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ON THE LOCAL CONVERGENCE OF NEWTON-LIKE METHODS WITH FOURTH AND FIFTH ORDER OF CONVERGENCE UNDER HYPOTHESES ONLY ON THE FIRST FRÉCHET DERIVATIVE

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Abstract. We present a local convergence analysis of several Newton-like methods with fourth and fifth order of convergence in order to approximate a locally unique solution of an equation in Banach space setting. Earlier studies have used hypotheses up to the fifth derivative although only the first derivative appears in the definition of these methods. In this study we only use the hypothesis on the first derivative. This way we expand the applicability of these methods. Moreover, we provide a radius of convergence, a uniqueness ball and computable error bounds based on Lipschitz constants. Numerical examples computing the radii of the convergence balls as well as examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

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1. Introduction

In this study, we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on an open, convex subset D of a Banach space X with values in a Banach space Y. Using mathematical modeling, many problems in computational sciences and other disciplines can be expressed as a nonlinear equation (1.1) $[2, 5, 14, 16]$. Closed form solutions of these nonlinear equations exist only for few special cases which may not be of much practical value. Therefore solutions of these nonlinear equations (1.1)

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are approximated by iterative methods. In particular, the practice of Numerical Functional Analysis for approximating solutions iteratively is essentially connected to Newton-like methods [1–28]. The study of convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semi-local convergence analysis of Newton-like methods such as [1–28].

Newton's method is undoubtedly the most popular method for approximating a locally unique solution x^* provided that the initial point is close enough to the solution. In order to obtain a higher order of convergence Newton-like methods have been studied such as Potra-Ptak, Chebyshev, Cauchy Halley and Ostrowski method [3, 6, 21, 26]. The number of function evaluations per step increases with the order of convergence. In the scalar case the efficiency index [3, 6, 16] $EI = p^{\frac{1}{m}}$ provides a measure of balance where p is the order of the method and m is the number of function evaluations.

It is well known that according to the Kung-Traub conjuncture the convergence of any multi-point method without memory cannot exceed the upper bound 2^{m-1} [26] (called the optimal order). Hence the optimal order for a method with three function evaluations per step is 4. The corresponding efficiency index is $EI = 4^{\frac{1}{3}} = 1.58740...$ which is better than Newtons method which is $EI = 2^{\frac{1}{2}} = 1.414....$ Therefore, the study of new optimal methods of order four is important.

We present the local convergence analysis of some fourth order of convergence and some fifth order of convergence methods under the same set of conditions. Many other similar methods can be studied in an analogous way. These methods are:

Fourth order Newton-Traub method (NT4) [26] defined for each $n = 0, 1, 2, ...$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= y_n - F'(x_n)^{-1}F(y_n) \\ x_{n+1} &= y_n - F'(z_n)^{-1}F(y_n); \end{aligned}$$

Fourth order Zhanlar et al. method (ZCA4) [28]:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= y_n - F'(x_n)^{-1}F(y_n) \\ x_{n+1} &= z_n - F'(x_n)^{-1}F(z_n); \end{aligned}$$

Fourth order Sharma method (S4) [25]

$$y_n = x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n)$$

$$x_{n+1} = x_n - \frac{1}{2}(-I + \frac{9}{4}F'(y_n)^{-1}F'(x_n) + \frac{3}{4}F'(x_n)^{-1}F'(y_n))F'(x_n)^{-1}F(x_n);$$

Fifth order Newton-Traub method (NT5) [26]:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= y_n - F'(x_n)^{-1}F(y_n) \\ x_{n+1} &= z_n - F'(y_n)^{-1}F(z_n); \end{aligned}$$

Fifth order Zhanlar et al. method (ZCA5) [28]:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= y_n - F'(y_n)^{-1}F(y_n) \\ x_{n+1} &= z_n - F'(y_n)^{-1}F(z_n); \end{aligned}$$

Fifth order Ezzati, Azadegan (EA5) methods [13]:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= y_n - F'(x_n)^{-1}F(y_n) \\ w_n &= z_n - F'(x_n)^{-1}F(z_n) \\ x_{n+1} &= z_n - F'(x_n)^{-1}(F(z_n) + F(w_n)). \end{aligned}$$

