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SOME INEQUALITIES FOR GENERALIZED NORMALIZED δ-CASORATI CURVATURES OF SLANT SUBMANIFOLDS IN GENERALIZED SASAKIAN SPACE FORM

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Abstract. In the present paper, we prove the inequality between the normalized scalar curvature and the generalized normalized δ-Casorati curvatures for the slant submanifolds of generalized Sasakian space form and also consider the equality case of the inequality.

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1. Introduction

The theory of Chen invariants, one of the most interesting research areas of differential geometry, is to establish the simple relationships between the main intrinsic invariants and the main extrinsic invariants of the submanifolds started in 1993 by Chen [7]. In the initial paper Chen established inequalities between the scalar curvature and the sectional curvature (intrinsic invariants) and the squared norm of the mean curvature (the main extrinsic invariant) of a submanifold in a real space form. The same author obtained the inequalities for submanifolds between the k-Ricci curvature, the squared mean curvature and the shape operator in the real space form with arbitrary codimension [6]. Since then, different geometers proved similar inequalities for different submanifolds and ambient spaces [4, 5, 13, 18, 19].

The Casorati curvature (extrinsic invariant) of a submanifold of a Riemannian manifold introduced by Casorati defined as the normalized square length of the second fundamental form [3]. The concept of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold [11]. The geometrical meaning and the importance of the Casorati curvature was discussed by some distinguished geometers [8, 9, 12, 21, 22]. Therefore, the attention of geometers turned to obtaining the optimal inequalities for the Casorati curvatures of the submanifolds of different ambient spaces [10, 14, 15, 16, 20].

In this paper, we will study the inequalities for the generalized normalized δ-Casorati curvature for the slant submanifolds of generalized Sasakian space form.

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2. Preliminaries

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure (ϕ, ξ, η, g) where ϕ is a tensor field of type $(1,1)$, ξ a vector field, η is a one form and g is the Riemannian metric on $\overline{M}$. Then they satisfy

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y).$$

These conditions also imply that

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi),$$

and

$$g(\phi X, Y) + g(X, \phi Y) = 0,$$

for all vector fields X, Y in $T\overline{M}$. Here $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$.

Let $(\overline{M}, \phi, \xi, \eta, g)$ be an almost contact metric manifold whose curvature tensor satisfies

$$\begin{aligned} \overline{R}(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \tag{2.1}$$

for all vector fields X, Y, Z, where f_1, f_2, f_3 are differentiable functions on $\overline{M}$, then $\overline{M}$ is said to be a generalized Sasakian space form and is denoted by $\overline{M}(f_1, f_2, f_3)$.

The generalized Sasakian space form introduced by Alegre et al. [1] generalizes the concept of Sasakian space form, Kenmotsu space form and cosymplectic space form. A Sasakian space form $\overline{M}(c)$ is the generalized Sasakian space form with $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$. A Kenmotsu space form $\overline{M}(c)$ is a generalized Sasakian space form with $f_1 = \frac{c-3}{4}$ and $f_2 = f_3 = \frac{c+1}{4}$. The cosymplectic space form $\overline{M}(c)$ is the generalized Sasakian space form with $f_1 = f_2 = f_3 = \frac{c}{4}$.

The equation of Gauss is given by

$$\begin{aligned} \overline{R}(X,Y,Z,W) = R(X,Y,Z,W) & - g(h(X,W), h(Y,Z)) \\ & + g(h(X,Z), h(Y,W)) \end{aligned} \tag{2.2}$$

for $X, Y, Z, W \in TM$, where $\overline{R}$ and R represent the curvature tensors of $\overline{M}(f_1, f_2, f_3)$ and M, respectively.

Let M be an n-dimensional submanifold of a generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$ of dimension $2m+1$. For any tangent vector field $X \in TM$, we can write $\phi X = PX + QX$, where PX and QX are the tangential and normal components of ϕX, respectively. If $P = 0$, the submanifold is said to be an anti-invariant submanifold and if $Q = 0$, the submanifold is said to be an invariant submanifold. The squared norm of P at $p \in M$ is defined as

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(\phi e_i, e_j), \tag{2.3}$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis of the tangent space T_pM.. The structural vector field ξ can be decomposed as

$$\xi = \xi^{\top} + \xi^{\perp},$$

where $\xi^{\top}$ and $\xi^{\perp}$ are the tangential and the normal components of the ξ.

