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EXPONENT OF CONVERGENCE OF SOLUTIONS OF A CLASS OF LINEAR DIFFERENTIAL EQUATIONS IN THE UNIT DISC

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Abstract. In this paper, we investigate the exponent of convergence of $f^{(i)} - \varphi$ where $f^{(0)} = f \not\equiv 0$ is a solution of a linear differential equation with meromorphic coefficients in the unit disc and φ is a small function of f. Our investigation is based on the behavior of the coefficients in a subset Γ of the unit disc such that the set $\Gamma_0 = \{|z| : z \in \Gamma\}$ is of infinite logarithmic measure. By this investigation we can deduce the fixed points of $f^{(i)}$ by taking $\varphi(z) = z$.

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1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic function on the complex plane $\mathbb{C}$ and in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ (see [10, 12, 16]). In addition, for $n \in \mathbb{N} - \{0\}$, the iterated $n-$order of meromorphic function $f(z)$ in D is defined by

$$\sigma_n(f) = \limsup_{r \to 1^-} \frac{\log_n^+ T(r,f)}{-\log(1-r)},$$

where $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$, $\log_{n+1}^+ x = \log^+ \log_n^+ x$ and $T(r,f)$ is the Nevanlinna characteristic function of f. For an analytic function $f(z)$ in D, we have also the iterated $n-$order defined by

$$\sigma_{M,n}(f) = \limsup_{r \to 1^-} \frac{\log_{n+1}^+ M(r,f)}{-\log(1-r)},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$. If f is analytic in D, it is well known that $\sigma_1(f)$ and $\sigma_{M,1}(f)$ satisfy the inequalities

$$\sigma_1(f) \le \sigma_{M,1}(f) \le \sigma_1(f) + 1,$$

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which are the best possible in the sense that there are analytic functions g and h such that $\sigma_{M,1}(g) = \sigma_1(g)$ and $\sigma_{M,1}(h) = \sigma_1(h) + 1$, see [6, 14].

For example, the function $g(z) = \exp\left\{\dfrac{1}{(1-z)^{\mu}}\right\}$ $(\mu \geq 1)$ satisfies $\sigma_1(g) = \mu - 1$ and $\sigma_{M,1}(g) = \mu$. We note that the principal branch of logarithm is used.

Obviously, we have

$$\sigma_1(f) < \infty \text{ if and only if } \sigma_{M,1}(f) < \infty.$$

However, it follows by Proposition 2.2.2 in [12] that $\sigma_{M,n}(f) = \sigma_n(f)$ for $n \geq 2$.

Definition 1. *[11] A meromorphic function f in D is called admissible if and only if*

$$\limsup_{r\to 1^-} \frac{T(r,f)}{-\log(1-r)} = \infty;$$

and f is called nonadmissible if and only if

$$\limsup_{r\to 1^-} \frac{T(r,f)}{-\log(1-r)} < \infty.$$

We will use the notation $\lambda_n(f)$ to denote the n-iterated exponent of convergence of the zero-sequence of meromorphic function $f(z)$ and $\overline{\lambda}_n(f)$ to denote the n-iterated exponent of convergence of distinct zero-sequence of $f(z)$, which are defined as the following:

$$\lambda_n(f) = \limsup_{r\to 1^-} \frac{\log_n N\left(r, \frac{1}{f}\right)}{-\log(1-r)} \quad \text{and} \quad \overline{\lambda}_n(f) = \limsup_{r\to 1^-} \frac{\log_n \overline{N}\left(r, \frac{1}{f}\right)}{-\log(1-r)}.$$

Many authors investigated the linear differential equation

$$f'' + A(z)e^{az}f' + B(z)e^{bz}f = 0, \tag{1.1}$$

where $A(z)$ and $B(z)$ are entire functions, see for example [1, 4, 5, 7]. Recently in [8], Hamouda investigated the linear differential equation

$$f'' + A(z)e^{\overline{\frac{a}{(z_0-z)^{\mu}}}}f' + B(z)e^{\overline{\frac{b}{(z_0-z)^{\mu}}}}f = 0,$$

where $A(z)$ and $B(z) \not\equiv 0$ are analytic functions in the unit disc by making use the behavior of the coefficients on a neighborhood of a point on the boundary of the unit disc. After that in [9], Hamouda proved the following results.

Theorem A. *[9] Let $A_0(z), ..., A_{k-1}(z)$ be meromorphic functions in the unit disc D. If there exist a point ω_0 on the boundary ∂D of the unit disc D and a curve $\gamma \subset D$ tending to ω_0 such that*

$$\lim_{\substack{z\to\omega_0\\ z\in\gamma}} \frac{\sum_{j=1}^{k-1}|A_j(z)| + 1}{|A_0(z)|(1-|z|)^{\mu}} = 0,$$

for any $\mu > 0$, *then every meromorphic solution* $f(z) \not\equiv 0$ *of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \dots + A_1(z) f' + A_0(z) f = 0, \tag{1.2}$$

is of infinite order.

