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ON A TWO-VARIABLES FRACTIONAL PARTIAL DIFFERENTIAL INCLUSION VIA RIEMANN-LIOUVILLE DERIVATIVE

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Abstract. We investigate the existence a solution to a two-variable fractional partial differential inclusion via Riemann-Liouville derivative. Also, we provide an example to illustrate our main result.

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1. Introduction

There are many published works about fractional partial differential equations by using the notions of delay or time-fractional (see for example, [1], [2], [9] and [10]). It is interesting to work on two variables fractional partial differential equations (see, for example, [3], [4], [6] and [11]).

Let $\theta = (0,0)$ and $\alpha = (\alpha_1, \alpha_2)$ where $0 < \alpha_1, \alpha_2 \leq 1$. Also, put $J_a \times J_b = [0,a] \times [0,b]$ where a and b are positive constants. The Riemann-Liouville fractional partial integral of $u \in L^1(J_a \times J_b)$ is defined by

$$(I_\theta^\alpha u)(x,y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1 - 1}(y-t)^{\alpha_2 - 1} u(s,t)dtds$$

whenever the integral exists (see, for example, [14], [15] and [16]). The Riemann-Liouville partial derivative of fractional order α for a function $u \in L^1(J_a \times J_b)$ is defined by

$$(D_\theta^\alpha u)(x,y) = D_{xy}^2 (I_\theta^{1-\alpha} u)(x,y) = \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y \frac{(x-s)^{-\alpha_1}(y-t)^{-\alpha_2}}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)} dtds$$

(see for more details [14], [15] and [16]). Note that

$$(I_\theta^\alpha I_\theta^\beta u)(x,y) = (I_\theta^{\alpha+\beta} u)(x,y)$$

whenever $\beta = (\beta_1, \beta_2) > 0$ ([16]).

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Let (X, d) be a metric space, $\mathcal{P}(X)$ the class of all nonempty subsets of X, $\mathcal{P}_{cl}(X)$ the class of all closed subsets of X, $\mathcal{P}_{bd}(X)$ the class of all bounded subsets of X, $\mathcal{P}_{cp}(X)$ the class of all compact subsets of X and $\mathcal{P}_{cv}(X)$ the class of all convex subsets of X. A multi-valued map $F : J_a \times J_b \to \mathcal{P}_{cl}(\mathbb{R})$ is measurable whenever the function $(x, y) \mapsto d(w, F(x, y)) = \inf\{\|w - v\| : v \in F(x, y)\}$ is measurable for all $w \in \mathbb{R}$, where $J_a \times J_b = [0, a] \times [0, b]$ ([12]). Also, the Pompeiu-Hausdorff metric $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, \infty)$ is defined by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where $d(A, b) = \inf_{a \in A} d(a, b)$ ([7]). Then $(\mathcal{P}_{cl,bd}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space ([7]). Recall that a multifunction $F : X \to \mathcal{P}(X)$ is said to be a contraction if there exists $k \in (0, 1)$ such that $H_d(F(u), F(v)) \leq kd(u, v)$ for all $u, v \in X$ ([13]). An element $u \in X$ is called endpoint of the multifunction $F : X \to \mathcal{P}(X)$ whenever $Fu = \{u\}$ ([8]). We say that the multifunction F has an approximate endpoint property whenever $\inf_{u \in X} \sup_{w \in Fu} d(u, w) = 0$ ([8]). A real-valued function f on $\mathbb{R}$ is called upper semi-continuous whenever $\limsup_{n\to\infty} f(\lambda_n) \leq f(\lambda)$ for all sequence $\{\lambda_n\}_{n\geq 1}$ with $\lambda_n \to \lambda$. In this paper, using the main idea of [5], [6] and [17], we investigate the existence of solutions for the two-variables fractional partial differential inclusion

$$(D_\theta^\alpha u)(x, y) \in F(x, y, u(x, y)), \tag{1.1}$$

with the partial integral boundary value conditions

$$(I_\theta^{1-\alpha} u)(x, 0) = \lambda_1 \varphi(x), \ (I_\theta^{1-\alpha} u)(0, y) = \lambda_2 \gamma(y), \tag{1.2}$$

where D_θ^α denotes the Riemann-Liouville fractional partial derivative of order α, $(x, y) \in J_a \times J_b$, $0 < \alpha_i \leq 1$, $\lambda_i \in \mathbb{R}^+$ $(i = 1, 2)$ and $F : J_a \times J_b \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a compact valued multi-valued map. Here, the functions $\varphi : J_a \to \mathbb{R}$ and $\gamma : J_b \to \mathbb{R}$ are absolutely continuous with $\varphi(0) = \gamma(0) = 0$. We need the following endpoint result.