Lotta introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold [17]. A submanifold M tangent to ξ is said to be a slant submanifold if for any $p \in M$ and a non zero vector $X \in T_pM$, linearly dependent on ξ, the angle between ϕX and T_pM is constant, i.e., the angle does not depend on the choice of $p \in M$ and $X \in T_pM$. The angle $\theta \in [0, \frac{\pi}{2}]$ is called the slant angle of M in $\overline{M}$.

Invariant and anti-invariant submanifolds are the slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively and when $0 < \theta < \frac{\pi}{2}$, then slant submanifold is called proper slant submanifold.

Suppose M be a slant submanifold of a generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$, if dim $M = (n = 2l + 1)$. Let $p \in M$ and let $\{e_1, \ldots, e_n = \xi\}$ be an orthonormal basis of T_pM, with

$$e_1, e_2 = \frac{1}{cos\theta} Pe_1, \ldots, e_{2l} = \frac{1}{cos\theta} Pe_{2l-1}, e_{2l+1} = \xi,$$

we have

$$g(\phi e_1, e_2) = g(\phi e_1, \frac{1}{cos\theta} Pe_1) = \frac{1}{cos\theta} g(Pe_1, Pe_1) = cos\theta.$$

Similarly, we have

$$g^2(\phi e_i, e_{i+1}) = cos^2\theta,$$

thus, we have

$$\sum_{i,j=1}^{n} g^2(\phi e_i, e_j) = (n-1)cos^2\theta. \tag{2.4}$$

Let M be a Riemannian manifold and $K(\pi)$ denotes the sectional curvature of M of the plane section $\pi \subset T_pM$ at a point $p \in M$. If $\{e_1, \ldots, e_n\}$ and let $\{e_{n+1}, \ldots, e_{2m+1}\}$ be the orthonormal basis of T_pM and $T_p^{\perp}M$ at any $p \in M$, then the scalar curvature τ at that point is given by

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j)$$

and the normalized scalar curvature ρ is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$

The mean curvature vector denoted by H is defined as

$$H = \frac{1}{n}\sum_{i,j=1}^{n} h(e_i, e_i)$$

and also we put

$$h_{ij}^{\gamma} = g(h(e_i, e_j), e_{\gamma}),\ i, j \in 1, 2, .., n,\ \gamma \in \{n+1, n+2, ..., m\}.$$

The norm of the squared mean curvature of the submanifold is defined by

$$\|H\|^2 = \frac{1}{n^2}\sum_{\gamma=n+1}^{m}\left(\sum_{i=1}^{n} h_{ii}^{\gamma}\right)^2$$

and the squared norm of second fundamental form h is denoted by $\mathcal{C}$ defined as

$$\mathcal{C} = \frac{1}{n}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{\gamma}\right)^2,$$

known as the Casorati curvature of the submanifold.

If we suppose that L is an r-dimensional subspace of TM, $r \geq 2$, and $\{e_1, e_2, \ldots, e_r\}$ is an orthonormal basis of L, then the scalar curvature of the r-plane section L is given as

$$\tau(L) = \sum_{1\leq\gamma<\beta\leq r} K(e_{\gamma} \wedge e_{\beta})$$

and the Casorati curvature $\mathcal{C}$ of the subspace L is as follows

$$\mathcal{C}(L) = \frac{1}{r}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{\gamma}\right)^2.$$

A point $p \in M$ is said to be an invariantly quasi-umbilical point if there exist $m-n$ mutually orthogonal unit normal vectors $\xi_{n+1}, \ldots, \xi_m$ such that the shape operators with respect to all directions ξ_{γ} have an eigenvalue of multiplicity $n-1$ and that for each ξ_{γ} the distinguished eigen direction is the same. The submanifold is said to be an invariantly quasi-umbilical submanifold if each of its points is an invariantly quasi-umbilical point [2].

