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A TWO-DIMENSIONAL SYSTEM OF DELTA-NABLA FRACTIONAL DIFFERENCE INCLUSIONS

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Abstract. We investigate the existence of a solution for a system of Delta-Nabla fractional difference inclusion via fractional boundary conditions with some boundary conditions. We provide an example to illustrate our main result.

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1. Introduction

There are many published works about distinct types of fractional finite difference equations (see for example, [1], [6], [7], [27], [29] and the references therein) and some equations including the delta and nabla operators (see for example, [4], [14]-[17], [19], [23] and [24]). Also, many papers have been published about two-dimensional systems of differential or difference equations (see for example, [10], [12], [13], [18] and [25]). As we know, each differential equation is a special case of a related differential inclusion and so it is suitable for us to investigate fractional inclusions. Recently, some results have been obtained on fractional finite difference inclusions ([8] and [9]). The Gamma function has some well-known properties, such as $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(n) = (n-1)!$ for all $n \geq 1$. Define $t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$ for all $t, \nu \in \mathbb{R}$ whenever the right-hand side is defined. If $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then we define $t^{\underline{\nu}} = 0$ (see [3]). One can verify that $\nu^{\underline{\nu}} = \nu^{\underline{\nu-1}} = \Gamma(\nu+1)$ and $t^{\underline{\nu+1}} = (t-\nu)t^{\underline{\nu}}$. We use the notations $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ for all $a \in \mathbb{R}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}$ for all real numbers a and b whenever $b-a$ is a natural number. Let $\nu > 0$ be such that $m-1 < \nu \leq m$ for some natural number m. Then the ν-th fractional sum of f based at a is defined by $\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} f(s)$ for all $t \in \mathbb{N}_{a+\nu}$ ([22]). We consider the trivial case $\Delta_a^{-0} f(t) = f(t)$ for $t \in \mathbb{N}_a$. Similarly, we define $\Delta_a^{\nu} f(t) = \frac{1}{\Gamma(-\nu)} \sum_{k=a}^{t+\nu} (t-\sigma(k))^{\underline{-\nu-1}} f(k)$ for all $t \in \mathbb{N}_{a+m-\nu}$, where $\sigma(k) = k+1$ is the forward jump operator (see [3] and [5]).

For using the Covitz and Nadler theorem in our main result, we need to introduce some notions about multifunctions on metric spaces. Let (X, d) be a

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metric space. Denote by $P(X)$, 2^X, $P_c(X)$ and $P_{cp}(X)$ the class of all subsets, the class of all nonempty subsets, the class of all closed subsets and the class of all compact subsets of X, respectively. A mapping $Q : X \to 2^X$ is called a multifunction on X and $u \in X$ is called a fixed point of Q whenever $u \in Qu$ ([8] and [9]). The (generalized) Pompeiu-Hausdorff metric H_d on $P_c(X)$ is defined $H_d(A,B) = \max\left\{ \sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\right\}$, where $d(A,b) = \inf_{a\in A} d(a,b)$ (see [2] and [11]). Denote by Ψ the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$. Let $\alpha : X \times X \to [0,\infty)$ be a map, $\psi \in \Psi$ and $T : X \to 2^X$ a multifunction. We say that T is α-admissible whenever $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$ ([26]). Also, we say that X has the condition (C_α) whenever for each sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k ([26]).

In 2011, Goodrich investigated the problem $-\Delta^\nu y(t) = f(t+\nu-1, y(t+\nu-1))$ with boundary conditions $\alpha y(\nu-2)-\beta\Delta y(\nu-2) = 0$ and $\gamma y(\nu+b)-\delta\Delta y(\nu+b) = 0$, where $t \in [0,b]_{\mathbb{N}_0}, \nu \in (1,2]$ and $\alpha\gamma + \alpha\delta + \beta\gamma \neq 0$ with $\alpha,\beta,\gamma,\delta \geq 0$ ([20]). In 2015 by using idea of [20], Baleanu, Rezapour and Salehi investigated the existence of solution for the fractional finite difference inclusion $\Delta^\nu x(t) \in F(t, x(t), \Delta x(t), \Delta^2 x(t))$ with boundary conditions $\xi x(\nu-3)+\beta\Delta x(\nu-3) = 0$, $x(\eta) = 0$ and $\gamma x(b+\nu) + \delta\Delta x(b+\nu) = 0$, where $\eta \in \mathbb{N}_{\nu-2}^{b+\nu-1}$, $2 < \nu < 3$ and $F : \mathbb{N}_{\nu-3}^{b+\nu+1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction ([8]). Also, they investigated the fractional finite difference inclusion $\Delta_{\mu-2}^{\mu} x(t) \in F(t, x(t), \Delta x(t))$ via the boundary conditions $\Delta x(b+\mu) = A$ and $x(\mu-2) = B$, where $1 < \mu \leq 2$, $A, B \in \mathbb{R}$ and $F : \mathbb{N}_{\mu-2}^{b+\mu+2} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction in 2016 ([9]). By mixing ideas of the mentioned papers on two-dimensional systems of differential equations, we investigate the existence of solutions for the two-dimensional system of Delta-Nabla fractional difference inclusions

$$\begin{cases} \Delta^\nu x(t) \in F_1\Big(t, x(t), y(t), \nabla x(t), \nabla y(t), \nabla^2 x(t), \nabla^2 y(t)\Big), \\ \Delta^\mu y(t) \in F_2\Big(t, x(t), y(t), \nabla x(t), \nabla y(t), \nabla^2 x(t), \nabla^2 y(t)\Big) \end{cases} \tag{1.1}$$

with the boundary conditions $x(\nu-3) + \Delta^{\alpha_1} x(\nu-1-\alpha_1) = 0$, $x(\nu+b+1) + \Delta^{\beta_1} x(\nu+b-\beta_1) = 0$, $\Delta x(\eta_1) = 0$ and $y(\mu-3) + \Delta^{\alpha_2} y(\mu-1-\alpha_2) = 0$, $y(\mu+b+1)+\Delta^{\beta_2} y(\mu+b-\beta_2) = 0$ and $\Delta y(\eta_2) = 0$, where $1 < \alpha_1 \leq 2$, $0 < \beta_1 \leq 1$, $2 < \nu < 3$, $\eta_1 \in \mathbb{N}_{\nu-1}^{\nu+b-2}$, $1 < \alpha_2 \leq 2$, $0 < \beta_2 \leq 1$, $2 < \mu < 3$, $\eta_2 \in \mathbb{N}_{\mu-1}^{\mu+b-2}$ and $F_1 : \mathbb{N}_{\nu-3}^{b+\nu+1} \times \mathbb{R}^6 \to 2^{\mathbb{R}}$ and $F_2 : \mathbb{N}_{\mu-3}^{b+\mu+1} \times \mathbb{R}^6 \to 2^{\mathbb{R}}$ are compact valued multifunctions. As commonly known, the nabla operator ∇ on a function f acts by $\nabla f(t) = f(t)-f(t-1)$ ([28]). It is easy to check that $\nabla_t (t-a)^{\overline{\nu}} = \nu(t-a)^{\overline{\nu-1}}$ and $\nabla_t (a-t)^{\overline{\nu}} = -\nu(a-\rho(t))^{\overline{\nu-1}}$ ([28]). Let $\nu > 0$ be such that $m-1 < \nu \leq m$ for some natural number m. Then the ν-th nabla fractional sum of f based at a is defined by $\nabla_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{k=a+1}^{t} (t-\rho(k))^{\overline{\nu-1}} f(k)$ for all $t \in \mathbb{N}_a$ ([28]). Similarly, one can define $\nabla_a^{\nu} f(t) = \frac{1}{\Gamma(-\nu)} \sum_{k=a+1}^{t} (t-\rho(k))^{\overline{-\nu-1}} f(k)$ for all

$t \in \mathbb{N}_{a+m}$ ([28]). By considering the details, it is easy now to know more about ∇u and $\nabla^2 u$ which we used in the problem (1). To prove of our main results, we need the following results.