The efficiency index for these methods is $4^{1/4} = 1.4142..$, $4^{1/4}$, $4^{1/3} = 1.5874..$, $5^{1/5} = 1.3797..$, $5^{1/5}$ and $5^{1/5}$, respectively. The local convergence in the scalar case is shown using Taylor expansion and hypothesis on the fourth derivative for the first five methods and the sixth derivative for the last method although only the first derivative appears in the definition of these methods. These hypotheses limit the applicability of the preceding methods. As a motivational example, let us define function F on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \; F'(1) = 3, \\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ F'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously, function F does not have bounded third derivative in D. In the present study we extend the applicability of these methods by using hypotheses up to the first derivative of function F and contractions. Moreover we avoid

Taylor expansions and use instead Lipschitz parameters. This way we do not have to use higher order derivatives to show the convergence of these methods.

The paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result not given in the earlier studies using Taylor expansions. Special cases and numerical examples are presented in the concluding Section 3.

2. Local convergence analysis

We present the local convergence analysis of the methods defined in Section 1. Let $U(v,\rho)$ and $\bar{U}(v,\rho)$ stand, respectively for the open and closed balls in X with center $v \in X$ and of radius $\rho > 0$. By $L(X,Y)$ we denote the space of bounded linear operators from X into Y. We shall use the conditions $(\mathcal{C})$:

$(\mathcal{C}_1)$ $F : D \subset X \to Y$ is Fréchet differentiable operator, where D is an open convex set and X, Y are Banach spaces;

There exists:

$(\mathcal{C}_2)$ $x^* \in D$ such that $F(x^*) = 0$ and $F'(x^*)^{-1} \in L(Y,X)$;

$(\mathcal{C}_3)$ $L > 0$ such that for each $x, y \in D$ $\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|$;

It follows from the first part of $(\mathcal{C}_2)$ and $(\mathcal{C}_3)$ that there exists $L_0 \in (0, L]$ such that for each $x \in D$ $\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|$. Moreover, by the regularity of $F'(x^*)$ and $(\mathcal{C}_3)$ there exists $M \geq 1$ such that for each $x \in D$ $\|F'(x^*)^{-1}F'(x)\| \leq M$.

We shall give the complete proofs for the convergence of S_4 and $EA5$. The proofs for the rest of the methods are given in a analogous way by simply changing the majorizing functions and in some cases the definition of the radius of convergence of the method.

Theorem 2.1. *Suppose that the $(\mathcal{C})$ conditions hold, $M \in [1,3)$ and $\bar{U}(x^*, r) \subseteq D$ where*

$$r = min\{r_1, r_2\},\ r_1 = \frac{2(1 - M/3)}{2L_0 + L} \tag{2.1}$$

and r_2 is the smallest zero of the function $h_2(t) = g_2(t) - 1$ on the interval $[0, \frac{1}{L_0})$, where

$$g_2(t) = \frac{1}{2(1 - L_0 t)}\left[L + 3L_0 M \frac{1 + g_1(t)}{1 - L_0 t}\right] t.$$

Then the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ by method S4 is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, ...$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r \tag{2.2}$$

and

$$\|x_{n+1} - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.3}$$

where the function g_1 *is defined by*

$$g_1(t) = \frac{1}{2(1-L_0 t)}\left(Lt + \frac{2M}{3}\right).$$

Furthermore, for $T \in [r, \frac{2}{L_0})$ *the limit point* x^* *is the only solution of equation* $F(x) = 0$ *in* $\bar{U}(x^*, T) \cap D$.