The normalized δ-Casorati curvature $\delta_c(n-1)$ and $\widehat{\delta}_c(n-1)$ are defined as

$$(2.5)\quad [\delta_c(n-1)]_p = \frac{1}{2}\mathcal{C}_p + \frac{n+1}{2n} inf\{\mathcal{C}(L)|L : \text{a hyperplane of } T_pM\}$$

and

$$(2.6)\,[\widehat{\delta}_c(n-1)]_p = 2\mathcal{C}_p + \frac{2n-1}{2n} sup\{\mathcal{C}(L)|L : \text{a hyperplane of } T_pM\}.$$

Some authors use the coefficient $\frac{n+1}{2n(n-1)}$ instead of $\frac{2n-1}{2n}$ in the equation(2.6). It was pointed out that the coefficient $\frac{n+1}{2n(n-1)}$ is not suitable and therefore modified by the coefficient $\frac{2n-1}{2n}$. For a positive real number $t \neq n(n-1)$, put

$$a(t) = \frac{1}{nt}(n-1)(n+t)(n^2-nt), \tag{2.7}$$

then the generalized normalized δ-Casorati curvatures $\delta_c(t;n-1)$ and $\widehat{\delta}_c(t;n-1)$ are given as

$$[\delta_c(t;n-1)]_p = t\mathcal{C}_p + a(t)inf\{\mathcal{C}(L)|L : \text{a hyperplane of } T_pM\}$$

if $0 < t < n^2 - n$, and

$$[\widehat{\delta}_c(t;n-1)]_p = r\mathcal{C}_p + a(t)sup\{\mathcal{C}(L)|L : \text{a hyperplane of } T_pM\}.$$

if $t > n^2 - n$.

3. Main Theorem

Theorem 3.1. *Let M be an n-dimensional slant submanifold of a generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$ of dimension $2m+1$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t;n-1)$ satisfies

$$\rho \leq \frac{\delta_c(t;n-1)}{n(n-1)} + f_1 + \frac{3f_2}{n}cos^2\theta + \frac{2f_3}{n}\|\xi^\top\|^2 \tag{3.1}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta}_c(t;n-1)$ satisfies

$$\rho \leq \frac{\widehat{\delta}_c(t;n-1)}{n(n-1)} + f_1 + \frac{3f_2}{n}cos^2\theta + \frac{2f_3}{n}\|\xi^\top\|^2 \tag{3.2}$$

for any real number $t > n(n-1)$. Moreover, the equality holds in (3.1) and (3.2) iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to a suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and a normal orthonormal frame $\{e_{n+1}, \cdots, e_{2m+1}\}$, the shape operators $S_r \equiv S_{e_r}$, $r \in \{n+1, \ldots, m\}$, take the following form

$$S_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0 \\ 0 & a & 0 & \ldots & 0 & 0 \\ 0 & 0 & a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{t}a \end{pmatrix}, \tag{3.3}$$

$$S_{n+2} = \cdots = S_m = 0.$$

Proof. Let $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_{2m+1}\}$ be the orthonormal basis of T_pM and $T_p^{\perp}M$, respectively, at any point $p \in M$. Putting $X = W = e_i$, $Y = Z = e_j$, $i \neq j$ from (2.1), we have

$$\begin{aligned}
&\overline{R}(e_i, e_j, e_j, e_i) = f_1\{g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\} \\
&+f_2\{g(e_i, \phi e_j)g(\phi e_j, e_i) - g(e_j, \phi e_j)g(\phi e_i, e_i) + 2g(e_i, \phi e_j)g(\phi e_j, e_i)\} \\
&+f_3\{\eta(e_i)\eta(e_j)g(e_i, e_i) - \eta(e_j)\eta(e_j)g(e_i, e_i) + \eta(e_j)\eta(e_i)g(e_i, e_j) \\
&-\eta(e_i)\eta(e_i)g(e_j, e_j)\}.
\end{aligned} \tag{3.4}$$