Lemma 1.1. *([21]) Let $a \in \mathbb{R}$, $\nu > 0$ with $m - 1 < \nu \leq m$ and $\mu > 0$. Then, $\Delta(t-a)^{\underline{\mu}} = \mu(t-a)^{\underline{\mu-1}}$ for all t for which both sides are well defined. Also, $\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}} = \mu^{\underline{-\nu}}(t-a)^{\underline{\mu+\nu}}$ for $t \in \mathbb{N}_{a+\mu+\nu}$ and $\Delta_{a+\mu}^{\nu}(t-a)^{\underline{\mu}} = \mu^{\underline{\nu}}(t-a)^{\underline{\mu-\nu}}$ for $t \in \mathbb{N}_{a+\mu+m-\nu}$.*

Lemma 1.2. *([29]) Let $\mu > 0$ with $m - 1 < \mu \leq m$, $a \in \mathbb{R}$ and $f : \mathbb{N}_a \to \mathbb{R}$ be a map. Then, $\Delta_{a+m-\mu}^{-\mu}\Delta_a^{\mu} f(t) = f(t) + c_1(t-a-m+\mu)^{\underline{\mu-1}} + c_2(t-a-m+\mu)^{\underline{\mu-2}} + \ldots + c_m(t-a-m+\mu)^{\underline{\mu-m}}$, where $c_1, \ldots, c_m \in \mathbb{R}$ are some constants.*

Lemma 1.3. *[26] Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, \infty)$ a map, $\psi \in \Psi$ a strictly increasing map and $T : X \to CB(X)$ an α-admissible multifunction such that $\alpha(x, y)H(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$, and there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$. If X satisfies the condition (C_α), then T has a fixed point.*

2. Main Results

Now, we are ready to state and prove our main results. First, we provide the following key Lemma.

Lemma 2.1. *Let $y : \mathbb{N}_0^{b+1} \to \mathbb{R}$ be a map and $2 < \nu < 3$. Then x_0 is a solution for the fractional difference equation $\Delta^\nu x(t) = y(t)$ with the boundary conditions $x(\nu-3) + \Delta^\alpha x(\nu-1-\alpha) = 0$, $x(\nu+b+1) + \Delta^\beta x(\nu+b-\beta) = 0$ and $\Delta x(\eta) = 0$ if and only if x_0 is a solution for the fractional sum equation $x(t) = \sum_{s=0}^{b+1} G(t, s)y(s)$, where*

$$G(t,s) = \left[\frac{-\delta(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta\delta+\lambda)\Gamma(\nu)} - \frac{\delta(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]t^{\underline{\nu-1}}$$

$$+\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta\delta+\lambda)\Gamma(\nu)} + \frac{(\nu-3)(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\nu-2)(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]t^{\underline{\nu-2}}$$

$$+\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta\delta+\lambda)\Gamma(\nu)} - \frac{(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]t^{\underline{\nu-3}} + \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}$$

whenever $0 \leq s \leq t-\nu \leq b+1$ and

$$G(t,s) = \left[\frac{-\delta(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta\delta+\lambda)\Gamma(\nu)} - \frac{\delta(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]t^{\underline{\nu-1}}$$

$$+\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta\delta+\lambda)\Gamma(\nu)} + \frac{(\nu-3)(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\nu-2)(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]t^{\underline{\nu-2}}$$

$$+\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta\delta+\lambda)\Gamma(\nu)} - \frac{(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]t^{\underline{\nu-3}}$$

whenever $0 \leq t-\nu+1 \leq s \leq b+1$. Here,

$$\theta = \frac{\Gamma(\nu+b+2)}{(b+2)!} + \frac{(\nu+b-\beta)^{\underline{\nu-3-\beta}}\Gamma(\nu)(b+2)(b+3)}{\Gamma(\nu-\beta)},$$

$$\delta = \frac{(\nu-\alpha-1)(\nu-4+\alpha)-2}{2(\nu-1)(\nu-2)}$$

and

$$\lambda = \frac{\Gamma(\nu+b+2)(10-3\nu-3b-b\nu)}{(\nu-2)(b+4)!} + \frac{(\nu+b-\beta)^{\underline{\nu-3-\beta}}\Gamma(\nu-2)(7-b\nu-2\nu+3b-\beta)}{\Gamma(\nu-\beta-1)}.$$

Proof. Let x_0 be a solution for the fractional difference equation $\Delta^\nu x(t) = y(t)$ with the boundary conditions $x(\nu-3) + \Delta^\alpha x(\nu-1-\alpha) = 0$, $x(\nu+b+1) + \Delta^\beta x(\nu+b-\beta) = 0$ and $\Delta x(\eta) = 0$. By using Lemma 1.2, we get $x_0(t) = c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}} + c_3 t^{\underline{\nu-3}} + \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}y(s)$ and so

$$\Delta^\alpha x_0(t) = c_1\frac{\Gamma(\nu)t^{\underline{\nu-1-\alpha}}}{\Gamma(\nu-\alpha)} + c_2\frac{\Gamma(\nu-1)t^{\underline{\nu-2-\alpha}}}{\Gamma(\nu-\alpha-1)} + c_3\frac{\Gamma(\nu-2)t^{\underline{\nu-3-\alpha}}}{\Gamma(\nu-\alpha-2)}$$

$$+\frac{1}{\Gamma(\nu-\alpha)}\sum_{s=0}^{t-\nu+\alpha}(t-\sigma(s))^{\underline{\nu-1-\alpha}}y(s).$$

By using the boundary condition $x(\nu-3) + \Delta^\alpha x(\nu-1-\alpha) = 0$, we obtain

$$c_1(\nu-3)^{\underline{\nu-1}} + c_2(\nu-3)^{\underline{\nu-2}} + c_3(\nu-3)^{\underline{\nu-3}} + \frac{1}{\Gamma(\nu)}\sum_{s=0}^{-3}((\nu-3)-\sigma(s))^{\underline{\nu-1}}y(s)$$

$$= -c_1\frac{\Gamma(\nu)(\nu-\alpha-1)^{\underline{\nu-1-\alpha}}}{\Gamma(\nu-\alpha)} - c_2\frac{\Gamma(\nu-1)(\nu-\alpha-1)^{\underline{\nu-2-\alpha}}}{\Gamma(\nu-\alpha-1)}$$

$$-c_3\frac{\Gamma(\nu-2)(\nu-\alpha-1)^{\underline{\nu-3-\alpha}}}{\Gamma(\nu-\alpha-2)} - \frac{1}{\Gamma(\nu-\alpha)}\sum_{s=0}^{(\nu-\alpha-1)-\nu+\alpha}((\nu-\alpha-1)-\sigma(s))^{\underline{\nu-1-\alpha}}y(s).$$

Since $(\nu-3)^{\underline{\nu-1}} = (\nu-3)^{\underline{\nu-2}} = 0$, $(\nu-3)^{\underline{\nu-3}} = \Gamma(\nu-2)$ and $\sum_{s=0}^{-3}(\nu-3-\sigma(s))^{\underline{\nu-1}}y(s) = 0$, we get $c_3\Gamma(\nu-2) = -c_1\Gamma(\nu) - c_2\Gamma(\nu-1)(\nu-\alpha-1) - \frac{1}{2}c_3\Gamma(\nu-2)(\nu-\alpha-1)(\nu-\alpha-2)$ and so

$$c_3 = -c_1(\nu-1)(\nu-2) - c_2(\nu-2)(\nu-\alpha-1) - \frac{1}{2}c_3(\nu-\alpha-1)(\nu-\alpha-2).$$

Now by using the condition $x(\nu+b+1) + \Delta^\beta x(\nu+b-\beta) = 0$, we obtain

$$c_1(\nu+b+1)^{\underline{\nu-1}} + c_2(\nu+b+1)^{\underline{\nu-2}} + c_3(\nu+b+1)^{\underline{\nu-3}}$$

$$+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{(\nu+b+1)-\nu}((\nu+b+1)-\sigma(s))^{\underline{\nu-1}}y(s) + c_1\frac{\Gamma(\nu)(\nu+b-\beta)^{\underline{\nu-1-\beta}}}{\Gamma(\nu-\beta)}$$

$$+c_2\frac{\Gamma(\nu-1)(\nu+b-\beta)^{\underline{\nu-2-\beta}}}{\Gamma(\nu-\beta-1)} + c_3\frac{\Gamma(\nu-2)(\nu+b-\beta)^{\underline{\nu-3-\beta}}}{\Gamma(\nu-\beta-2)}$$

$$+\frac{1}{\Gamma(\nu-\beta)}\sum_{s=0}^{(\nu+b-\beta)-\nu+\beta}((\nu+b-\beta)-\sigma(s))^{\underline{\nu-1-\beta}}y(s) = 0$$

and so

$$c_1 \frac{(b+3)\Gamma(\nu+b+2)}{\Gamma(\nu+b+1-\beta)} + c_2 \frac{\Gamma(\nu+b+2)}{\Gamma(\nu+b+1-\beta)} + c_3 \frac{\Gamma(\nu+b+2)}{(b+4)\Gamma(\nu+b+1-\beta)}$$

$$+ \frac{1}{(\nu+b-\beta)^{\underline{\nu-3-\beta}}\Gamma(\nu)} \sum_{s=0}^{b+1} (\nu+b+1-\sigma(s))^{\underline{\nu-1}} y(s)$$

$$+ c_1 \frac{\Gamma(\nu)(b+2)(b+3)}{\Gamma(\nu-\beta)} + c_2 \frac{\Gamma(\nu-1)(b+3)}{\Gamma(\nu-\beta-1)} + c_3 \frac{\Gamma(\nu-2)}{\Gamma(\nu-\beta-2)}$$

$$+ \frac{1}{(\nu+b-\beta)^{\underline{\nu-3-\beta}}\Gamma(\nu-\beta)} \sum_{s=0}^{b} (\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}} y(s) = 0.$$