Proof. It follows from the hypothesis $M \in [1, 3)$, the definition of function g_1 and parameter r_1 that $r_1 > 0$, $g_1(r_1) = 1$ and $0 \leq g_1(t) < 1$ for each $t \in [0, r_1)$ let $h_2(t) = g_2(t) - 1$. We have by the definition of the function h_2 that $h_2(0) = -1$ and $h_2(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. Then, using the intermediate value theorem we deduce that function h_2 has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_2 the smallest such zero. Then, it follows from (2.1) that

$$0 \leq g_1(t) < 1 \tag{2.4}$$

and

$$0 \leq g_2(t) < 1 \text{ for each } t \in [0, r). \tag{2.5}$$

We shall show estimates (2.2) and (2.3) using mathematical induction. Using the hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, (2.1), $(\mathcal{C}_2)$ and $(\mathcal{C}_3)$, we get that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0\|x_0 - x^*\| < L_0 r < 1. \tag{2.6}$$

It follows from (2.6) and the Banach lemma on invertible operators [3,6,17,19, 23] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|}. \tag{2.7}$$

Hence y_0 is well defined by the first sub-step of the method S4 for $n = 0$. We can write

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.8}$$

Then, since $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| < r$, we obtain from the estimate $\|F'(x^*)^{-1}F(x)\| \leq M$ and (2.8) that

(2.9)

$$\|F'(x^*)^{-1}F(x_0)\| = \|\int_0^1 F'(x^*)^{-1}F'(x^*+\theta(x_0-x^*))(x_0-x^*)d\theta\| \leq M\|x_0-x^*\|.$$

Using the first sub-step of method S4 for $n=0$,($\mathcal{C}_2$), (2.4),(2.7) and (2.9) we get, in turn, that

$$\begin{aligned}
\|y_0-x^*\| &= \|x_0-x^*-F'(x_0)^{-1}F(x_0)+1/3F'(x_0)^{-1}F(x_0)\|\\
&\leq \|F'(x_0)^{-1}F'(x^*)\|\\
&\times \|\int_0^1 F'(x^*)^{-1}(F'(x^*+\theta(x_0-x^*))-F'(x_0))(x_0-x^*)d\theta\|\\
&+\frac{1}{3}\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\|\\
&\leq \frac{L\|x_0-x^*\|^2}{2(1-L_0\|x_0-x^*\|)}+\frac{M\|x_0-x^*\|}{3(1-L_0\|x_0-x^*\|)}\\
&= g_1(\|x_0-x^*\|)\|x_0-x^*\| < \|x_0-x^*\| < r,
\end{aligned}$$

which shows (2.3) for $n=0$ and $y_0\in U(x^*,r)$. As in (2.7) we have that $F'(y_0)^{-1}\in L(Y,X)$ and

$$\begin{aligned}
\|F'(y_0)^{-1}F'(x^*)\| &\leq \frac{1}{1-L_0\|y_0-x^*\|}\\
&\leq \frac{1}{1-L_0g_1(\|x_0-x^*\|)\|x_0-x^*\|} \quad (2.10)\\
&\leq \frac{1}{1-L_0\|x_0-x^*\|}. \quad (2.11)
\end{aligned}$$

Hence x_1 is well defined by the second sub-step of method $S4$ for $n=0$. Notice that we can write from the second sub-step of the method $S4$ for $n=0$ that

$$x_1-x^*=x_0-x^*-F'(x_0)^{-1}F(x_0)+\frac{1}{2}A(x_0)F'(x_0)^{-1}F(x_0), \quad (2.12)$$

where

$$A(x_0)=3I-\frac{9}{4}F'(y_0)^{-1}F'(x_0)-\frac{3}{4}F'(x_0)^{-1}F'(y_0). \quad (2.13)$$

We need an upper bound on $\|A(x_0)\|$. Notice that $A(x_0)$ can also be written as

$$\begin{aligned}
A(x_0) &= 3/4[3F'(y_0)^{-1}(F'(y_0)-F'(x^*))\\
&+F'(x_0)^{-1}(F'(x_0)-F'(x^*))-3F'(y_0)^{-1}(F'(x_0)-F'(x^*))\\
&-F'(x_0)^{-1}(F'(y_0)-F'(x^*))]. \quad (2.14)
\end{aligned}$$