Theorem B. *[9] Let* $A_0(z), \dots, A_{k-1}(z)$ *be meromorphic functions in the unit disc* D. *If there exist* $\omega_0 \in \partial D$ *and a curve* $\gamma \subset D$ *tending to* ω_0 *such that*

$$\lim_{\substack{z \to \omega_0 \\ z \in \gamma}} \frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|} \exp_n \left\{ \frac{\lambda}{(1-|z|)^{\mu}} \right\} = 0,$$

where $n \geq 1$ *is an integer, (*$\exp_1(z) = \exp(z)$, $\exp_{n+1}(z) = \exp\{\exp_n(z)\}$*), and* $\lambda > 0$, $\mu > 0$ *are real constants, then every meromorphic solution* $f(z) \not\equiv 0$ *of the differential equation (1.2) satisfies* $\sigma_n(f) = \infty$, *and furthermore* $\sigma_{n+1}(f) \geq \mu$.

Remark 1. *Theorem A and Theorem B remain valid if in their assumptions, instead of taking limit along a curve* $\gamma \subset D$ *we take limit as* $|z| \to 1^-$ *along a subset* $\Gamma \subset D$ *such that the set* $\Gamma_0 = \{|z| : z \in \Gamma\}$ *is of infinite logarithmic measure, that is* $\int_{\Gamma_0} \frac{dr}{1-r} = +\infty$. *We note that, for a curve* $\gamma \subset D$ *tending to a point* $\omega_0 \in \partial D$, *the set* $\{|z| : z \in \gamma\}$ *is of infinite logarithmic measure; which means that the curve* $\gamma \subset D$ *is a particular case of the subset* $\Gamma \subset D$.

Theorem B has been generalized recently by Semochko in [13] by making use a new approach on the growth, namely φ-orders.

In [15], Xu, Tu and Zheng investigated the relationship between small function and derivatives of solutions of (1.2) where $A_j(z)$ are entire or meromorphic functions in the complex plane, and obtained the following results.

Theorem C. *[15] Let* $A_j(z)$ $(j = 0, 1, \dots, k-1)$ *be entire functions with finite order and satisfy one of the following conditions:*
(i) $\max\{\sigma(A_j) : j = 1, 2, \dots, k-1\} < \sigma(A_0) < \infty$;
(ii) $0 < \sigma(A_{k-1}) = \dots = \sigma(A_0) < \infty$ *and* $\max\{\tau(A_j) : j = 1, 2, \dots, k-1\} = \tau_1 < \tau(A_0) = \tau$,
then for every solution $f \not\equiv 0$ *of (1.2) and for any entire function* $\varphi(z) \not\equiv 0$ *satisfying* $\sigma_2(\varphi) < \sigma(A_0)$, *we have*

$$\overline{\lambda_2}(f - \varphi) = \overline{\lambda_2}(f' - \varphi) = \overline{\lambda_2}(f'' - \varphi) = \overline{\lambda_2}\left(f^{(i)} - \varphi\right) = \sigma_2(f) = \sigma(A_0)\,(i \in \mathbb{N})$$

Theorem D. *[15] Let* $A_j(z)$ $(j = 1, 2, \dots, k-1)$ *be polynomials,* $A_0(z)$ *be a transcendental entire function, then for every solution* $f \not\equiv 0$ *of (1.2) and for any entire function* $\varphi(z)$ *of finite order, we have*
(i) $\overline{\lambda}(f - \varphi) = \lambda(f - \varphi) = \sigma(f) = \infty$;
(ii) $\overline{\lambda}\left(f^{(i)} - \varphi\right) = \lambda\left(f^{(i)} - \varphi\right) = \sigma\left(f^{(i)} - \varphi\right) = \infty \quad (i \geq 1, i \in \mathbb{N})$.

Theorem E. *[15] Let $A_j(z)$ $(j=0,1,...,k-1)$ be meromorphic functions satisfying* $\max\{\sigma(A_j) : j=1,2,...,k-1\} < \sigma(A_0)$ *and* $\delta(\infty, A_0) > 0$. *Then, for every meromorphic solution $f \not\equiv 0$ of (1.2) and for any meromorphic function $\varphi(z) \not\equiv 0$ satisfying $\sigma_2(\varphi) < \sigma(A_0)$, we have*

$$\overline{\lambda_2}\left(f^{(i)} - \varphi\right) = \lambda_2\left(f^{(i)} - \varphi\right) \geq \sigma(A_0) \quad (i \in \mathbb{N}), \text{ where } f^{(0)} = f.$$

In this paper, we will investigate the exponent of convergence of $f^{(i)} - \varphi$ where f is a solution of (1.2) and the coefficients are meromorphic in the unit disc and satisfy assumptions more general than those of Theorem A and Theorem B. In fact, we will prove the following results.