On the other hand, we have

$$\Delta x_0(t) = c_1(\nu-1)t^{\underline{\nu-2}} + c_2(\nu-2)t^{\underline{\nu-3}} + c_3(\nu-3)t^{\underline{\nu-4}}$$

$$+ \frac{1}{\Gamma(\nu-1)} \sum_{s=0}^{t-\nu+1} (t-\sigma(s))^{\underline{\nu-2}} y(s).$$

By using the condition $\Delta x_0(\nu-3) = 0$, we get

$$0 = c_1(\nu-1)(\nu-3)^{\underline{\nu-2}} + c_2(\nu-2)(\nu-3)^{\underline{\nu-3}} + c_3(\nu-3)(\nu-3)^{\underline{\nu-4}}$$

$$+ \frac{1}{\Gamma(\nu-1)} \sum_{s=0}^{\nu-3-\nu+1} (\nu-3-\sigma(s))^{\underline{\nu-2}} y(s)$$

and so $c_2(\nu-2) + c_3(\nu-3) = 0$. Hence, we obtain

$$c_1 = \frac{-\delta}{(\theta\delta+\lambda)\Gamma(\nu)} \sum_{s=0}^{b+1} (\nu+b+1-\sigma(s))^{\underline{\nu-1}} y(s)$$

$$- \frac{\delta}{(\theta\delta+\lambda)\Gamma(\nu-\beta)} \sum_{s=0}^{b} (\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}} y(s),$$

$c_2 = \frac{\nu-3}{(\nu-2)(\theta\delta+\lambda)\Gamma(\nu)} \sum_{s=0}^{b+1} (\nu+b+1-\sigma(s))^{\underline{\nu-1}} y(s) + \frac{(\nu-3)}{(\nu-2)(\theta\delta+\lambda)\Gamma(\nu-\beta)} \sum_{s=0}^{b} (\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}} y(s)$ and $c_3 = \frac{-1}{(\theta\delta+\lambda)\Gamma(\nu)} \sum_{s=0}^{b+1} (\nu+b+1-\sigma(s))^{\underline{\nu-1}} y(s) - \frac{1}{(\theta\delta+\lambda)\Gamma(\nu-\beta)} \sum_{s=0}^{b} (\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}} y(s)$. We can replace $\sum_{s=0}^{b} (\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}} y(s)$ by $\sum_{s=0}^{b+1} (\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}} y(s)$, because $(\nu+b-\beta-\sigma(b+1))^{\underline{\nu-1-\beta}} = 0$. Thus, we obtain

$$x_0(t) = \sum_{s=0}^{b+1} \left[\frac{-\delta(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta\delta+\lambda)\Gamma(\nu)} - \frac{\delta(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\theta\delta+\lambda)\Gamma(\nu-\beta)} \right] t^{\underline{\nu-1}} y(s)$$

$$+\sum_{s=0}^{b+1}\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta\delta+\lambda)\Gamma(\nu)}+\frac{(\nu-3)(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\nu-2)(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]t^{\underline{\nu-2}}y(s)$$

$$+\sum_{s=0}^{b+1}\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta\delta+\lambda)\Gamma(\nu)}-\frac{(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]t^{\underline{\nu-3}}y(s)$$

$$+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}y(s)=\sum_{s=0}^{b+1}G(t,s)y(s).$$

Now let x_0 be a solution for the equation $x(t)=\sum_{s=0}^{b+1}G(t,s)y(s)$. Then, we have

$$x_0(t)=\sum_{s=0}^{b+1}\left[\frac{-\delta(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta\delta+\lambda)\Gamma(\nu)}-\frac{\delta(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]t^{\underline{\nu-1}}y(s)$$

$$+\sum_{s=0}^{b+1}\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta\delta+\lambda)\Gamma(\nu)}+\frac{(\nu-3)(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\nu-2)(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]t^{\underline{\nu-2}}y(s)$$

$$+\sum_{s=0}^{b+1}\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta\delta+\lambda)\Gamma(\nu)}-\frac{(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]t^{\underline{\nu-3}}y(s)$$

$$+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}y(s).$$

Since $(\nu-3)^{\underline{\nu-1}}=(\nu-3)^{\underline{\nu-2}}=0$, $(\nu-3)^{\underline{\nu-3}}=\Gamma(\nu-2)$ and $\sum_{s=0}^{-3}(\nu-3-\sigma(s))^{\underline{\nu-1}}y(s)=0$, we get $x_0(\nu-3)=\Gamma(\nu-2)\sum_{s=0}^{b+1}\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta\delta+\lambda)\Gamma(\nu)}-\frac{(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]y(s)$. On the other hand, some simple calculations show us that $(\nu-1-\alpha)^{\underline{\nu-1-\alpha}}=(\nu-1-\alpha)^{\underline{\nu-2-\alpha}}=\Gamma(\nu-\alpha)$, $(\nu-1-\alpha)^{\underline{\nu-3-\alpha}}=\frac{1}{2}\Gamma(\nu-\alpha)$ and $\sum_{s=0}^{-1}(t-\sigma(s))^{\underline{\nu-1-\alpha}}y(s)=0$. Since

$$\Delta^{\alpha}x_0(\nu-1-\alpha)=\frac{\Gamma(\nu)\Gamma(\nu-\alpha)}{\Gamma(\nu-\alpha)}$$

$$\times\sum_{s=0}^{b+1}\left[\frac{-\delta(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta\delta+\lambda)\Gamma(\nu)}-\frac{\delta(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]y(s)$$

$$+\frac{\Gamma(\nu-1)\Gamma(\nu-\alpha)}{\Gamma(\nu-\alpha-1)}$$

$$\times\sum_{s=0}^{b+1}\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta\delta+\lambda)\Gamma(\nu)}+\frac{(\nu-3)(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\nu-2)(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]y(s)$$

$$+\frac{\Gamma(\nu-2)\Gamma(\nu-\alpha)}{2\Gamma(\nu-\alpha-2)}\sum_{s=0}^{b+1}\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta\delta+\lambda)\Gamma(\nu)}-\frac{(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\theta\delta+\lambda)\Gamma(\nu-\beta)}\right]y(s),$$

we get $x_0(\nu-3)+\Delta^{\alpha}x_0(\nu-1-\alpha)=0$. One can check that $x_0(\nu+b+1)+\Delta^{\beta}x_0(\nu+b-\beta)=0$ and $\Delta x_0(\eta)=0$. Thus, $x_0(t)=c_1t^{\underline{\nu-1}}+c_2t^{\underline{\nu-2}}+c_3t^{\underline{\nu-3}}+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}y(s)$ is a solution for the equation $\Delta^{\nu}x(t)=y(t)$. This completes the proof. □

Now, we are ready to prove our result about the existence of solutions for the two-dimensional system of Delta-Nabla fractional difference inclusions (1.1). Let $\mathcal{X}_1$ be the set of all functions $x : \mathbb{N}_{\nu-3}^{b+\nu+1} \to \mathbb{R}$ endowed with the norm $\|x\|_1 = \max_{t\in\mathbb{N}_{\nu-3}^{b+\nu+1}} |x(t)| + \max_{t\in\mathbb{N}_{\nu-3}^{b+\nu+1}} |\nabla x(t)| + \max_{t\in\mathbb{N}_{\nu-3}^{b+\nu+1}} |\nabla^2 x(t)|$ and $\mathcal{X}_2$ be the set of all functions $y : \mathbb{N}_{\mu-3}^{b+\mu+1} \to \mathbb{R}$ endowed with the norm

$$\|y\|_2 = \max_{t\in\mathbb{N}_{\mu-3}^{b+\mu+1}} |y(t)| + \max_{t\in\mathbb{N}_{\mu-3}^{b+\nu+1}} |\nabla y(t)| + \max_{t\in\mathbb{N}_{\mu-3}^{b+\mu+1}} |\nabla^2 y(t)|.$$