From Gauss equation and (3.4), we have

$$\begin{aligned}
&f_1\{g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\} + f_2\{g(e_i, \phi e_j)g(\phi e_j, e_i) \\
&-g(e_j, \phi e_j)g(\phi e_i, e_i) + 2g(e_i, \phi e_j)g(\phi e_j, e_i)\} + f_3\{\eta(e_i)\eta(e_j)g(e_i, e_i) \\
&-\eta(e_j)\eta(e_j)g(e_i, e_i) + \eta(e_j)\eta(e_i)g(e_i, e_j) - \eta(e_i)\eta(e_i)g(e_j, e_j)\} \\
&= R(e_i, e_j, e_j, e_i) - g(h(e_i, e_i), h(e_j, e_j)) + g(h(e_i, e_j), h(e_j, e_i))
\end{aligned} \tag{3.5}$$

By taking summation $1 \leq i, j \leq n$, we have

$$\begin{aligned}
2\tau = n^2\|H\|^2 - n\mathcal{C} + n(n-1)f_1 + 3f_2 \sum_{i,j=1}^{n} g^2(\phi e_i, e_j) \\
+2(n-1)f_3\|\xi^{\top}\|^2.
\end{aligned} \tag{3.6}$$

Using (2.4), we get

$$\begin{aligned}
2\tau = n^2\|H\|^2 - n\mathcal{C} + n(n-1)f_1 + 3(n-1)f_2 cos^2\theta \\
+2(n-1)f_3\|\xi^{\top}\|^2.
\end{aligned} \tag{3.7}$$

Define the following function, denoted by $\mathcal{Q}$, a quadratic polynomial in the components of the second fundamental form

$$\begin{aligned}
\mathcal{Q} = t\mathcal{C} + a(t)\mathcal{C}(L) - 2\tau + n(n-1)f_1 + 3(n-1)f_2 cos^2\theta \\
+2(n-1)f_3\|\xi^{\top}\|^2,
\end{aligned} \tag{3.8}$$

where L is the hyperplane of T_pM. Without loss of generality, we suppose that L is spanned by $e_1, \ldots, e_{n-1}$. It follows from (3.8) that

$$\mathcal{Q} = \frac{n+t}{n} \sum_{\gamma=n+1}^{m} \sum_{i,j=1}^{n} (h_{ij}^{\gamma})^2 + \frac{a(t)}{n-1} \sum_{\gamma=n+1}^{m} \sum_{i,j=1}^{n-1} (h_{ij}^{\gamma})^2 - \sum_{\gamma=n+1}^{m} \left(\sum_{i=1}^{n} h_{ii}^{\gamma}\right)^2$$

which can be easily written as

$$\mathcal{Q} = \sum_{\gamma=n+1}^{m}\sum_{i=1}^{n-1}\left[\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right)(h_{ii}^{\gamma})^2+\frac{2(n+t)}{n}(h_{in}^{\gamma})^2\right]+$$

$$\sum_{n+1}^{m}\left[2\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right)\sum_{(i<j)=1}^{n}(h_{ij}^{\gamma})^2-2\sum_{(i<j)=1}^{n}h_{ii}^{\gamma}h_{jj}^{\gamma}+\frac{t}{n}(h_{nn}^{\gamma})^2\right] \tag{3.9}$$

From(3.9), we can see that the critical points

$$h^c = (h_{11}^{n+1}, h_{12}^{n+1}, \ldots, h_{nn}^{n+1}, \ldots, h_{11}^{m}, \ldots, h_{nn}^{m})$$

of $\mathcal{Q}$ are the solutions of the following system of homogenous equations:

$$\begin{cases} \frac{\partial \mathcal{Q}}{\partial h_{ii}^{\gamma}} = 2\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right)(h_{ii}^{\gamma}) - 2\sum_{k=1}^{n} h_{kk}^{\gamma} = 0 \\ \frac{\partial \mathcal{Q}}{\partial h_{nn}^{\gamma}} = \frac{2t}{n}h_{nn}^{\gamma} - 2\sum_{k=1}^{n-1} h_{kk}^{\gamma} = 0 \\ \frac{\partial \mathcal{Q}}{\partial h_{ij}^{\gamma}} = 4\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right)(h_{ij}^{\gamma}) = 0 \\ \frac{\partial \mathcal{Q}}{\partial h_{in}^{\gamma}} = 4(\frac{n+t}{n}(h_{in}^{\gamma}) = 0, \end{cases} \tag{3.10}$$

where $i, j = \{1, 2, \ldots, n-1\}, i \neq j$, and $\gamma \in \{n+1, \ldots, m\}$.