Theorem 1. *Let $A_0(z), ..., A_{k-1}(z)$ be meromorphic functions in the unit disc D of finite order. If there exist a subset $\Gamma \subset D$ such that the set $\Gamma_0 = \{|z| : z \in \Gamma\}$ is of infinite logarithmic measure, that is $\int\limits_{\Gamma_0} \frac{dr}{1-r} = +\infty$, and for every fixed $\mu > 0$ we have*

$$\lim_{\substack{|z|\to 1^-\\ z\in\Gamma}} \frac{\sum\limits_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|\,(1-|z|)^{\mu}} = 0, \tag{1.3}$$

then for every meromorphic solution $f \not\equiv 0$ of (1.2) and for any meromorphic function $\varphi(z) \not\equiv 0$ in the unit disc D of finite order, we have

$$\overline{\lambda}(f-\varphi) = \overline{\lambda}\left(f^{(i)} - \varphi\right) = \lambda\left(f^{(i)} - \varphi\right) = \sigma(f) = \infty \quad (i \in \mathbb{N}). \tag{1.4}$$

Corollary 1. *Let $A_0(z) \not\equiv 0, ..., A_{k-1}(z)$ be meromorphic functions in the unit disc of zero order such that $A_1, ..., A_{k-1}$ are nonadmissible while A_0 is admissible. Then for every solution $f(z) \not\equiv 0$ of (1.2) and for any meromorphic function $\varphi(z) \not\equiv 0$ in the unit disc D of finite order, we have*

$$\overline{\lambda}(f-\varphi) = \overline{\lambda}\left(f^{(i)} - \varphi\right) = \lambda\left(f^{(i)} - \varphi\right) = \sigma(f) = \infty \quad (i \in \mathbb{N}).$$

Theorem 2. *Let $A_0(z), ..., A_{k-1}(z)$ be meromorphic functions in the unit disc D with $\sigma_n(A_j) < \infty$. If there exist a subset $\Gamma \subset D$ such that the set $\Gamma_0 = \{|z| : z \in \Gamma\}$ is of infinite logarithmic measure and*

$$\lim_{\substack{|z|\to 1^-\\ z\in\Gamma}} \frac{\sum\limits_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|} \exp_n\left\{\frac{\beta}{(1-|z|)^{\mu}}\right\} = 0, \tag{1.5}$$

where $n \geq 1$ is an integer and $\beta > 0$, $\mu > 0$ are real constants, then for every meromorphic solution $f \not\equiv 0$ of (1.2) and for any meromorphic function $\varphi(z) \not\equiv 0$ in the unit disc D satisfying $\sigma_{n+1}(\varphi) < \mu$, we have

$$\overline{\lambda_{n+1}}(f-\varphi) = \overline{\lambda_{n+1}}\left(f^{(i)} - \varphi\right) = \lambda_{n+1}\left(f^{(i)} - \varphi\right) = \sigma_{n+1}(f) \geq \mu \quad (i \in \mathbb{N}). \tag{1.6}$$

Corollary 2. *Let $A_0(z), ..., A_{k-1}(z)$ be meromorphic functions in the unit disc D with $\sigma_n(A_j) < \infty$. If there exist a subset $\Gamma \subset D$ such that the set $\Gamma_0 = \{|z| : z \in \Gamma\}$ is of infinite logarithmic measure and*

$$|A_0(z)| \geq \exp_n \left\{ \frac{\alpha}{(1-|z|)^{\mu}} \right\}, \tag{1.7}$$

$$|A_j(z)| \leq \exp_n \left\{ \frac{\beta}{(1-|z|)^{\mu}} \right\}, \ (j = 1, ..., k-1), \tag{1.8}$$

as $|z| \to 1^-$ with $z \in \Gamma$, where $n \geq 1$ is an integer, $0 \leq \beta < \alpha$, $\mu > 0$ are real constants, then for every meromorphic solution $f \not\equiv 0$ of (1.2) and for any meromorphic function $\varphi(z) \not\equiv 0$ in the unit disc D satisfying $\sigma_{n+1}(\varphi) < \mu$, we have

$$\overline{\lambda_{n+1}}(f-\varphi) = \overline{\lambda_{n+1}}\left(f^{(i)} - \varphi\right) = \lambda_{n+1}\left(f^{(i)} - \varphi\right) = \sigma_{n+1}(f) \geq \mu \quad (i \in \mathbb{N}).$$

Corollary 3. *Let $A_0(z) \not\equiv 0, A_1(z), ..., A_{k-1}(z)$ be analytic functions in the unit disc with $\sigma_n(A_j) < \infty$. Suppose that $\alpha > 1$ is a real constant, a and ω_0 are complex numbers such that $a \neq 0$, $|\omega_0| = 1$. If $A_0(z), ..., A_{k-1}(z)$ are analytic at ω_0 then for every solution $f(z) \not\equiv 0$ of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + ... + A_1(z) f' + A_0(z) \exp_n \left\{ \frac{a}{(\omega_0 - z)^{\alpha}} \right\} f = 0,$$

and for any analytic function $\varphi(z) \not\equiv 0$ in the unit disc D satisfying $\sigma_{n+1}(\varphi) < \alpha$, we have

$$\overline{\lambda_{n+1}}(f-\varphi) = \overline{\lambda_{n+1}}\left(f^{(i)} - \varphi\right) = \lambda_{n+1}\left(f^{(i)} - \varphi\right) = \sigma_{n+1}(f) \geq \alpha \quad (i \in \mathbb{N}).$$

Remark 2. *The following example shows that the assumption (1.5) of Theorem 2 does not require that the iterated order of growth of A_0 is greater than those of the other coefficients; (the same applies for Theorem 1).*