It is clear that $(\mathcal{X}_1, \|.\|_1)$ and $(\mathcal{X}_2, \|.\|_2)$ are Banach spaces. Thus, the space $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ with the norm $\|(x,y)\|_{\mathcal{X}} = \|x\|_1 + \|y\|_2$ is a Banach space. Let $(x,y) \in \mathcal{X}$ be given. Define the set of selections of F_i at (x,y) by $S_{F_i,(x,y)} = \{z : \mathbb{N}_0^{b+1} \to \mathbb{R} : \ z(t) \in F_i(t, x(t), y(t), \nabla x(t), \nabla y(t), \nabla^2 x(t), \nabla^2 y(t))$ for all $t \in \mathbb{N}_0^{b+1}\}$ for $i = 1,2$. We say that $(x,y) \in \mathcal{X}$ is a solution for the two-dimensional system (1.1) whenever there exists functions $z_1, z_2 : \mathbb{N}_0^{b+1} \to \mathbb{R}$ such that $z_i(t) \in F_i(t, x(t), y(t), \nabla x(t), \nabla y(t), \nabla^2 x(t), \nabla^2 y(t))$ for all $t \in \mathbb{N}_0^{b+1}$,

$$\begin{aligned} x(t) = &\sum_{s=0}^{b+1}\left[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right] t^{\underline{\nu-1}} z_1(s) \\ &+\sum_{s=0}^{b+1}\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} + \frac{(\nu-3)(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right] t^{\underline{\nu-2}} z_1(s) \\ &+\sum_{s=0}^{b+1}\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right] t^{\underline{\nu-3}} z_1(s) \\ &+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}} z_1(s) \end{aligned}$$

for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$,

$$\begin{aligned} y(t) = &\sum_{s=0}^{b+1}\left[\frac{-\delta_2(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{\delta_2(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right] t^{\underline{\mu-1}} z_2(s) \\ &+\sum_{s=0}^{b+1}\left[\frac{(\mu-3)(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} + \frac{(\mu-3)(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right] t^{\underline{\mu-2}} z_2(s) \\ &+\sum_{s=0}^{b+1}\left[\frac{-(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right] t^{\underline{\mu-3}} z_2(s) \\ &+\frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}} z_2(s) \end{aligned}$$

for all $t \in \mathbb{N}_{\mu-3}^{b+\mu+1}$, $x(\nu-3)+\Delta^{\alpha_1}x(\nu-1-\alpha_1) = 0$, $x(\nu+b+1)+\Delta^{\beta_1}x(\nu+b-\beta_1) = 0$, $\Delta x(\eta_1) = 0$, $y(\mu-3)+\Delta^{\alpha_2}y(\mu-1-\alpha_2) = 0$, $y(\mu+b+1)+\Delta^{\beta_2}y(\mu+b-\beta_2) = 0$ and $\Delta y(\eta_2) = 0$. Since $F_i(t, x(t), y(t), \nabla x(t), \nabla y(t), \nabla^2 x(t), \nabla^2 y(t)) \neq \emptyset$ for $i = 1,2$, the selection principle implies that $S_{F_i,(x,y)}$ is non empty. Now, we prove our main result.

Theorem 2.2. *Suppose that $\psi \in \Psi$, $F_1 : \mathbb{N}_{\nu-3}^{b+\nu+1} \times \mathbb{R}^6 \to P_{cp}(\mathbb{R})$ and $F_2 : \mathbb{N}_{\mu-3}^{b+\mu+1} \times \mathbb{R}^6 \to P_{cp}(\mathbb{R})$ are multifunctions such that*

$$H_d\Big(F_i(t, x_1, x_2, x_3, x_4, x_5, x_6), F_i(t, y_1, y_2, y_3, y_4, y_5, y_6)\Big) \leq \psi(\sum_{j=1}^{6} |x_j - y_j|)$$

for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$, $x_1, \ldots, x_6, y_1, \ldots, y_6 \in \mathbb{R}$ and $i = 1, 2$. Then the problem (1.1) has a solution.

Proof. Let $(x, y) \in \mathcal{X}$ be given. Choose $z_1 \in S_{F_1,(x,y)}$ and $z_2 \in S_{F_2,(x,y)}$. Define

$$h_1(t) = \sum_{s=0}^{b+1} \left[\frac{-\delta_1(\nu + b + 1 - \sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1 + \lambda_1)\Gamma(\nu)} - \frac{\delta_1(\nu + b - \beta_1 - \sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1 + \lambda_1)\Gamma(\nu - \beta_1)} \right] t^{\underline{\nu-1}} z_1(s)$$

$$+ \sum_{s=0}^{b+1} \left[\frac{(\nu - 3)(\nu + b + 1 - \sigma(s))^{\underline{\nu-1}}}{(\nu - 2)(\theta_1\delta_1 + \lambda_1)\Gamma(\nu)} + \frac{(\nu - 3)(\nu + b - \beta_1 - \sigma(s))^{\underline{\nu-1-\beta_1}}}{(\nu - 2)(\theta_1\delta_1 + \lambda_1)\Gamma(\nu - \beta_1)} \right] t^{\underline{\nu-2}} z_1(s)$$

$$+ \sum_{s=0}^{b+1} \left[\frac{-(\nu + b + 1 - \sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1 + \lambda_1)\Gamma(\nu)} - \frac{(\nu + b - \beta_1 - \sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1 + \lambda_1)\Gamma(\nu - \beta_1)} \right] t^{\underline{\nu-3}} z_1(s)$$

$$+ \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - \sigma(s))^{\underline{\nu-1}} z_1(s)$$

for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$ and

$$h_2(t) = \sum_{s=0}^{b+1} \left[\frac{-\delta_2(\mu + b + 1 - \sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2 + \lambda_2)\Gamma(\mu)} - \frac{\delta_2(\mu + b - \beta_2 - \sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2 + \lambda_2)\Gamma(\mu - \beta_2)} \right] t^{\underline{\mu-1}} z_2(s)$$

$$+ \sum_{s=0}^{b+1} \left[\frac{(\mu - 3)(\mu + b + 1 - \sigma(s))^{\underline{\mu-1}}}{(\mu - 2)(\theta_2\delta_2 + \lambda_2)\Gamma(\mu)} + \frac{(\mu - 3)(\mu + b - \beta_2 - \sigma(s))^{\underline{\mu-1-\beta_2}}}{(\mu - 2)(\theta_2\delta_2 + \lambda_2)\Gamma(\mu - \beta_2)} \right] t^{\underline{\mu-2}} z_2(s)$$

$$+ \sum_{s=0}^{b+1} \left[\frac{-(\mu + b + 1 - \sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2 + \lambda_2)\Gamma(\mu)} - \frac{(\mu + b - \beta_2 - \sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2 + \lambda_2)\Gamma(\mu - \beta_2)} \right] t^{\underline{\mu-3}} z_2(s)$$

$$+ \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu} (t - \sigma(s))^{\underline{\mu-1}} z_2(s)$$

for all $t \in \mathbb{N}_{\mu-3}^{b+\mu+1}$, where

$$\lambda_1 = \frac{\Gamma(\nu + b + 2)(10 - 3\nu - 3b - b\nu)}{(\nu - 2)(b + 4)!}$$

$$+ \frac{(\nu + b - \beta_1)^{\underline{\nu-3-\beta_1}}\Gamma(\nu - 2)(7 - b\nu - 2\nu + 3b - \beta_1)}{\Gamma(\nu - \beta_1 - 1)},$$