Then using ($\mathcal{C}_3$), (2.7), (2.11) and (2.14), we obtain, in turn, that:

$$
\begin{aligned}
\|A(x_0)\| &\leq \frac{3}{4}\left[\frac{3L_0\|y_0-x^*\|}{1-L_0\|y_0-x^*\|}+\frac{L_0\|x_0-x^*\|}{1-L_0\|x_0-x^*\|}\right]\\
&+\frac{3}{4}\left[\frac{3L_0\|x_0-x^*\|}{1-L_0\|y_0-x^*\|}+\frac{L_0\|y_0-x^*\|}{1-L_0\|x_0-x^*\|}\right]\\
&=\frac{3L_0}{4}(\|x_0-x^*\|+\|y_0-x^*\|)\\
&\times\left(\frac{3}{1-L_0\|y_0-x^*\|}+\frac{1}{1-L_0\|x_0-x^*\|}\right)\\
&\leq\frac{3L_0}{4}(1+g_1(\|x_0-x^*\|))(\frac{3}{1-L_0\|x_0-x^*\|}\\
&+\frac{1}{1-L_0\|x_0-x^*\|})\|x_0-x^*\|:=\alpha_0.
\end{aligned}\tag{2.15}
$$

Then using ($\mathcal{C}_2$), ($\mathcal{C}_3$), (2.1), (2.5), (2.7), (2.9), (2.10) and (2.15), we get that

$$
\begin{aligned}
\|x_1-x^*\| &\leq \frac{L\|x_0-x^*\|^2}{2(1-L_0\|x_0-x^*\|)}\\
&+\frac{1}{2}\|A(x_0)\|\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\|\\
&\leq\frac{L\|x_0-x^*\|^2}{2(1-L_0\|x_0-x^*\|)}+\frac{1}{2}\alpha_0\frac{M\|x_0-x^*\|}{1-L_0\|x_0-x^*\|}\\
&=g_2(\|x_0-x^*\|)\|x_0-x^*\|\\
&\leq c\|x_0-x^*\|<\|x_0-x^*\|<r,
\end{aligned}
$$

where $c=g_2(\|x_0-x^*\|)\in[0,1)$ which shows (2.3) for $n=0$ and $x_1\in U(x^*,r)$. By simply replacing x_0,y_0,x_1 by x_k,y_k,x_{k+1} in the preceding estimates we arrive at estimates (2.2) and (2.3). Then, from the estimate $\|x_{k+1}-x^*\|\leq c\|x_k-x^*\|<r$, we deduce that $\lim_{k\to\infty}x_k=x^*$ and $x_{k+1}\in U(x^*,r)$. To show the uniqueness part, let $Q=\int_0^1F'(y^*+\theta(x^*-y^*))d\theta$ for some $y^*\in\bar{U}(x^*,T)$ with $F(y^*)=0$. Using ($\mathcal{C}_3$) we get that

$$
\begin{aligned}
\|F'(x^*)^{-1}(Q-F'(x^*))\| &\leq \int_0^1 L_0\|y^*+\theta(x^*-y^*)-x^*\|d\theta\\
&\leq \int_0^1(1-\theta)\|x^*-y^*\|d\theta\leq\frac{L_0}{2}T<1.
\end{aligned}\tag{2.16}
$$

It follows from (2.16) and the Banach Lemma on invertible functions that Q is invertible. Finally, from the identity $0=F(x^*)-F(y^*)=Q(x^*-y^*)$, we deduce that $x^*=y^*$. □

Theorem 2.2. *Suppose that the ($\mathcal{C}$) conditions hold and $\bar{U}(x^*,r)\subset D$, where $r=min\{r_2,r_3\}$, r_2,r_3 are the smallest zeros of functions $h_2(t)=g_2(t)-1$ and $h_3(t)=g_3(t)-1$, respectively on the interval $(0,1/L_0)$,*

$$g_1(t)=\frac{Lt}{2(l-L_0t)},$$

$$g_2(t) = \left(1 + \frac{M}{1 - L_0 t}\right) g_1(t)$$

and

$$g_3(t) = \left(1 + \frac{M}{1 - L_0 g_2(t) t}\right) g_1(t).$$

Then the conclusions of Theorem 2.1 hold as follows, but for method NT4:

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|.$$

Theorem 2.3. *Suppose that the ($\mathcal{C}$) conditions hold and $\bar{U}(x^*, r) \subset D$, where $r = r_3, r_3$ is the smallest zero of the function $h_3(t) = g_3(t) - 1$ on the interval* $(0, 1/L_0)$

$$g_3(t) = \left(1 + \frac{M}{1 - L_0 t}\right) g_2(t),$$

$$g_2(t) = \left(1 + \frac{M}{1 - L_0 t}\right) g_1(t)$$

and

$$g_1(t) = \frac{Lt}{2(1 - L_0 t)}.$$

Then the conclusion of Theorem 2.2 holds, but for method ZCA4.