Hence, every solution h^c has $h_{ij}^{\gamma} = 0$ for $i \neq j$ and the corresponding determinant to the first two equations of the above system is zero. Moreover, the Hessian matrix of $\mathcal{Q}$ is of the following form

$$\mathcal{H}(\mathcal{Q}) = \begin{pmatrix} H_1 & O & O \\ O & H_2 & O \\ O & O & H_3 \end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} 2\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right)-2 & -2 & \cdots & -2 & -2 \\ -2 & 2\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right)-2 & \cdots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \cdots & 2\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right)-2 & -2 \\ -2 & -2 & \cdots & -2 & \frac{2t}{n} \end{pmatrix},$$

H_2 and H_3 are the diagonal matrices and O is the null matrix of the respective dimensions. H_2 and H_3 are respectively given as

$$H_2 = diag\left(4\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right), 4\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right), \ldots, 4\left(\frac{n+t}{n}+\frac{a(t)}{n-1}\right)\right),$$

and

$$H_3 = diag\left(\frac{4(n+t)}{n}, \frac{4(n+t)}{n}, \ldots, \frac{4(n+t)}{n}\right).$$

Hence, we find that $\mathcal{H}(\mathcal{Q})$ has the following eigenvalues

$$\lambda_{11} = 0,\ \lambda_{22} = 2\left(\frac{2t}{n} + \frac{a(t)}{n-1}\right),\ \lambda_{33} = \cdots = \lambda_{nn} = 2\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right),$$

$$\lambda_{ij} = 4\left(\frac{n+t}{n} + \frac{a(t)}{n-1}\right), \lambda_{in} = \frac{4(n+t)}{n},\ \forall\ i,j \in \{1,2,\ldots,n-1\},\ i \neq j.$$

Thus, $\mathcal{Q}$ is parabolic and reaches at minimum $\mathcal{P}(h^c) = 0$ for the solution h^c of the system (3.10). Hence $\mathcal{P} \geq 0$ and hence

$$2\tau \leq t\mathcal{C} + a(t)\mathcal{C}(L) + n(n-1)f_1 + 3(n-1)f_2cos^2\theta + 2(n-1)f_3\|\xi^\top\|^2,$$

whereby, we obtain

$$\rho \leq \frac{t}{n(n-1)}\mathcal{C} + \frac{a(t)}{n(n-1)}\mathcal{C}(L) + f_1 + \frac{3f_2}{n}cos^2\theta + \frac{2f_3}{n}\|\xi^\top\|^2.$$

for every tangent hyperplane L of M. If we take the infimum over all tangent hyperplanes L, the result trivially follows. Moreover the equality sign holds iff

$$h^\gamma_{ij} = 0,\ \forall\ i,j \in \{1,\ldots,n\},\ i \neq j \text{ and } \gamma \in \{n+1,\ldots,m\} \tag{3.11}$$

and

$$h^\gamma_{nn} = \frac{n(n-1)}{t}h^\gamma_{11} = \cdots = \frac{n(n-1)}{t}h^\gamma_{n-1n-1}, \quad \forall\gamma \in \{n+1,\ldots,m\}. \tag{3.12}$$

From (3.11) and (3.12), we obtain that the equality holds if and only if the submanifold is invariantly quasi-umbilical with normal connections in $\overline{M}$, such that the shape operator takes the form (3.3) with respect to the orthonormal tangent and orthonormal normal frames.

In the same way, we can prove (ii).

□

Corollary 3.2. *Let M be an n-dimensional slant submanifold of a generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$. Then*

(i) The normalized δ-Casorati curvature $\delta_c(n-1)$ satisfies

$$\rho \leq \delta_c(n-1) + f_1 + \frac{3f_2}{n}cos^2\theta + \frac{2f_3}{n}\|\xi^\top\|^2.$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to a suitable tangent orthonormal frame $\{e_1,\ldots,e_n\}$ and a normal orthonormal frame