$\theta_1 = \frac{\Gamma(\nu+b+2)}{(b+2)!} + \frac{(\nu+b-\beta_1)^{\underline{\nu-3-\beta_1}}\Gamma(\nu)(b+2)(b+3)}{\Gamma(\nu-\beta_1)}$, $\delta_1 = \frac{(\nu-\alpha_1-1)(\nu-4+\alpha_1)-2}{2(\nu-1)(\nu-2)}$, $\theta_2 = \frac{\Gamma(\mu+b+2)}{(b+2)!} + \frac{(\mu+b-\beta_2)^{\underline{\mu-3-\beta_2}}\Gamma(\mu)(b+2)(b+3)}{\Gamma(\nu-\beta_2)}$, $\delta_2 = \frac{(\mu-\alpha_2-1)(\mu-4+\alpha_2)-2}{2(\mu-1)(\mu-2)}$ and $\lambda_2 =$

$\frac{\Gamma(\mu+b+2)(10-3\mu-3b-b\mu)}{(\mu-2)(b+4)!}+\frac{(\mu+b-\beta_2)^{\underline{\mu-3-\beta_2}}\Gamma(\mu-2)(7-b\mu-2\mu+3b-\beta_2)}{\Gamma(\mu-\beta_2-1)}$. It is easy to see that $h_1 \in \mathcal{X}_1, h_2 \in \mathcal{X}_2$ and so the sets

$$
\begin{aligned}
\Omega_1 = \Big\{ & h_1 \in \mathcal{X}_1 : \text{ there exists } y \in S_{F_1,(x,y)} \text{ such that} \\
h_1(t) =& \sum_{s=0}^{b+1}\left[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right]t^{\underline{\nu-1}}z_1(s) \\
&+\sum_{s=0}^{b+1}\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} + \frac{(\nu-3)(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right]t^{\underline{\nu-2}}z_1(s) \\
&+\sum_{s=0}^{b+1}\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right]t^{\underline{\nu-3}}z_1(s) \\
&+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}z_1(s) \text{ for all } t \in \mathbb{N}_{\nu-3}^{\nu+b+1}\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
\Omega_2 = \Big\{ & h_2 \in \mathcal{X}_2 : \text{ there exists } y \in S_{F_2,(x,y)} \text{ such that} \\
h_2(t) =& \sum_{s=0}^{b+1}\left[\frac{-\delta_2(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{\delta_2(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-1}}z_2(s) \\
&+\sum_{s=0}^{b+1}\left[\frac{(\mu-3)(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} + \frac{(\mu-3)(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-2}}z_2(s) \\
&+\sum_{s=0}^{b+1}\left[\frac{-(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-3}}z_2(s) \\
&+\frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}z_2(s) \text{ for all } t \in \mathbb{N}_{\mu-3}^{\mu+b+1}\Big\}
\end{aligned}
$$

are nonempty. Define the operator $T : \mathcal{X} \to 2^{\mathcal{X}}$ by

$$
T(x,y)(t_1,t_2) = \begin{pmatrix} T_1(x,y)(t_1) \\ T_2(x,y)(t_2) \end{pmatrix},
$$

where

$$T_1(x,y) = \Big\{ g_1 \in \mathcal{X}_1 : \text{ there exists } \ z_1 \in S_{F_{1,(x,y)}} \text{such that}$$
$$g_1(t) = \sum_{s=0}^{b+1} \left[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)} \right] t^{\underline{\nu-1}} z_1(s)$$
$$+ \sum_{s=0}^{b+1} \left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} + \frac{(\nu-3)(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)} \right] t^{\underline{\nu-2}} z_1(s)$$
$$+ \sum_{s=0}^{b+1} \left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)} \right] t^{\underline{\nu-3}} z_1(s)$$
$$+ \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} z_1(s) \text{ for all } t \in \mathbb{N}_{\nu-3}^{\nu+b+1} \Big\}$$

and

$$T_2(x,y) = \Big\{ g_2 \in \mathcal{X}_2 : \text{ there exists } \ z_2 \in S_{F_{2,(x,y)}} \text{such that}$$
$$g_2(t) = \sum_{s=0}^{b+1} \left[\frac{-\delta_2(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{\delta_2(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)} \right] t^{\underline{\mu-1}} z_2(s)$$
$$+ \sum_{s=0}^{b+1} \left[\frac{(\mu-3)(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} + \frac{(\mu-3)(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)} \right] t^{\underline{\mu-2}} z_2(s)$$
$$+ \sum_{s=0}^{b+1} \left[\frac{-(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)} \right] t^{\underline{\mu-3}} z_2(s)$$
$$+ \frac{1}{\Gamma(\mu)} \sum_{s=0}^{t-\mu} (t-\sigma(s))^{\underline{\mu-1}} z_2(s) \text{ for all } t \in \mathbb{N}_{\mu-3}^{\mu+b+1} \Big\}.$$

We show that the multifunction T has a fixed point. First, we prove that $T(x,y)$ is a closed subset of $\mathcal{X}$ for all $(x,y) \in \mathcal{X}$. Let $(x,y) \in \mathcal{X}$ and $\{(x^n,y^n)\}_{n\geq 1}$ be a sequence in $T(x,y)$ with $(x^n,y^n) \to (x^0,y^0)$. For each n, choose $(z_1^n, z_2^n) \in S_{F_{1,(x,y)}} \times S_{F_{2,(x,y)}}$ such that

$$x^n = \sum_{s=0}^{b+1} \left[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)} \right] t^{\underline{\nu-1}} z_1^n(s)$$
$$+ \sum_{s=0}^{b+1} \left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} + \frac{(\nu-3)(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)} \right] t^{\underline{\nu-2}} z_1^n(s)$$
$$+ \sum_{s=0}^{b+1} \left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)} \right] t^{\underline{\nu-3}} z_1^n(s)$$
$$+ \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} z_1^n(s)$$

for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$,

$$y^n = \sum_{s=0}^{b+1}\left[\frac{-\delta_2(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{\delta_2(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-1}}z_2^n(s)$$
$$+\sum_{s=0}^{b+1}\left[\frac{(\mu-3)(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} + \frac{(\mu-3)(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-2}}z_2^n(s)$$
$$+\sum_{s=0}^{b+1}\left[\frac{-(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-3}}z_2^n(s)$$
$$+\frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}z_2^n(s)$$

for all $t \in \mathbb{N}_{\mu-3}^{b+\mu+1}$ and $n \geq 1$. Since the multifunctions F_1 and F_2 are compact valued, $\{z_i^n\}_{n\geq 1}$ has a subsequence which converges to some $z_i^0 : \mathbb{N}_0^{b+1} \to \mathbb{R}$ for $i = 0, 1$. We denote this subsequence again by $\{z_i^n\}_{n\geq 1}$. It is easy to check that $z_1^0 \in S_{F_1,(x,y)}$, $z_2^0 \in S_{F_2,(x,y)}$,

$$x^n(t) \to x^0(t) =$$
$$\sum_{s=0}^{b+1}\left[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right]t^{\underline{\nu-1}}z_1^0(s)$$
$$+\sum_{s=0}^{b+1}\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} + \frac{(\nu-3)(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right]t^{\underline{\nu-2}}z_1^0(s)$$
$$+\sum_{s=0}^{b+1}\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right]t^{\underline{\nu-3}}z_1^0(s)$$
$$+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}z_1^0(s)$$

for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$ and

$$y^n(t) \to y^0(t) =$$
$$\sum_{s=0}^{b+1}\left[\frac{-\delta_2(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{\delta_2(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-1}}z_2^0(s)$$
$$+\sum_{s=0}^{b+1}\left[\frac{(\mu-3)(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} + \frac{(\mu-3)(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-2}}z_2^0(s)$$
$$+\sum_{s=0}^{b+1}\left[\frac{-(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-3}}z_2^0(s)$$
$$+\frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}z_2^0(s)$$

for all $t \in \mathbb{N}_{\mu-3}^{b+\mu+1}$. This implies that $x^0 \in T_1(x,y)$ and $y^0 \in T_2(x,y)$. Hence, $(x^0,y^0) \in T(x,y)$ and so the multifunction T has closed values. Since T is a compact valued multifunction, it is easy to see that $T(x,y)$ is a bounded set in $\mathcal{X}$ for all $(x,y) \in \mathcal{X}$. Let $(u_1,u_2),(v_1,v_2) \in \mathcal{X}$, $(h_1,h_2) \in T(u_1,u_2)$ and $(h_1',h_2') \in T(v_1,v_2)$. Choose $(z_1,z_2) \in S_{F_1,(u_1,u_2)} \times S_{F_2,(u_1,u_2)}$ and $(z_1',z_2') \in S_{F_1,(v_1,v_2)} \times S_{F_2,(v_1,v_2)}$ such that