Notice also that

$$r_3 < r_2 < r_1 = \frac{2}{2L_0 + L}. \tag{2.17}$$

Indeed, we have $g_1(r) = 1$ and if $h_2(t) = g_2(t) - 1$, $h_3(t) = g_3(t) - 1$, then $h_2(0) = h_3(0) = -1 < 0$ and

$$\begin{aligned}
h_2(r_1) &< \left(1 + \frac{M}{1 - L_0 r}\right) g_1(r) - 1 \\
&= \frac{M}{1 - L_0 r_1} > 0, \implies r_2 < r_1, \\
h_3(r_2) &= \left(1 + \frac{M}{1 - L_0 r_2}\right) g_2(r_2) - 1 = \frac{M}{1 - L_0 r_2} > 0,
\end{aligned}$$

$$(\text{since } g_2(r_2) = 1) \implies r_2 < r_1$$

and

$$\begin{aligned}
h_3(r_1) &= \left(1 + \frac{M}{1 - L_0 r_1}\right) g_2(r_1) - 1 \\
&= \left(1 + \frac{M}{1 - L_0 r_1}\right)^2 g_1(r_1) - 1 \\
&= \left(1 + \frac{M}{1 - L_0 r_1}\right)^2 - 1 > 0 \implies r_3 < r_1.
\end{aligned}$$

Let us define functions on the interval $(0, 1/L_0]$ by

$$\begin{aligned}
g_1(t) &= \frac{Lt}{2(1-L_0t)},\\
h_1(t) &= g_1(t) - 1,\\
g_2(t) &= \left(1 + \frac{M}{1-L_0t}\right) g_1(t)\\
h_2(t) &= g_2(t) - 1,\\
g_3(t) &= \left(1 + \frac{M}{1-L_0t}\right)^2 g_1(t)\\
h_3(t) &= g_3(t) - 1, g_4(t) = \left[1 + \frac{M(2+\frac{M}{1-L_0t})}{1-L_0t}\right]\left(1+\frac{M}{1-L_0t}\right) g_1(t)
\end{aligned}$$

and

$$h_4(t) = g_4(t) - 1.$$

Then, as in Theorem 2.3 we have that $r_3 < r_2 < r_1$. Moreover, we have $h_4(0) = -1 < 0$ and since

$$\begin{aligned}
g_4(t) = g_2(t) + g_3(t) + M(1 + \frac{M}{1-L_0t})^2 \frac{g_1(t)}{1-L_0t}\\
+ M\left(1 + \frac{M}{1-L_0t}\right)^2 \left(\frac{M+L_0t-1}{1-L_0t}\right)^2 - 1,
\end{aligned}$$

we get that

$$\begin{aligned}
g_4(r_3) = g_2(r_3) + M(1 + \frac{M}{1-L_0r_3})\frac{g_1(r_3)}{1-L_0r_3}\\
+\left(1+\frac{M}{1-L_0r_3}\right)^2 (\frac{M+L_0r_3-1}{1-L_0r_3}) > 0,
\end{aligned}$$

since $g_3(r_3) = 1$, $g_1(r_3) > 0$, $1 - L_0r_3 > 0$ and $M \geq 1$.

It follows that function h_4 has zeros in the interval $(0, r_3)$. Denote by r_4 the smallest such zero and set

$$r = r_4. \tag{2.18}$$

Then, we can show the following local convergence theorem for method EA5.

Theorem 2.4. *Suppose that the $(\mathcal{C})$ conditions hold and $\bar{U}(x^*, r) \subset D$, where r is given by* (2.18). *Then the conclusions of Theorem 2.1, but for method EA5, hold as follows:*

$$\begin{aligned}
\|y_n - x^*\| &\leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r,\\
\|z_n - x^*\| &\leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|\\
\|w_n - x^*\| &\leq g_3(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|
\end{aligned}$$

and

$$\|x_{n+1}-x^*\| \leq g_4(\|x_n-x^*\|)\|x_n-x^*\| < \|x_n-x^*\|,$$

where the "g'' function are defined above Theorem 2.4.