Example 1. *Consider the differential equation*

$$f'' + A_1(z) f' + A_0(z) f = 0.$$

(i) If $A_1(z) = \exp_3\left\{\frac{1}{1-z}\right\} \exp_4\left\{\frac{1}{1-z}\right\}$ and $A_0(z) = \exp_3\left\{\frac{2}{1-z}\right\} \exp_4\left\{\frac{1}{1-z}\right\}$, then $\sigma_4(A_1) = \sigma_4(A_0) = 1$; and it is clear that the assumption (1.5) holds by taking $n = 3$, $\beta = 1$, $\mu = 1$, $\Gamma = \{z \in D : \arg z = 0\}$; and so $\sigma_4(f) \geq 1$.
(ii) If $A_1(z) = \exp_3\left\{\frac{1}{1-z}\right\} \exp_4\left\{\frac{1}{i-z}\right\}$ and $A_0(z) = \exp_3\left\{\frac{2}{1-z}\right\} \exp_4\left\{\frac{1}{2+z}\right\}$, then $\sigma_4(A_1) = 1$ while $\sigma_4(A_0) = 0$; and it is clear that the assumption (1.5) holds by taking $n = 3$, $\beta = 1$, $\mu = 1$, $\Gamma = \{z \in D : \arg z = 0\}$; and so $\sigma_4(f) \geq 1$.
And for any meromorphic function $\varphi(z) \not\equiv 0$ in the unit disc D satisfying $\sigma_4(\varphi) < 1$, (1.6) holds.

Now, we give an example to prove the existence of meromorphic solutions in the case of meromorphic coefficients that satisfy the assumptions of Theorem 1 and Theorem 2.

Example 2. *The differential equation*

$$f'' + A_1(z) f' + A_0(z) f = 0,$$

admits the two linearly independent solutions

$$f_1(z) = \frac{1}{z} \exp_2\left\{\frac{1}{1-z}\right\}, \; f_2(z) = \frac{1}{z} \exp_2\left\{\frac{2}{1-z}\right\},$$

where

$$A_1(z) = -\frac{\frac{f_2''}{f_2} - \frac{f_1''}{f_1}}{\frac{f_2'}{f_2} - \frac{f_1'}{f_1}}, \; A_0(z) = -\frac{f_1''}{f_1} - A_1(z) \frac{f_1'}{f_1}.$$

$z = 0$ *is a pole for* $A_0(z)$ *and* $A_1(z)$. *By taking*

$$\Gamma = \left\{z \in D : \arg z = 0 \text{ and } |z| \geq \frac{1}{2}\right\} = \left[\frac{1}{2}, 1\right)$$

and after some computations, we obtain that

$$|A_1(z)| = \frac{2}{(1-|z|)^2} \exp\left\{\frac{2}{1-|z|}\right\}(1 + o(1)),$$

$$|A_0(z)| = \frac{2}{(1-|z|)^4} \exp\left\{\frac{3}{1-|z|}\right\}(1 + o(1)),$$

as $|z| \to 1^-$ *with* $z \in \Gamma$. *It is clear that the assumption (1.5) holds by taking* $n = 1, \; \mu = 1, \; \beta = \frac{1}{2}$. *Also the assumption (1.3) holds for any fixed* $\mu > 0$.

2. Preliminary lemmas

Throughout this paper, we use the following notations that are not necessarily the same at each occurrence:

$E \subset (0, 1)$ is a set of finite logarithmic measure, that is $\int_E \frac{dr}{1-r} < \infty$.

$\varepsilon > 0, \; \alpha > 0, \; \beta > 0, \; 0 \leq r_0 < 1$, are real constants.

Lemma 1. *[15] Assume that* $f \not\equiv 0$ *is a solution of (1.2). Set* $g = f - \varphi$; *then* g *satisfies the equation*

$$g^{(k)} + A_{k-1} g^{(k-1)} + \ldots + A_0 g = -\left[\varphi^{(k)} + A_{k-1} \varphi^{(k-1)} + \ldots + A_0 \varphi\right]. \tag{2.1}$$

Lemma 2. *[15] Assume that* $f \not\equiv 0$ *is a solution of (1.2). Set* $g_i = f^{(i)} - \varphi, \; (i \in \mathbb{N} - \{0\})$; *then* g_i *satisfies the equation*

$$g_i^{(k)} + U_{k-1}^i g_i^{(k-1)} + \ldots + U_0^i g_i = -\left[\varphi^{(k)} + U_{k-1}^i \varphi^{(k-1)} + \ldots + U_0^i \varphi\right], \tag{2.2}$$

where

$$U_j^i = \left(U_{j+1}^{i-1}\right)' + U_j^{i-1} - \frac{\left(U_0^{i-1}\right)'}{U_0^{i-1}} U_{j+1}^{i-1}, \tag{2.3}$$

$j = 0, 1, ..., k-1,\ U_j^0 = A_j$ *and* $U_k^i \equiv 1$.

