{span}\{f(\lambda) : \lambda \in \Omega\}$ ,  $i \in W$ . Then  $\vec{x}_\lambda \in \mathfrak{M}_{\mathfrak{D}}$ ,  $\lambda \in \Omega$  and the abstract Cauchy problem (5.1) is  $\mathfrak{D}_{\mathfrak{P}}$ -topologically mixing provided that  $\sum_{j=1}^q e^{\lambda_j} f(\lambda_j) \in \mathfrak{P}(\sum_{j=1}^q \vec{x}_{\lambda_j})$  for any  $\sum_{j=1}^q \vec{x}_{\lambda_j} \in E_0$  ( $q \in \mathbb{N}$ ;  $\lambda_j \in \Omega$ ,  $1 \leq j \leq q$ ).

As observed in [24, Remark 2(iii)], Theorem 6.3 cannot be so simply reformulated for abstract degenerate multi-term inclusion (5.1), provided that there exists an index  $i \in \mathbb{N}_n$  such that  $\alpha_i \notin \mathbb{N}$ .

Examples already given in [19, Chapter 3] and [22]-[26] can serve for illustration of our theoretical results. Now we would like to present some new elaborate examples in support of Theorem 3.3, Theorem 5.2 and Theorem 6.3.

**Example 6.4.** (i) Suppose that  $E := C^2(\mathbb{R})$  is equipped with the usual Fréchet topology,  $0 < \alpha < 2$ ,  $m \in C(\mathbb{R})$  and  $m(x) > 0$ ,  $x \in \mathbb{R}$ . For any  $\lambda \in \mathbb{C}$ , we denote by  $\{f_\lambda^1(x), f_\lambda^2(x)\}$  the fundamental set of solutions of ordinary differential equation  $y'' = \lambda m(x)y$ . Using the elementary theory of linear ordinary differential equations, and direct computation of the matrix exponential

$$e^x \begin{bmatrix} 0 & 1 \\ \lambda m(x) & 0 \end{bmatrix}, \quad x \in \mathbb{R}, \lambda \in \mathbb{C} \setminus (-\infty, 0],$$

we can simply prove that for any arbitrarily chosen open connected subset  $\Omega$  of  $\mathbb{C} \setminus (-\infty, 0]$  satisfying that  $\Omega \cap \{e^{\pm it\alpha/2} : t \geq 0\} \neq \emptyset$ , the mappings  $\lambda \mapsto f_\lambda^1(x) \in E$  and  $\lambda \mapsto f_\lambda^2(x) \in E$  are analytic. Let  $\Omega$  be such a set.

Denote  $\tilde{E} := \overline{\text{span}\{f_\lambda^i(x) : \lambda \in \Omega, i = 1, 2\}}$ . Then we can apply [23, Theorem 5] or Theorem 3.3 in order to see that the abstract time-fractional Poisson heat equation (cf. [15] for more details):

$$\begin{aligned} \mathbf{D}_t^\alpha[m(x)u(t, x)] &= \Delta u(t, x), \quad t \geq 0, \quad x \in \mathbb{R}; \\ m(x)u(0, x) &= \phi(x); \quad \left(\frac{\partial}{\partial t}[m(x)u(t, x)]\right)_{t=0} = 0, \quad \text{if } \alpha > 1, \end{aligned}$$

is  $\tilde{E}$ -topologically mixing, with the meaning clear.

- (ii) Suppose that  $n = 3$ ,  $\frac{1}{3} < a < \frac{1}{2}$ ,  $\alpha_3 = 3a$ ,  $\alpha_2 = 2a$ ,  $\alpha_1 = 0$ ,  $\alpha = a$ ,  $c_1 < 0$ ,  $c_2 > 0$  and  $i = 1$ . Then the analysis given in [19, Example 3.3.12(iii)], in combination with Theorem 5.2, enables one to deduce some results on topologically mixing properties of the following abstract degenerate multi-term inclusion:

$$\begin{aligned} 0 \in \mathbf{D}_t^{3a}u(t) + c_2\mathbf{D}_t^{2a}u(t) + c_1\mathbf{D}_t^a u(t) - \mathcal{A}u(t), \quad t > 0, \\ u(0) = 0, \quad u'(0) = x, \quad u''(0) = 0, \end{aligned}$$

where  $\mathcal{A}$  is an MLO and satisfies certain conditions.

- (iii) Suppose that  $\mathbf{A}$  is an MLO,  $\Omega$  is an open non-empty subset of  $\mathbb{C}$  intersecting the imaginary axis,  $f : \Omega \rightarrow E$  is an analytic mapping,  $\lambda f(\lambda) \in \mathbf{A}f(\lambda)$ ,  $\lambda \in \Omega$ ,  $P_i(z)$  is a non-zero complex polynomial ( $0 \leq i \leq n$ ) and

$$(6.1) \quad z^n P_n(z) + \sum_{i=1}^{n-1} z^i P_i(z) - P_0(z) \equiv 0.$$

Set  $\mathcal{A}_i := P_i(\mathbf{A})$ ,  $i \in \mathbb{N}_n^0$ . Then for any non-zero complex polynomial  $P(z)$  we have  $P(\lambda)f(\lambda) \in P(\mathbf{A})f(\lambda)$ ,  $\lambda \in \Omega$  so that (6.1) implies

$$0 \in \left( \lambda^n \mathcal{B} + \sum_{i=1}^{n-1} \lambda^i \mathcal{A}_i - \mathcal{A} \right) f(\lambda), \quad \lambda \in \Omega.$$

Hence, Theorem 6.3 is susceptible to applications.

Besides hypercyclicity and topologically mixing property, which have been analyzed in this paper, there exist a great number of other known concepts in the theory of topological dynamics of linear operators, like Li-Yorke chaos, distributional chaos and frequent hypercyclicity. We refer the interested reader to [1, 4, 7, 8, 10, 26, 29, 33] for some references in this direction, closing the paper with the observation that it could be very interesting to further analyze the above-mentioned concepts from the point of view of the theory of abstract degenerate Volterra integro-differential equations.

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