Theorem 1.1. *([8]) Suppose that (X, d) is a complete metric space, $\psi : [0, \infty) \to [0, \infty)$ is an upper semi-continuous function such that $\psi(t) < t$ and $\liminf_{t\to\infty}(t - \psi(t)) > 0$ for all $t > 0$ and $T : X \to CB(X)$ is a multifunction such that $H_d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$. Then T has a unique endpoint if and only if T has approximate endpoint property.*

2. Main results

Now we are ready to state and prove our main results. First, we give the following key result.

Lemma 2.1. *Let $f \in L(J_a \times J_b)$ and $\alpha = (\alpha_1, \alpha_2) \in (0, 1] \times (0, 1]$. Then the continuous function $u_0 \in L(J_a \times J_b)$ is a solution for the fractional partial differential equation*

$$D_\theta^\alpha u(x, y) = f(x, y) \tag{2.1}$$

with boundary conditions $(I_\theta^{1-\alpha}u)(x,0)=\lambda_1\varphi(x)$ *and* $(I_\theta^{1-\alpha}u)(0,y)=\lambda_2\gamma(y)$ *if and only if* u_0 *is a solution for the fractional integral equation*

$$(2.2)\; u(x,y)=\frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_\theta^{\alpha_2}\dot\gamma)(y)+\frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_\theta^{\alpha_1}\dot\varphi)(x)+(I_\theta^{\alpha}f)(x,y).$$

Proof. Let u_0 be a solution for the fractional partial differential equation (2.1). Then, we have $D^2_{xy}(I_\theta^{1-\alpha}u_0)(x,y)=f(x,y)$ and so

$$(I_\theta^{1-\alpha}u_0)(x,y)-(I_\theta^{1-\alpha}u_0)(x,0)-(I_\theta^{1-\alpha}u_0)(0,y)+(I_\theta^{1-\alpha}u_0)(0,0)=(I_\theta^{1}f)(x,y).$$

Using the boundary conditions we get

$$(I_\theta^{1-\alpha}u_0)(x,y)-\lambda_1\varphi(x)-\lambda_2\gamma(y)+\lambda_1\varphi(0)=(I_\theta^{1}f)(x,y)$$

and so $(I_\theta^{1-\alpha}u_0)(x,y)-(I_\theta^{1}f)(x,y)=\lambda_1\varphi(x)+\lambda_2\gamma(y)$. Since $(I_\theta^{0}u_0)(x,y)=u_0(x,y)$, we obtain $\Big(I_\theta^{1-\alpha}(u_0(x,y)-(I_\theta^{\alpha}f)(x,y))\Big)(x,y)=\lambda_1\varphi(x)+\lambda_2\gamma(y)$. On the other hand, we have

$$I_\theta^{\alpha}\Big[I_\theta^{1-\alpha}(u_0(x,y)-(I_\theta^{\alpha}f)(x,y))(x,y)\Big](x,y)=I_\theta^{\alpha}\Big(\lambda_1\varphi(x)+\lambda_2\gamma(y)\Big)$$

and so

$$I_\theta^{1}\Big(u_0(x,y)-(I_\theta^{\alpha}f)(x,y)\Big)(x,y)=I_\theta^{\alpha}\Big(\lambda_1\varphi(x)+\lambda_2\gamma(y)\Big)=(I_\theta^{\alpha}p)(x,y). \quad (2.3)$$

But we have

$$\begin{aligned}(I_\theta^{\alpha}p)(x,y) &= \frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}\varphi(s)dtds\\ &+ \frac{\lambda_2}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}\gamma(t)dtds.\end{aligned}$$