Lemma 3. *[6] Let f be a meromorphic function in the unit disc D such that $f^{(j)}$ does not vanish identically. Let $\varepsilon > 0$ be a constant; k and j be integers satisfying $k > j \geq 0$ and $d \in (0,1)$. Then there exists a set $E \subset [0,1)$ which satisfies $\int\limits_E \frac{1}{1-r} dr < \infty$, such that for all $z \in D$ satisfying $|z| \notin E$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq \left(\left(\frac{1}{1-|z|} \right)^{(2+\varepsilon)} \max \left\{ \log \frac{1}{1-|z|}, T\left(s\left(|z|\right), f\right) \right\} \right)^{k-j},$$

where $s(|z|) = 1 - d(1-|z|)$. As particular cases:
i) if $\sigma_1(f) = \sigma_1 < \infty$, then

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq \left(\frac{1}{1-|z|} \right)^{(k-j)(\sigma_1 + 2 + \varepsilon)}, \ |z| \notin E;$$

ii) if $\sigma_n(f) = \sigma_n < \infty$ for $n \geq 2$, then

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq \exp_{n-1} \left\{ \left(\frac{1}{1-|z|} \right)^{(\sigma_n + \varepsilon)} \right\}, \ |z| \notin E.$$

Lemma 4. *[3] If f and g are meromorphic functions in the unit disc D, $n \in \mathbb{N} - \{0\}$, then we have*
(i) $\sigma_n(f) = \sigma_n\left(\frac{1}{f}\right),\ \sigma_n(a \cdot f) = \sigma_n(f)\ \ (a \in \mathbb{C} - \{0\})$;
(ii) $\sigma_n(f) = \sigma_n(f')$;
(iii) $\max\{\sigma_n(f+g),\ \sigma_n(f \cdot g)\} \leq \max\{\sigma_n(f),\ \sigma_n(g)\}$;
(iv) If $\sigma_n(f) < \sigma_n(g)$, then $\sigma_n(f+g) = \sigma_n(g)$ and $\sigma_n(f \cdot g) = \sigma_n(g)$.

Lemma 5. *Let $A_0(z), ..., A_{k-1}(z)$ be meromorphic functions in the unit disc D of finite $n-$order $\sigma_n(A_j) < \infty$ and satisfying the assumption (1.5). Then the sequence $\left\{U_j^i\right\}$ defined by (2.3) satisfies*

$$\lim_{\substack{|z| \to 1^- \\ z \in \Gamma}} \frac{\sum\limits_{j=1}^{k-1} \left|U_j^i(z)\right| + 1}{\left|U_0^i(z)\right|} \exp_n \left\{ \frac{\beta - \varepsilon}{(1-|z|)^{\mu}} \right\} = 0, \ \ |z| \notin E, \tag{2.4}$$

where $0 < \varepsilon < \beta$.

Proof. By induction on $i \in \mathbb{N}$. For $i = 0$, we have $U_j^0 = A_j$; this case is trivial because we have (1.5); we mention here that $\Gamma_0 \backslash E$ remains of infinite logarithmic measure. For $i=1$, we have $U_j^1 = A'_{j+1} + A_j - \frac{A'_0}{A_0} A_{j+1} = A_j + A_{j+1} \left(\frac{A'_{j+1}}{A_{j+1}} - \frac{A'_0}{A_0} \right)$ $(j = 0, 1, ..., k-1)$ and $A_k \equiv 1$. So

$$\left| U_0^1 \right| \geq |A_0| - |A_1| \left(\left| \frac{A'_1}{A_1} \right| + \left| \frac{A'_0}{A_0} \right| \right) \tag{2.5}$$

and

$$\left| U_j^1 \right| \leq |A_j| + |A_{j+1}| \left(\left| \frac{A'_{j+1}}{A_{j+1}} \right| + \left| \frac{A'_0}{A_0} \right| \right). \tag{2.6}$$

From (2.5) and (2.6), we get

$$\left| \frac{U_j^1}{U_0^1} \right| \leq \frac{|A_j| + |A_{j+1}| \left(\left| \frac{A'_{j+1}}{A_{j+1}} \right| + \left| \frac{A'_0}{A_0} \right| \right)}{|A_0| \left(1 - \frac{|A_1|}{|A_0|} \left(\left| \frac{A'_1}{A_1} \right| + \left| \frac{A'_0}{A_0} \right| \right) \right)}. \tag{2.7}$$

Set $\sigma_n = \max_{0 \leq j \leq k-1} \{\sigma_n (A_j)\}$. Then from Lemma 3, we have

$$\left| \frac{A'_j}{A_j} \right| \leq \exp_{n-1} \left\{ \frac{1}{(1-|z|)^{\sigma_n + \varepsilon}} \right\}, \ |z| \notin E. \tag{2.8}$$

From the assumption (1.5), we have

$$\lim_{\substack{|z| \to 1^- \\ z \in \Gamma}} \frac{|A_j(z)|}{|A_0(z)|} \exp_n \left\{ \frac{\beta}{(1-|z|)^{\mu}} \right\} = 0, \quad (j = 1, ..., k)$$

which means that for every $\varepsilon > 0$ there exists $0 < r_0 < 1$ such that for all $z \in \Gamma$ with $r_0 < |z| = r < 1$ we have

$$\frac{|A_j(z)|}{|A_0(z)|} \exp_n \left\{ \frac{\beta}{(1-|z|)^{\mu}} \right\} < \varepsilon.$$

In particular for $\varepsilon = 1$, we have

$$\left| \frac{A_j}{A_0} \right| \leq \left(\exp_n \left\{ \frac{\beta}{(1-|z|)^{\mu}} \right\} \right)^{-1}. \tag{2.9}$$