Notice that we have, in turn, the estimates

$$\begin{aligned}
\|y_n-x^*\| &\leq g_1(\|x_n-x^*\|)\|x_n-x^*\| < \|x_n-x^*\| \\
\|z_n-x^*\| &\leq \|y_n-x^*\| + \frac{M\|y_n-x^*\|}{1-L_0\|x_n-x^*\|} \\
&= (1+\frac{M}{1-L_0\|x_n-x^*\|})\|y_n-x^*\| \\
&\leq g_2(\|x_n-x^*\|)\|x_n-x^*\|, \\
\|w_n-x^*\| &\leq \|z_n-x^*\| + \frac{M\|z_n-x^*\|}{1-L_0\|x_n-x^*\|} \\
&= (1+\frac{M}{1-L_0\|x_n-x^*\|})\|z_n-x^*\| \\
&\leq g_3(\|x_n-x^*\|)\|x_n-x^*\|
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1}-x^*\| &\leq \|z_n-x^*\| + \frac{M(\|z_n-x^*\|+\|w_n-x^*\|)}{1-L_0\|x_n-x^*\|} \\
&\leq \|z_n-x^*\| + \frac{M(\|z_n-x^*\|+(1+\frac{M}{1-L_0\|\|z_n-x^*\|})\|z_n-x^*\|)}{1-L_0\|x_n-x^*} \\
&\leq \left[1+\frac{M(2+\frac{M}{1-L_0\|x_n-x^*})}{1-L_0\|x_n-x^*\|}\right] \\
&\times \left(1+\frac{M}{1-L_0\|x_n-x^*\|}\right) g_1(\|x_n-x^*\|) \\
&= g_4(\|x_n-x^*\|)\|x_n-x^*\| < \|x_n-x^*\|.
\end{aligned}$$

Let us define functions on the interval $(0, 1/L_0)$ by

$$\begin{aligned}
g_1(t) &= \frac{Lt}{2(1-L_0t)}, \\
h_1(t) &= g_1(t)-1, \\
g_2(t) &= (1+\frac{M}{1-L_0t})g_1(t), \\
h_2(t) &= g_2(t)-1, \\
g_3(t) &= (1+\frac{M}{1-L_0g_1(t)t})g_2(t),
\end{aligned}$$

and

$$h_3(t) = g_3(t)-1.$$

Then we have that $r_3 < r_2 < r_1$. Indeed, $h_2(0) = -1 < 0$ and $h_2(r_1) = (1+\frac{M}{1-L_0r_1})g_1(r_1) - 1 = \frac{M}{1-L_0r_1} > 0 \implies r_2 < r_1$ $h_3(0) = -1 < 0$,

$$\begin{aligned} h_3(r_1) &= (1+\frac{M}{1-L_0g_1(r_1)r_1})g_2(r_1) - 1 \\ &= (1+\frac{M}{1-L_0r_1})^2 - 1 > 0 \implies r_3 < r_1 \end{aligned}$$

and

$$h_3(r_2) = \frac{M}{1-L_0g_1(r_2)r_2} > 0,$$

since

$$\begin{aligned} L_0g_1(r_2)r_2 < L_0g_1(r_1)r_1 &\implies 1 - L_0g_1(r_2)r_2 \\ &> 1 - L_0g_1(r_1)r_1 = 1 - L_0r_10. \end{aligned}$$

Set

$$r = r_3 \tag{2.19}$$

Theorem 2.5. *Suppose that the ($\mathcal{C}$) conditions hold and $\bar{U}(x^*, r) \subset D$, where r is given by* (2.19). *Then the conclusion of Theorem 2.2, but for the method NT5, holds, provided that the definition of r and "g'' functions as given above Theorem 2.5 are used.*