$\{e_{n+1},\ldots,e_{2m+1}\}$, the shape operators $S_r \equiv S_{e_r}$, $r\in\{n+1,\ldots,m\}$, take the following form

$$S_{n+1}=\begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0\\ 0 & a & 0 & \ldots & 0 & 0\\ 0 & 0 & a & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & a & 0\\ 0 & 0 & 0 & \ldots & 0 & 2a \end{pmatrix},$$

$$S_{n+2}=\cdots=S_m=0. \tag{3.13}$$

(ii) The normalized δ-Casorati curvature $\widehat{\delta}_c(n-1)$ satisfies

$$\rho\le\widehat{\delta}_c(n-1)+f_1+\frac{3f_2}{n}cos^2\theta+\frac{2f_3}{n}\|\xi^\top\|^2.$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to a suitable tangent orthonormal frame $\{e_1,\ldots,e_n\}$ and a normal orthonormal frame $\{e_{n+1},\ldots,e_{2m+1}\}$, the shape operators $S_r\equiv S_{e_r}$, $r\in\{n+1,\ldots,m\}$, take the following form

$$S_{n+1}=\begin{pmatrix} 2a & 0 & 0 & \ldots & 0 & 0\\ 0 & 2a & 0 & \ldots & 0 & 0\\ 0 & 0 & 2a & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 2a & 0\\ 0 & 0 & 0 & \ldots & 0 & a \end{pmatrix},$$

$$S_{n+2}=\cdots=S_m=0. \tag{3.14}$$

Theorem 3.3. *Let M be an n-dimensional slant submanifold of a Sasakian space form $\overline{M}(c)$ of dimension $2m+1$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t;n-1)$ satisfies

$$\rho\le\frac{\delta_c(t;n-1)}{n(n-1)}+\frac{c+3}{4}+\frac{3(c-1)}{4n}cos^2\theta+\frac{(c-1)}{2n}\|\xi^\top\|^2 \tag{3.15}$$

for any real number t such that $0<t<n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta}_c(t;n-1)$ satisfies

$$\rho\le\frac{\widehat{\delta}_c(t;n-1)}{n(n-1)}+\frac{c+3}{4}+\frac{3(c-1)}{4n}cos^2\theta+\frac{(c-1)}{2n}\|\xi^\top\|^2 \tag{3.16}$$

for any real number $t>n(n-1)$. Moreover, the equality holds in (3.15) and (3.16) iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to a suitable tangent orthonormal frame $\{e_1,\ldots,e_n\}$ and a normal orthonormal frame $\{e_{n+1},\ldots,e_{2m+1}\}$, the shape operators $S_r\equiv S_{e_r}$, $r\in\{n+1,\ldots,m\}$, take the form of (3.3).

Corollary 3.4. *Let M be an n-dimensional slant submanifold of a Sasakian space form $\overline{M}(c)$. Then*

(i) The normalized δ-Casorati curvature $\delta_c(n-1)$ satisfies

$$\rho \leq \delta_c(n-1) + \frac{c+3}{4} + \frac{3(c-1)}{4n} cos^2\theta + \frac{(c-1)}{2n}\|\xi^\top\|^2.$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to a suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and a normal orthonormal frame $\{e_{n+1}, \ldots, e_{2m+1}\}$, the shape operators $S_r \equiv S_{e_r}$, $r \in \{n+1, \ldots, m\}$, take the form of (3.13)

(ii) The normalized δ-Casorati curvature $\widehat{\delta}_c(n-1)$ satisfies

$$\rho \leq \widehat{\delta}_c(n-1) + \frac{c+3}{4} + \frac{3(c-1)}{4n} cos^2\theta + \frac{(c-1)}{2n}\|\xi^\top\|^2.$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to a suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and a normal orthonormal frame $\{e_{n+1}, \ldots, e_{2m+1}\}$, the shape operators $S_r \equiv S_{e_r}$, $r \in \{n+1, \ldots, m\}$,take the form of (3.14)