$$
\begin{aligned}
h_1(t) = &\sum_{s=0}^{b+1}\left[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right]t^{\underline{\nu-1}}z_1(s)\\
&+\sum_{s=0}^{b+1}\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} + \frac{(\nu-3)(\nu+b-\beta_{1_1}-\sigma(s))^{\underline{\nu-1-\beta_{1_1}}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_{1_1})}\right]t^{\underline{\nu-2}}z_1(s)\\
&+\sum_{s=0}^{b+1}\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{(\nu+b-\beta_{1_1}-\sigma(s))^{\underline{\nu-1-\beta_{1_1}}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right]t^{\underline{\nu-3}}z_1(s)\\
&+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}z_1(s),
\end{aligned}
$$

$$
\begin{aligned}
h_1'(t) = &\sum_{s=0}^{b+1}\left[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right]t^{\underline{\nu-1}}z_1'(s)\\
&+\sum_{s=0}^{b+1}\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} + \frac{(\nu-3)(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right]t^{\underline{\nu-2}}z_1'(s)\\
&+\sum_{s=0}^{b+1}\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right]t^{\underline{\nu-3}}z_1'(s)\\
&+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}z_1'(s)
\end{aligned}
$$

for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$ and

$$
\begin{aligned}
h_2 = &\sum_{s=0}^{b+1}\left[\frac{-\delta_2(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{\delta_2(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-1}}z_2(s)\\
&+\sum_{s=0}^{b+1}\left[\frac{(\mu-3)(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} + \frac{(\mu-3)(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-2}}z_2(s)\\
&+\sum_{s=0}^{b+1}\left[\frac{-(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-3}}z_2(s)\\
&+\frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}z_2(s),
\end{aligned}
$$

$$h_2' = \sum_{s=0}^{b+1}\left[\frac{-\delta_2(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{\delta_2(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-1}}z_2'(s)$$

$$+\sum_{s=0}^{b+1}\left[\frac{(\mu-3)(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} + \frac{(\mu-3)(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-2}}z_2'(s)$$

$$+\sum_{s=0}^{b+1}\left[\frac{-(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-3}}z_2'(s)$$

$$+\frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}z_2'(s)$$

for all $t \in \mathbb{N}_{\mu-3}^{b+\mu+1}$. Since

$$H_d(F_1(t,u_1(t),v_1(t),\nabla u_1(t),\nabla v_1(t),\nabla^2 u_1(t),\nabla^2 v_1(t)),$$

$$F_1(t,u_2(t),v_2(t),\nabla u_2(t),\nabla v_2(t),\nabla^2 u_2(t),\nabla^2 v_2(t)))$$

$$\leq \psi\Big(|u_1(t)-u_2(t)|+|v_1(t)-v_2(t)|+|\nabla u_1(t)-\nabla u_2(t)|$$

$$+|\nabla v_1(t)-\nabla v_2(t)|+|\nabla^2 u_1(t)-\nabla^2 u_2(t)|+|\nabla^2 v_1(t)-\nabla^2 v_2(t)|\Big)$$

for all $(u_1,u_2),(v_1,v_2)\in\mathcal{X}$ and $t\in\mathbb{N}_{\nu-3}^{b+\nu+1}$, we get

$$|z_1(t)-z_2(t)| \leq \psi\Big(|u_1(t)-u_2(t)|+|v_1(t)-v_2(t)|+|\nabla u_1(t)-\nabla u_2(t)|$$
$$+|\nabla v_1(t)-\nabla v_2(t)|+|\nabla^2 u_1(t)-\nabla^2 u_2(t)|+|\nabla^2 v_1(t)-\nabla^2 v_2(t)|\Big)$$

For all $t\in\mathbb{N}_0^{b+1}$. Now, put

$$K_1 = \max_{t\in\mathbb{N}_{\nu-3}^{b+1+\nu}}\Bigg|\sum_{s=0}^{b+1}\left[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right]t^{\underline{\nu-1}}$$

$$+\sum_{s=0}^{b+1}\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} + \frac{(\nu-3)(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right]t^{\underline{\nu-2}}$$

$$+\sum_{s=0}^{b+1}\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right]t^{\underline{\nu-3}}$$

$$+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}\Bigg|,$$

$$K_2 = \max_{t\in\mathbb{N}_{\nu-3}^{b+1+\nu}} \Bigg| \sum_{s=0}^{b+1}(\nu-1)\left[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right](t-1)^{\underline{\nu-2}}$$
$$+\sum_{s=0}^{b+1}(\nu-2)\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} + \frac{(\nu-3)(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right](t-1)^{\underline{\nu-3}}$$
$$+\sum_{s=0}^{b+1}(\nu-3)\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right](t-1)^{\underline{\nu-4}}$$
$$+\frac{1}{\Gamma(\nu-1)}\sum_{s=0}^{t-\nu+1}(t-1-\sigma(s))^{\underline{\nu-2}}\Bigg|,$$

$$K_3 = \max_{t\in\mathbb{N}_{\nu-3}^{b+1+\nu}}$$
$$\times\Bigg|\sum_{s=0}^{b+1}(\nu-1)(\nu-2)\left[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right](t-2)^{\underline{\nu-3}}$$
$$+\sum_{s=0}^{b+1}(\nu-2)(\nu-3)$$
$$\times\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} + \frac{(\nu-3)(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right](t-2)^{\underline{\nu-4}}$$
$$+\sum_{s=0}^{b+1}(\nu-3)(\nu-4)\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)} - \frac{(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right](t-2)^{\underline{\nu-5}}$$
$$+\frac{1}{\Gamma(\nu-2)}\sum_{s=0}^{t-\nu+2}(t-2-\sigma(s))^{\underline{\nu-3}}\Bigg|,$$

$$K_4 = \max_{t\in\mathbb{N}_{\mu-3}^{b+1+\mu}}\Bigg|\sum_{s=0}^{b+1}\left[\frac{-\delta_2(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{\delta_2(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-1}}$$
$$+\sum_{s=0}^{b+1}\left[\frac{(\mu-3)(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} + \frac{(\mu-3)(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-2}}$$
$$+\sum_{s=0}^{b+1}\left[\frac{-(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right]t^{\underline{\mu-3}}$$
$$+\frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-\sigma(s))^{\underline{\mu-1}}\Bigg|,$$

$$
\begin{aligned}
K_5 = \max_{t\in\mathbb{N}_{\mu-3}^{b+1+\mu}} \Bigg| \sum_{s=0}^{b+1} (\mu-1) \\
\times \left[\frac{-\delta_2(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{\delta_2(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right](t-1)^{\underline{\mu-2}} \\
+\sum_{s=0}^{b+1}(\mu-2)\left[\frac{(\mu-3)(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} + \frac{(\mu-3)(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right](t-1)^{\underline{\mu-3}} \\
+\sum_{s=0}^{b+1}(\mu-3)\left[\frac{-(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right](t-1)^{\underline{\mu-4}} \\
+\frac{1}{\Gamma(\mu-1)}\sum_{s=0}^{t-\mu+1}(t-1-\sigma(s))^{\underline{\mu-2}}\Bigg|
\end{aligned}
$$