So, from (2.8) and (2.9), we get

$$\frac{|A_1|}{|A_0|} \left(\left| \frac{A'_1}{A_1} \right| + \left| \frac{A'_0}{A_0} \right| \right) \leq \left(\exp_n \left\{ \frac{\beta - \frac{\varepsilon}{2}}{(1-|z|)^{\mu}} \right\} \right)^{-1},$$

as $|z| \to 1^-$ with $z \in \Gamma$ and $|z| \notin E$; and so we can put

$$1 - \frac{|A_1|}{|A_0|} \left(\left| \frac{A'_1}{A_1} \right| + \left| \frac{A'_0}{A_0} \right| \right) > \frac{1}{2} \ \text{ as } |z| \to 1^- \text{ with } z \in \Gamma \text{ and } |z| \notin E. \tag{2.10}$$

By combining (2.7)-(2.10), we obtain
(2.11)
$$\left|\frac{U_j^1}{U_0^1}\right| \leq \left(\exp_n\left\{\frac{\lambda-\frac{\varepsilon}{2}}{(1-|z|)^{\mu}}\right\}\right)^{-1} \quad \text{as } |z|\to 1^- \text{ with } z\in\Gamma \text{ and } |z|\notin E.$$

Now from (2.5) we have

$$\frac{1}{|U_0^1|} \leq \frac{1}{|A_0|\left(1-\frac{|A_1|}{|A_0|}\left(\left|\frac{A_1'}{A_1}\right|+\left|\frac{A_0'}{A_0}\right|\right)\right)}. \tag{2.12}$$

By the assumption (1.5), we have

$$\lim_{\substack{|z|\to 1^-\\ z\in\Gamma}} \frac{1}{|A_0(z)|}\exp_n\left\{\frac{\beta}{(1-|z|)^{\mu}}\right\}=0;$$

and by the same previous method for$|z|\to 1^-$ with $z\in\Gamma$, we get

$$\frac{1}{|A_0|} \leq \left(\exp_n\left\{\frac{\beta}{(1-|z|)^{\mu}}\right\}\right)^{-1}. \tag{2.13}$$

Using (2.10) and (2.13) in (2.12), we obtain
(2.14)
$$\frac{1}{|U_0^1|} \leq \left(\exp_n\left\{\frac{\beta-\frac{\varepsilon}{2}}{(1-|z|)^{\mu}}\right\}\right)^{-1} \quad \text{as } |z|\to 1^- \text{ with } z\in\Gamma \text{ and } |z|\notin E.$$

(2.11) and (2.14) imply that (2.4) is satisfied for $i=1$. Now, suppose that (2.4) is satisfied for $i=m$; which implies that, as $|z|\to 1^-$ with $z\in\Gamma$ and $|z|\notin E$, we have

$$\left|\frac{U_j^m}{U_0^m}\right| \leq \left(\exp_n\left\{\frac{\beta-\varepsilon}{(1-|z|)^{\mu}}\right\}\right)^{-1}$$

and

$$\frac{1}{|U_0^m|} \leq \left(\exp_n\left\{\frac{\beta-\varepsilon}{(1-|z|)^{\mu}}\right\}\right)^{-1}.$$

From (2.3), we get

$$\left|U_j^{m+1}\right| \leq \left|U_j^m\right|+\left|U_{j+1}^m\right|\left(\left|\frac{U_{j+1}^{m\prime}}{U_{j+1}^m}\right|+\left|\frac{U_0^{m\prime}}{U_0^m}\right|\right)$$

and

$$\frac{1}{\left|U_0^{m+1}\right|} \leq \frac{1}{\left|U_0^m\right|\left(1-\frac{|U_1^m|}{|U_0^m|}\left(\left|\frac{U_1^{m\prime}}{U_1^m}\right|+\left|\frac{U_0^{m\prime}}{U_0}\right|\right)\right)}.$$

By the same method used in (2.5)-(2.14) and taking into account that $\sigma_n\left(U_j^i\right)\leq \max_{0\leq j\leq k-1}\{\sigma_n(A_j)\}$ (this follows from Lemma 4 and by induction on i), we obtain

$$\frac{\left|U_j^{m+1}\right|}{\left|U_0^{m+1}\right|} \leq \left(\exp_n\left\{\frac{\beta-\frac{\varepsilon}{2}}{(1-|z|)^{\mu}}\right\}\right)^{-1}$$

and

$$\frac{1}{\left|U_0^{m+1}\right|} \leq \left(\exp_n\left\{\frac{\beta-\frac{\varepsilon}{2}}{(1-|z|)^{\mu}}\right\}\right)^{-1}$$

as $|z| \to 1^-$ with $z \in \Gamma$ and $|z| \notin E$; which imply that (2.4) is satisfied for $i = m+1$. Thus the proof is completed. □

Lemma 6. *Let $A_0(z), ..., A_{k-1}(z)$ be meromorphic functions in the unit disc D of finite order satisfying the condition (1.3). Then, for every fixed $\mu > 0$, the sequence $\left\{U_j^i\right\}$ defined by (2.3) satisfies*

$$\lim_{\substack{|z|\to 1^- \\ z\in\Gamma}} \frac{\sum_{j=1}^{k-1}\left|U_j^i(z)\right|+1}{\left|U_0^i(z)\right|(1-|z|)^{\mu}} = 0, \quad |z| \notin E. \tag{2.15}$$