Theorem 3.5. *Let M be an n-dimensional slant submanifold of a Kenmotsu space form $\overline{M}(c)$ of dimension $2m+1$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t; n-1)$ satisfies

$$\rho \leq \frac{\delta_c(t;n-1)}{n(n-1)} + \frac{c-3}{4} + \frac{3(c+1)}{4n} cos^2\theta + \frac{(c+1)}{2n}\|\xi^\top\|^2 \tag{3.17}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta}_c(t; n-1)$ satisfies

$$\rho \leq \frac{\widehat{\delta}_c(t;n-1)}{n(n-1)} + \frac{c-3}{4} + \frac{3(c+1)}{4n} cos^2\theta + \frac{(c+1)}{2n}\|\xi^\top\|^2 \tag{3.18}$$

for any real number $t > n(n-1)$. Moreover, the equality holds in (3.17) and (3.18) iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to a suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and a normal orthonormal frame $\{e_{n+1}, \ldots, e_{2m+1}\}$, the shape operators $S_r \equiv S_{e_r}$, $r \in \{n+1, \ldots, m\}$, take the form of (3.3)

Corollary 3.6. *Let M be an n-dimensional slant submanifold of a Kenmotsu space form $\overline{M}(c)$. Then*

(i) The normalized δ-Casorati curvature $\delta_c(n-1)$ satisfies

$$\rho \leq \delta_c(n-1) + \frac{c-3}{4} + \frac{3(c+1)}{4n} cos^2\theta + \frac{(c+1)}{2n}\|\xi^\top\|^2.$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to a suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and a normal orthonormal frame $\{e_{n+1}, \ldots, e_{2m+1}\}$, the shape operators $S_r \equiv S_{e_r}$, $r \in \{n+1, \ldots, m\}$, take the form of (3.13).

(ii) The normalized δ-Casorati curvature $\widehat{\delta}_c(n-1)$ satisfies

$$\rho \leq \widehat{\delta}_c(n-1) + \frac{c-3}{4} + \frac{3(c+1)}{4n} cos^2\theta + \frac{(c+1)}{2n} \|\xi^\top\|^2.$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to a suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and a normal orthonormal frame $\{e_{n+1}, \ldots, e_{2m+1}\}$, the shape operators $S_r \equiv S_{e_r}$, $r \in \{n+1, \ldots, m\}$, take the form of (3.14)

Theorem 3.7. *Let M be an n-dimensional slant submanifold of a cosymplectic space form $\overline{M}(c)$ of dimension $2m+1$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t; n-1)$ satisfies

$$\rho \leq \frac{\delta_c(t; n-1)}{n(n-1)} + \frac{c}{4} + \frac{3c}{4n} cos^2\theta + \frac{c}{2n} \|\xi^\top\|^2 \tag{3.19}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta}_c(t; n-1)$ satisfies

$$\rho \leq \frac{\widehat{\delta}_c(t; n-1)}{n(n-1)} + \frac{c}{4} + \frac{3c}{4n} cos^2\theta + \frac{c}{2n} \|\xi^\top\|^2 \tag{3.20}$$

for any real number $t > n(n-1)$. Moreover, the equality holds in (3.19) and (3.20) iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to a suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and a normal orthonormal frame $\{e_{n+1}, \ldots, e_{2m+1}\}$, the shape operators $S_r \equiv S_{e_r}$, $r \in \{n+1, \ldots, m\}$, take the form of (3.3).

Corollary 3.8. *Let M be an n-dimensional slant submanifold of a cosymplectic space form $\overline{M}$. Then*

(i) The normalized δ-Casorati curvature $\delta_c(n-1)$ satisfies

$$\rho \leq \delta_c(n-1) + \frac{c}{4} + \frac{3c}{4n} cos^2\theta + \frac{c}{2n} \|\xi^\top\|^2.$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to a suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and a normal orthonormal frame $\{e_{n+1}, \ldots, e_{2m+1}\}$, the shape operators $S_r \equiv S_{e_r}$, $r \in \{n+1, \ldots, m\}$, take the form of (3.13).

(ii) The normalized δ-Casorati curvature $\widehat{\delta}_c(n-1)$ satisfies

$$\rho \leq \widehat{\delta}_c(n-1) + \frac{c}{4} + \frac{3c}{4n} cos^2\theta + \frac{c}{2n} \|\xi^\top\|^2.$$

Moreover, the equality sign holds iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}$, such that with respect to a suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and a normal orthonormal frame $\{e_{n+1}, \ldots, e_{2m+1}\}$, the shape operators $S_r \equiv S_{e_r}$, $r \in \{n+1, \ldots, m\}$, take the form of (3.14).

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