and

$$
\begin{aligned}
K_6 = \max_{t\in\mathbb{N}_{\mu-3}^{b+1+\mu}} \Bigg| \sum_{s=0}^{b+1} (\mu-1)(\mu-2) \\
\times \left[\frac{-\delta_2(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{\delta_2(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right](t-2)^{\underline{\mu-3}} \\
+\sum_{s=0}^{b+1}(\mu-2)(\mu-3) \\
\times\left[\frac{(\mu-3)(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} + \frac{(\mu-3)(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\mu-2)(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right](t-2)^{\underline{\mu-4}} \\
+\sum_{s=0}^{b+1}(\mu-3)(\mu-4)\left[\frac{-(\mu+b+1-\sigma(s))^{\underline{\mu-1}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu)} - \frac{(\mu+b-\beta_2-\sigma(s))^{\underline{\mu-1-\beta_2}}}{(\theta_2\delta_2+\lambda_2)\Gamma(\mu-\beta_2)}\right](t-2)^{\underline{\mu-5}} \\
+\frac{1}{\Gamma(\mu-2)}\sum_{s=0}^{t-\mu+2}(t-2-\sigma(s))^{\underline{\mu-3}}\Bigg|.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&|h_1(t)-h_1'(t)| = \\
&\Big|\sum_{s=0}^{b+1}\Big[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}-\frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\Big]t^{\underline{\nu-1}}(z_1(s)-z_1'(s)) \\
&+\sum_{s=0}^{b+1}\Big[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}+\frac{(\nu-3)(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\Big]t^{\underline{\nu-2}}(z_1(s)-z_1'(s)) \\
&+\sum_{s=0}^{b+1}\Big[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}-\frac{(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\Big]t^{\underline{\nu-3}}(z_1(s)-z_1'(s)) \\
&+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}(z_1(s)-z_1'(s))\Big| \\
&\le \max_{t\in\mathbb{N}_0^{b+1}}|z_1(s)-z_1'(s)| \\
&\times\max_{t\in\mathbb{N}_{\nu-3}^{b+1+\nu}}\Big|\sum_{s=0}^{b+1}\Big[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}-\frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\Big]t^{\underline{\nu-1}} \\
&+\sum_{s=0}^{b+1}\Big[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}+\frac{(\nu-3)(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\Big]t^{\underline{\nu-2}} \\
&+\sum_{s=0}^{b+1}\Big[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}-\frac{(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\Big]t^{\underline{\nu-3}}+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}\Big| \\
&\le \psi\Big(|u_1(t)-u_2(t)|+|v_1(t)-v_2(t)|+|\nabla u_1(t)-\nabla u_2(t)|+ \\
&|\nabla v_1(t)-\nabla v_2(t)|+|\nabla^2 u_1(t)-\nabla^2 u_2(t)|+|\nabla^2 v_1(t)-\nabla^2 v_2(t)|\Big)\times K_1.
\end{aligned}
$$

Also by using some similar calculations, we obtain

$$
\begin{aligned}
&\nabla h_1(t) = \sum_{s=0}^{b+1}(\nu-1) \\
&\times\Big[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}-\frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\Big](t-1)^{\underline{\nu-2}}z_1(s) \\
&+\sum_{s=0}^{b+1}(\nu-2) \\
&\times\Big[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}+\frac{(\nu-3)(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta)}\Big](t-1)^{\underline{\nu-3}}z_1(s) \\
&+\sum_{s=0}^{b+1}(\nu-3)\Big[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}-\frac{(\nu+b-\beta-\sigma(s))^{\underline{\nu-1-\beta}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta)}\Big](t-1)^{\underline{\nu-4}}z_1(s) \\
&+\frac{1}{\Gamma(\nu-1)}\sum_{s=0}^{t-\nu+1}(t-1-\sigma(s))^{\underline{\nu-2}}z_1(s),
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&|\nabla h_1(t)-\nabla h_1'(t)| \le \max_{t\in\mathbb{N}_0^{b+1}} |z_1(s)-z_1'(s)|\times \\
&\max_{t\in\mathbb{N}_{\nu-3}^{b+1+\nu}} \Bigg| \sum_{s=0}^{b+1}(\nu-1)\left[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}-\frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right](t-1)^{\underline{\nu-2}} \\
&+\sum_{s=0}^{b+1}(\nu-2)\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}+\frac{(\nu-3)(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right](t-1)^{\underline{\nu-3}} \\
&+\sum_{s=0}^{b+1}(\nu-3)\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}-\frac{(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right](t-1)^{\underline{\nu-4}} \\
&+\frac{1}{\Gamma(\nu-1)}\sum_{s=0}^{t-\nu+1}(t-1-\sigma(s))^{\underline{\nu-2}}\Bigg| \\
&\le \psi\Big(|u_1(t)-u_2(t)|+|v_1(t)-v_2(t)|+|\nabla u_1(t)-\nabla u_2(t)|+ \\
&|\nabla v_1(t)-\nabla v_2(t)|+|\nabla^2 u_1(t)-\nabla^2 u_2(t)|+|\nabla^2 v_1(t)-\nabla^2 v_2(t)|\Big)\times K_2
\end{aligned}
$$

and

$$
\begin{aligned}
&|\nabla^2 h_1(t)-\nabla^2 h'1(t)| \le \max_{t\in\mathbb{N}_0^{b+1}} |z_1(s)-z_1'(s)|\times \\
&\max_{t\in\mathbb{N}_{\nu-3}^{b+1+\nu}} \\
&\times\Bigg| \sum_{s=0}^{b+1}(\nu-1)(\nu-2)\left[\frac{-\delta_1(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}-\frac{\delta_1(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right](t-2)^{\underline{\nu-3}} \\
&+\sum_{s=0}^{b+1}(\nu-2)(\nu-3) \\
&\times\left[\frac{(\nu-3)(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}+\frac{(\nu-3)(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\nu-2)(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right](t-2)^{\underline{\nu-4}} \\
&+\sum_{s=0}^{b+1}(\nu-3)(\nu-4)\left[\frac{-(\nu+b+1-\sigma(s))^{\underline{\nu-1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu)}-\frac{(\nu+b-\beta_1-\sigma(s))^{\underline{\nu-1-\beta_1}}}{(\theta_1\delta_1+\lambda_1)\Gamma(\nu-\beta_1)}\right](t-2)^{\underline{\nu-5}} \\
&+\frac{1}{\Gamma(\nu-2)}\sum_{s=0}^{t-\nu+2}(t-2-\sigma(s))^{\underline{\nu-3}}\Bigg| \\
&\le \psi\Big(|u_1(t)-u_2(t)|+|v_1(t)-v_2(t)|+|\nabla u_1(t)-\nabla u_2(t)|+ \\
&|\nabla v_1(t)-\nabla v_2(t)|+|\nabla^2 u_1(t)-\nabla^2 u_2(t)|+|\nabla^2 v_1(t)-\nabla^2 v_2(t)|\Big)\times K_3.
\end{aligned}
$$

By using some simple calculations, we obtain

$$
\begin{aligned}
&|h_2(t)-h_2'(t)| \le \psi\Big(|u_1(t)-u_2(t)|+|v_1(t)-v_2(t)|+|\nabla u_1(t)-\nabla u_2(t)| \\
&+|\nabla v_1(t)-\nabla v_2(t)|+|\nabla^2 u_1(t)-\nabla^2 u_2(t)|+|\nabla^2 v_1(t)-\nabla^2 v_2(t)|\Big)\times K_4,
\end{aligned}
$$

$$|\nabla h_2(t) - \nabla h_2'(t)| \leq \psi\Big(|u_1(t) - u_2(t)| + |v_1(t) - v_2(t)| + |\nabla u_1(t) - \nabla u_2(t)| + |\nabla v_1(t) - \nabla v_2(t)| + |\nabla^2 u_1(t) - \nabla^2 u_2(t)| + |\nabla^2 v_1(t) - \nabla^2 v_2(t)|\Big) \times K_5$$

and

$$|\nabla^2 h_2(t) - \nabla^2 h_2'(t)| \leq \psi\Big(|u_1(t) - u_2(t)| + |v_1(t) - v_2(t)| + |\nabla u_1(t) - \nabla u_2(t)| + |\nabla v_1(t) - \nabla v_2(t)| + |\nabla^2 u_1(t) - \nabla^2 u_2(t)| + |\nabla^2 v_1(t) - \nabla^2 v_2(t)|\Big) \times K_6.$$