Let us define the functions over the interval $[0, 1/L_0)$ by

$$\begin{aligned} g_1(t) &= \frac{Lt}{2(1-L_0t)}, \\ h_1(t) &= g_1(t) - 1, \\ g_2(t) &= \frac{Lg_1(t)^2t}{2(1-L_0g_1(t)t)}, \\ h_2(t) &= g_2(t) - 1, \\ g_3(t) &= (1+\frac{M}{1-L_0g_1(t)t})\frac{Lg_1(t)^2t}{2(1-L_0g_1(t)t)}, \end{aligned}$$

and

$$h_3(t) = g_3(t) - 1.$$

Then we have that $r_3 < r_2 = r_1$. Indeed, we get that

$$\begin{aligned} h_2(r_1) &= g_2(r_1) - 1 \\ &= \frac{Lg_1(r_1)^2r_1}{2(1-L_0g_1(r_1)r_1)} \\ &= \frac{Lr_1}{2(1-L_0r_1)} - 1 = 0 \implies r_2 = r_1, \end{aligned}$$

$$\begin{aligned}
h_3(0) &= -1 < 0,\\
h_3(r_2) &= h_3(r_1)\\
&= (1 + \frac{M}{1 - L_0 g_1(r_1) r_1}) \frac{L g_1^2(r_1) r_1}{2(1 - L_0 g_1(r_1) r_1)}\\
&= (1 + \frac{M}{1 - L_0 r_1}) \frac{L r_1}{2(1 -_0 r_1)} - 1\\
&= 1 + \frac{M}{1 - L_0 r_1} - 1 = \frac{M}{1 - L_0 r_1} > 0,\\
&\Longrightarrow r_3 < r_2 = r_1.
\end{aligned}$$

Set

$$r = r_3. \tag{2.20}$$

Theorem 2.6. *Suppose that the ($\mathcal{C}$) conditions hold and $\bar{U}(x^*, r) \subset D$, where r is given by* (2.20) *and the "g" functions are as defined above Theorem 2.6. Then the conditions of Theorem 2.5, but for method ZCA5, hold.*

Remark 2.7.

1. In view of the estimate

$$\begin{aligned}
\|F'(x^*)^{-1} F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\|\\
&\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\|\\
&\leq 1 + L_0 \|x - x^*\|,
\end{aligned}$$

   we can set

$$M(t) = 1 + L_0 t$$

   or $M = M(t) = 2$, since $t \in [0, \frac{1}{L_0})$.

2. The results obtained here can be used for operators F satisfying autonomous differential equations [3, 6, 17] of the form

$$F'(x) = G(F(x))$$

   where T is a continuous operator. Then, since $F'(x^*) = G(F(x^*)) = G(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then we can choose: $G(x) = x + 1$.

3. The local results obtained here can be used for projection methods such as the Arnoldi method, the generalized minimum residual method (GMRES), the generalized conjugate method(GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [2, 5].

4. The parameter $r_A = \frac{2}{2L_0+L}$ was shown by us to be the convergence radius of Newton's method [3,6]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{2.21}$$

under the conditions $(\mathcal{C}_1)$–$(\mathcal{C}_3)$. It follows from the definitions of radii r that the convergence radius r of these preceding methods cannot be larger than the convergence radius r_A of the second order Newton's method (2.21). As already noted in [3,6] r_A is at least as large as the convergence ball given by Rheinboldt [24]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have that

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \quad as \quad \frac{L_0}{L} \to 0.$$

That is, our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [19].

5. It is worth noticing that the studied methods are not changing when we use the conditions of the preceding Theorems instead of the stronger conditions used in [1,2,8–28]. Moreover, the preceding Theorems we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence.

3. Numerical examples

We present a numerical example in this section.

EXAMPLE 3.1. Let $X = Y = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then the Fréchet-derivative is given by

$$F'(w) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the $(\mathcal{C})$ conditions, we get $L_0 = e - 1, L = e, M = 2$. The parameters are given in Table 1.

Table 1: Parameters of methods S4, NT4, ZCA4, NT5 EA5 and ZCA5

parameters/ Methods	r	ξ
S4	0.0875	3.631507
NT4	0.1569	3.3555374
ZCA4	0.0625	3.639839
NT5	0.0662	2.727489
EA5	0.0277	3.101458
ZCA5	0.0662	1.981468

Table 1

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