Proof. By induction on $i \in \mathbb{N}$. For $i = 0$, we have $U_j^0 = A_j$; this case is trivial because we have (1.3). For $i = 1$; from the assumption (1.3), we have

$$\lim_{\substack{|z|\to 1^- \\ z\in\Gamma}} \frac{|A_j(z)|}{|A_0(z)|(1-|z|)^{\mu}} = 0, \quad (j = 1, ..., k) \tag{2.16}$$

and

$$\lim_{\substack{|z|\to 1^- \\ z\in\Gamma}} \frac{1}{|A_0(z)|(1-|z|)^{\mu}} = 0. \tag{2.17}$$

(2.16) means that for every $\varepsilon > 0$ there exists $0 < r_0 < 1$ such that for all $z \in \Gamma$ with $r_0 < |z| = r < 1$ we have

$$\frac{|A_j(z)|}{|A_0(z)|(1-|z|)^{\mu}} < \varepsilon.$$

In particular for $\varepsilon = 1$, we have

$$\frac{|A_j(z)|}{|A_0(z)|} < (1-|z|)^{\mu}, \quad \text{for every fixed } \mu > 0. \tag{2.18}$$

The same applies for (2.17); from which we get

$$\frac{1}{|A_0(z)|} < (1-|z|)^{\mu}, \quad \text{for every fixed } \mu > 0. \tag{2.19}$$

Set $\sigma_1 = \max_{0\leq j\leq k-1}\{\sigma_1(A_j)\}$. From Lemma 3, we have

$$\left|\frac{A_j'}{A_j}\right| \leq \frac{1}{(1-|z|)^{\sigma_1+2+\varepsilon}}, \ |z| \notin E. \tag{2.20}$$

So, from (2.18) and (2.20), we get

$$\frac{|A_1|}{|A_0|}\left(\left|\frac{A_1'}{A_1}\right|+\left|\frac{A_0'}{A_0}\right|\right) \leq \frac{(1-|z|)^{\mu}}{(1-|z|)^{\sigma_1+2+\varepsilon}};$$

and since we have for every fixed $\mu > 0$, we can choose $\mu > \sigma_1 + 2 + \varepsilon$; thus, as $|z| \to 1^-$ with $z \in \Gamma$ and $|z| \notin E$, we get

$$\frac{|A_1|}{|A_0|}\left(\left|\frac{A_1'}{A_1}\right| + \left|\frac{A_0'}{A_0}\right|\right) \to 0;$$

and then, we can put

$$(2.21) \quad 1 - \frac{|A_1|}{|A_0|}\left(\left|\frac{A_1'}{A_1}\right| + \left|\frac{A_0'}{A_0}\right|\right) > \frac{1}{2} \quad \text{as } |z| \to 1^- \text{ with } z \in \Gamma \text{ and } |z| \notin E.$$

By combining (2.18)-(2.21) with (2.7), we obtain

$$(2.22) \qquad \left|\frac{U_j^1}{U_0^1}\right| \le (1 - |z|)^{\mu'} \quad \text{as } |z| \to 1^- \text{ with } z \in \Gamma \text{ and } |z| \notin E,$$

for every fixed $\mu' > 0$. Also the combining of (2.19)-(2.21) with (2.12) gives

$$(2.23) \qquad \left|\frac{1}{U_0^1}\right| \le (1 - |z|)^{\mu'} \quad \text{as } |z| \to 1^- \text{ with } z \in \Gamma \text{ and } |z| \notin E,$$

for every fixed $\mu' > 0$. (2.22) and (2.23) imply that (2.15) holds for $i = 1$. Now, suppose that (2.15) is satisfied for $i = m$; which implies that, for every fixed $\mu > 0$, as $|z| \to 1^-$ with $z \in \Gamma$ and $|z| \notin E$, we have

$$\left|\frac{U_j^m}{U_0^m}\right| \le (1 - |z|)^{\mu}$$

and

$$\frac{1}{|U_0^m|} \le (1 - |z|)^{\mu}$$

By the same method used above, we obtain that, for every fixed $\mu' > 0$,

$$\frac{|U_j^{m+1}|}{|U_0^{m+1}|} \le (1 - |z|)^{\mu'}$$

and

$$\frac{1}{|U_0^{m+1}|} \le (1 - |z|)^{\mu'}$$

as $|z| \to 1^-$ with $z \in \Gamma$ and $|z| \notin E$; which imply that (2.15) holds for $i = m+1$. Thus, the proof is completed. $\square$

Lemma 7. *Let $G \not\equiv 0, H_j(z)$ $j = 0, 1, ..., k-1$ be meromorphic functions in the unit disc D. If f is a meromorphic solution of the differential equation*

$$(2.24) \qquad f^{(k)} + H_{k-1}(z) f^{(k-1)} + ... + H_1(z) f' + H_0(z) f = G(z),$$

satisfying $\max\{\sigma_n(G), \sigma_n(H_j);\ j = 0, 1, ..., k-1\} < \sigma_n(f)$, *then*

$$\overline{\lambda_n}(f) = \lambda_n(f) = \sigma_n(f),\ (n \in \mathbb{N} - \{0\}).$$

Proof. The same reasoning of the proof of Lemma 3.5 in [2] when $G \not\equiv 0,\ H_j(z)$ are analytic in the unit disc D. $\square$