Hence,

$$\begin{aligned}\|(h_1, h_2) - (h_1', h_2')\|_{\mathcal{X}} &= \max_{t \in \mathbb{N}_{\nu-3}^{b+1+\nu}} |h_1(t) - h_1'(t)| + \max_{t \in \mathbb{N}_{\nu-3}^{b+1+\nu}} |\nabla h_1(t) - \nabla h_1'(t)| \\ &+ \max_{t \in \mathbb{N}_{\nu-3}^{b+1+\nu}} |\nabla^2 h_1(t) - \nabla^2 h_1'(t)| + \max_{t \in \mathbb{N}_{\nu-3}^{b+1+\nu}} |h_2(t) - h_2'(t)| \\ &+ \max_{t \in \mathbb{N}_{\nu-3}^{b+1+\nu}} |\nabla h_2(t) - \nabla h_2'(t)| + \max_{t \in \mathbb{N}_{\nu-3}^{b+1+\nu}} |\nabla^2 h_2(t) - \nabla^2 h_2'(t)| \\ &\leq \psi(|u_1(t) - u_2(t)| + |v_1(t) - v_2(t)| + |\nabla u_1(t) - \nabla u_2(t)| + |\nabla v_1(t) - \nabla v_2(t)| \\ &+ |\nabla^2 u_1(t) - \nabla^2 u_2(t)| + |\nabla^2 v_1(t) - \nabla^2 v_2(t)|\Big) \times (K_1 + \cdots + K_6) \\ &\leq (K_1 + \cdots + K_6)\psi(\|(u_1, u_2) - (v_1, v_2)\|)\end{aligned}$$

for all $(u_1, u_2), (v_1, v_2) \in \mathcal{X}$, $(h_1, h_2) \in T(u_1, u_2)$ and $(h_1', h_2') \in T(v_1, v_2)$. This implies that

$$H_d(T(u_1, u_2), T(v_1, v_2)) \leq (K_1 + \cdots + K_6)\psi(\|(u_1, u_2) - (v_1, v_2)\|)$$

for all $(u_1, u_2), (v_1, v_2) \in \mathcal{X}$. Now, define the function $\alpha : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by $\alpha((u_1, u_2), (v_1, v_2)) = 1$ whenever $K_1 + \cdots + K_6 < 1$ and $\alpha((u_1, u_2), (v_1, v_2)) = \frac{1}{K_1 + \cdots + K_6}$ otherwise. Thus, we conclude that

$$\alpha((u_1, u_2), (v_1, v_2)) H_d(T(u_1, u_2), T(v_1, v_2)) \leq \psi(\|(u_1, u_2) - (v_1, v_2)\|)$$

for all $(u_1, u_2), (v_1, v_2) \in \mathcal{X}$. One can easily check that $\mathcal{X}$ has the condition (C_α) and T is α-admissible. By using Theorem 1.3, there exists $(u^*, v^*) \in \mathcal{X}$ such that $(u^*, v^*) \in T((u^*, v^*))$. One can check that (u^*, v^*) is a solution for the problem (1.1). □

Example 2.3. Consider the two-dimensional system Delta-Nabla fractional difference inclusions

(2.1)
$$\begin{cases} \Delta^e x(t) \in [1, \cosh(t^3) + \frac{\sin x(t)}{40|t|} + \frac{\nabla \sin x(t)}{e^{5|t|}} + \frac{|\nabla^2 x(t)| + |y(t)|}{5\pi^3 t^6} + \frac{|\nabla y(t)|}{20|t|} + \frac{|\nabla^2 y(t)|}{80t^2}], \\ \Delta^{2.5} y(t) \in [1, 10\pi + \frac{|x(t)|}{4\pi} + \frac{\nabla \sin x(t)}{e^8} + \frac{|\nabla^2 x(t)|}{8\cosh t^2} + \frac{|y(t)|}{5e^{\pi|t|}} + \frac{|\nabla y(t)|}{30|t|} + \frac{|\nabla^2 y(t)|}{10e^{|t|}}] \end{cases}$$

with the boundary conditions $x(e-3)+\Delta^{e-1}x(0)=0$, $x(13+e)+\Delta^{e-2}x(14)=0$, $\Delta x(e+1)=0$, $y(-0.5)+\Delta^{\alpha_2}y(\mu-1-\alpha_2)=0$, $y(\mu+b+1)+\Delta^{\beta_2}y(\mu+b-\beta_2)=0$ and $\Delta y(3.5)=0$. Put $b=12$, $\alpha_1=e-1$, $\beta_1=e-2$,$\eta_1=e+1$, $\alpha_2=1.5$, $\beta_2=0.5$ and $\eta_2=3.5$. Consider the multifunctions $F_1:\mathbb{N}_{e-3}^{e+13}\times\mathbb{R}^6\to 2^{\mathbb{R}}$ and $F_2:\mathbb{N}_{-0.5}^{15.5}\times\mathbb{R}^6\to 2^{\mathbb{R}}$ defined by

$$F_1(t,x_1,x_2,x_3,x_4,x_5,x_6)=[1,\cosh(t^3)+\frac{\sin x_1}{40|t|}+\frac{\sin x_2}{e^{5|t|}}+\frac{|x_3|+|x_4|}{5\pi^3t^6}+\frac{|x_5|}{20|t|}+\frac{|x_6|}{80t^2}]$$

and $F_2(t,x_1,x_2,x_3,x_4,x_5,x_6)=[1,10\pi+\frac{|x_1|}{4\pi}+\frac{\sin x_2}{e^8}+\frac{|x_3|}{8\cosh t^2}+\frac{|x_4|}{5e^{\pi|t|}}+\frac{|x_5|}{30|t|}+\frac{|x_6|}{10e^{|t|}}]$. One can check that $\cosh(t^3)+\frac{\sin x_1}{40|t|}+\frac{\sin x_2}{e^{5|t|}}+\frac{|x_3|+|x_4|}{5\pi^3t^6}+\frac{|x_5|}{20|t|}+\frac{|x_6|}{80t^2}>1$ for $t\in\mathbb{N}_{e-3}^{e+13}$ and $x_1,\dots,x_6\in\mathbb{R}$ and so $F_1:\mathbb{N}_{e-3}^{e+13}\times\mathbb{R}^6\to 2^{\mathbb{R}}$ is a nonempty valued multifunction. Similarly, we have $10\pi+\frac{|x_1|}{4\pi}+\frac{\sin x_2}{e^8}+\frac{|x_3|}{8\cosh t^2}+\frac{|x_4|}{5e^{\pi|t|}}+\frac{|x_5|}{30|t|}+\frac{|x_6|}{10e^{|t|}}>1$ for $t\in\mathbb{N}_{-0.5}^{15.5}$ and $x_1,\dots,x_6\in\mathbb{R}$ and so $F_2:\mathbb{N}_{-0.5}^{15.5}\times\mathbb{R}^6\to 2^{\mathbb{R}}$ is a nonempty valued multifunction. Now, consider the map $\psi\in\Psi$ defined by $\psi(r)=\frac{r}{6}$ for all $r\geq 0$. Then, we have

$$\begin{aligned}
&H_d\left(F_1(t,x_1,\dots,x_6),F_1(t,y_1,\dots,y_6)\right)\\
&\leq\Bigg|\frac{\sin x_1}{40|t|}+\frac{\sin x_2}{e^{5|t|}}+\frac{|x_3|+|x_4|}{5\pi^3t^6}+\frac{|x_5|}{20|t|}+\frac{|x_6|}{80t^2}\\
&\quad-\frac{\sin y_1}{40|t|}-\frac{\sin y_2}{e^{5|t|}}-\frac{|y_3|+|y_4|}{5\pi^3t^6}-\frac{|y_5|}{20|t|}-\frac{|y_6|}{80t^2}\Bigg|\\
&\leq\frac{1}{6}\Big|\sin x_1-\sin y_1+\sin x_2-\sin y_2+|x_3-y_3|+|x_4-y_4|+|x_5-y_5|+|x_6-y_6|\Big|\\
&\leq\frac{1}{6}\sum_{k=1}^{6}|x_k-y_k|=\psi(\sum_{k=1}^{6}|x_k-y_k|)
\end{aligned}$$

for all $t\in\mathbb{N}_{e-3}^{13+e}$ and $x_1,\dots,x_6,y_1,\dots,y_6\in\mathbb{R}$. Similarly, we obtain

$$\begin{aligned}
&H_d\left(F_2(t,x_1,\dots,x_6),F_2(t,y_1,\dots,y_6)\right)\\
&\leq\Bigg|\frac{|x_1|}{4\pi}+\frac{\sin x_2}{e^8}+\frac{|x_3|}{8\cosh t^2}+\frac{|x_4|}{5e^{\pi|t|}}+\frac{|x_5|}{30|t|}+\frac{|x_6|}{10e^{|t|}}\\
&\quad-\frac{|y_1|}{4\pi}-\frac{\sin y_2}{e^8}-\frac{|y_3|}{8\cosh t^2}-\frac{|y_4|}{5e^{\pi|t|}}-\frac{|y_5|}{30|t|}-\frac{|y_6|}{10e^{|t|}}\Bigg|\\
&\leq\frac{1}{6}\sum_{k=1}^{6}|x_k-y_k|=\psi(\sum_{k=1}^{6}|x_k-y_k|)
\end{aligned}$$

for all $t\in\mathbb{N}_{-0.5}^{15.5}$ and $x_1,\dots,x_6,y_1,\dots,y_6\in\mathbb{R}$. Now by using Theorem 2.2, we conclude that the problem (2.1) has a solution.

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