3. Proof of theorems

Proof of Theorem 1. Suppose that $f \not\equiv 0$ is a solution of (1.2) and $\varphi(z) \not\equiv 0$ is a meromorphic function of finite order in the unit disc D. We start to prove (1.4) for $i = 0$, i.e. $\overline{\lambda_1}(f-\varphi) = \lambda_1(f-\varphi) = \sigma_1(f) = \infty$. From Theorem A and Remark 1, we have $\sigma_1(f) = \infty$. Set $g = f - \varphi$. Since $\sigma_1(\varphi) < \infty$, we have $\sigma_1(g) = \sigma_1(f) = \infty$. By Lemma 1, g satisfies (2.1). Set $G(z) = \varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \dots + A_0\varphi$. If $G \equiv 0$, then by Theorem A and Remark 1, we have $\sigma_1(\varphi) = \infty$, a contradiction; thus $G \not\equiv 0$. We have $\sigma_1(g) = \sigma_1(f) = \infty > \max\{\sigma_1(G), \sigma_1(A_j)\}$; so the assumption of Lemma 7 holds for the differential equation (2.1); and then we have $\overline{\lambda_1}(g) = \lambda_1(g) = \sigma_1(g)$. Then, we conclude that $\overline{\lambda_1}(f-\varphi) = \lambda_1(f-\varphi) = \sigma_1(f) = \infty$. Now, we prove (1.4) for $i \geq 1$. Set $g_i = f^{(i)} - \varphi$. By Lemma 4, we have $\sigma_1\left(f^{(i)}\right) = \sigma_1(f) = \infty$, and since $\sigma_1(\varphi) < \infty$, we obtain $\sigma_1(g_i) = \sigma_1(f) = \infty$. By Lemma 2, g_i satisfies (2.2). Set $G_i = \varphi^{(k)} + U^i_{k-1}\varphi^{(k-1)} + \dots + U^i_0\varphi$. If $G_i \equiv 0$, by Lemma 6, Theorem A and Remark 1, we get $\sigma_1(\varphi) = \infty$, a contradiction; so $G_i \not\equiv 0$. Now, by Lemma 7, as above, we obtain $\overline{\lambda_1}(g_i) = \lambda_1(g_i) = \sigma_1(g_i)$ i.e. $\overline{\lambda_1}\left(f^{(i)} - \varphi\right) = \lambda_1\left(f^{(i)} - \varphi\right) = \sigma_1(f) = \infty$. □

Proof of Theorem 2. Suppose that $f \not\equiv 0$ is a solution of (1.2) and $\varphi(z) \not\equiv 0$ is a meromorphic function in the unit disc D satisfying $\sigma_{n+1}(\varphi) < \mu$. We start to prove (1.6) for $i = 0$, i.e. $\overline{\lambda_{n+1}}(f-\varphi) = \lambda_{n+1}(f-\varphi) = \sigma_{n+1}(f) \geq \mu$. From Theorem B and Remark 1, we have $\sigma_{n+1}(f) \geq \mu$. Set $g = f - \varphi$. From $\sigma_{n+1}(\varphi) < \mu$, we get $\sigma_{n+1}(g) = \sigma_{n+1}(f)$. By Lemma 1, g satisfies (2.1). Set $G(z) = \varphi^{(k)} + A_{k-1}\varphi^{(k-1)} + \dots + A_0\varphi$. If $G \equiv 0$, then by Theorem B and Remark 1, we obtain $\sigma_{n+1}(\varphi) \geq \mu$, a contradiction; thus $G \not\equiv 0$. Now, from $\sigma_{n+1}(g) = \sigma_{n+1}(f) \geq \mu > \max\{\sigma_{n+1}(G), \sigma_{n+1}(A_j)\}$, the assumption of Lemma 7 holds; and then we have $\overline{\lambda_{n+1}}(g) = \lambda_{n+1}(g) = \sigma_{n+1}(g)$. Then, we conclude that $\overline{\lambda_{n+1}}(f-\varphi) = \lambda_{n+1}(f-\varphi) = \sigma_{n+1}(f) \geq \mu$. Now we prove (1.6) for $i \geq 1$. Set $g_i = f^{(i)} - \varphi$. From $\sigma_{n+1}\left(f^{(i)}\right) = \sigma_{n+1}(f) \geq \mu$ and $\sigma_{n+1}(\varphi) < \mu$, we have $\sigma_{n+1}(g_i) = \sigma_{n+1}(f) \geq \mu$. By Lemma 2, g_i satisfies (2.2). Set $G_i = \varphi^{(k)} + U^i_{k-1}\varphi^{(k-1)} + \dots + U^i_0\varphi$. If $G_i \equiv 0$, by Lemma 5, Theorem B and Remark 1, we get $\sigma_{n+1}(\varphi) \geq \mu$, a contradiction; so $G_i \not\equiv 0$. Now, by Lemma 7, as above, we obtain $\overline{\lambda_{n+1}}(g_i) = \lambda_{n+1}(g_i) = \sigma_{n+1}(g_i)$ i.e. $\overline{\lambda_{n+1}}\left(f^{(i)} - \varphi\right) = \lambda_{n+1}\left(f^{(i)} - \varphi\right) = \sigma_{n+1}(f) \geq \mu$. □

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