Since $\varphi(x)=\int_0^x\dot\varphi(s)ds$ and $\gamma(y)=\int_0^y\dot\gamma(t)dt$, we obtain

$$\begin{aligned}(I_\theta^{\alpha}p)(x,y) &= \frac{\lambda_1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}\Big(\int_0^s\dot\varphi(\tau)d\tau\Big)dtds\\ &+ \frac{\lambda_2}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}\Big(\int_0^t\dot\gamma(\tau)d\tau\Big)dtds\\ &= \frac{\lambda_1 y^{\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2+1)}\int_0^x (x-s)^{\alpha_1-1}(\int_0^s\dot\varphi(\tau)d\tau)ds\\ &+ \frac{\lambda_2 x^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2)}\int_0^y (y-t)^{\alpha_2-1}(\int_0^t\dot\gamma(\tau)d\tau)dt\\ &= \frac{\lambda_1 y^{\alpha_2}}{\Gamma(\alpha_2+1)}\int_0^x\Big(\frac{1}{\Gamma(\alpha_1)}\int_0^s (s-\tau)^{\alpha_1-1}\dot\varphi(\tau)d\tau\Big)ds\\ &+ \frac{\lambda_2 x^{\alpha_1}}{\Gamma(\alpha_1+1)}\int_0^y\Big(\frac{1}{\Gamma(\alpha_2)}\int_0^t (t-\tau)^{\alpha_2-1}\dot\gamma(\tau)d\tau\Big)dt\\ &= \frac{\lambda_1 y^{\alpha_2}}{\Gamma(\alpha_2+1)}\int_0^x (I_\theta^{\alpha_1}\dot\varphi)(s)ds+\frac{\lambda_2 x^{\alpha_1}}{\Gamma(\alpha_1+1)}\int_0^y (I_\theta^{\alpha_2}\dot\gamma)(t)dt.\end{aligned}$$

Since $(I_\theta^{\alpha_1}\dot{\varphi})(x) \in L(J_a)$ and $(I_\theta^{\alpha_2}\dot{\gamma})(y) \in L(J_b)$, the functions $\int_0^x (I_\theta^{\alpha_1}\dot{\varphi})(s)ds$ and $\int_0^y (I_\theta^{\alpha_2}\dot{\gamma})(t)dt$ are absolutely continuous and so there exists $D_{xy}^2(I_\theta^\alpha p)(x,y)$ for almost all $(x,y) \in J_a \times J_b$. By applying the operator D_{xy}^2 on both sides of (2.3), we get $D_{xy}^2\Big[I_\theta^1\Big(u_0(x,y) - (I_\theta^\alpha f)(x,y)\Big)(x,y)\Big] = D_{xy}^2\Big[(I_\theta^\alpha p)(x,y)\Big]$. Thus,

$$u_0(x,y) - (I_\theta^\alpha f)(x,y)$$

$$= D_{xy}^2\Big[\frac{\lambda_1 y^{\alpha_2}}{\Gamma(\alpha_2+1)}\int_0^x (I_\theta^{\alpha_1}\dot{\varphi})(s)ds + \frac{\lambda_2 x^{\alpha_1}}{\Gamma(\alpha_1+1)}\int_0^y (I_\theta^{\alpha_2}\dot{\gamma})(t)dt\Big]$$

$$= D_x\Big[\frac{\lambda_1 \alpha_2 y^{\alpha_2-1}}{\Gamma(\alpha_2+1)}\int_0^x (I_\theta^{\alpha_1}\dot{\varphi})(s)ds + \frac{\lambda_2 x^{\alpha_1}}{\Gamma(\alpha_1+1)}(I_\theta^{\alpha_2}\dot{\gamma})(y)\Big]$$

$$= \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_\theta^{\alpha_1}\dot{\varphi})(x) + \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_\theta^{\alpha_2}\dot{\gamma})(y).$$

Hence, $u_0(x,y) = \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_\theta^{\alpha_1}\dot{\varphi})(x) + \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_\theta^{\alpha_2}\dot{\gamma})(y) + (I_\theta^\alpha f)(x,y)$. This shows that u_0 is a solution of the fractional integral equation (2.2). Now, let u_0 be a solution for the fractional integral equation (2.2). Then, $(I_\theta^{1-\alpha}u_0)(x,y) = I_\theta^{1-\alpha}\Big[\frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_\theta^{\alpha_1}\dot{\varphi})(x) + \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_\theta^{\alpha_2}\dot{\psi})(y)\Big](x,y) + (I_\theta^1 f)(x,y)$. On the other hand by using $\mathcal{B}(z,w) = \int_0^1 (1-x)^{w-1}x^{z-1}dx = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$, we get

$$I_\theta^{1-\alpha}\Big[\frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_\theta^{\alpha_2}\dot{\gamma})(y)\Big](x,y)$$

$$= \lambda_2\int_0^x\int_0^y \frac{(x-s)^{-\alpha_1}(y-t)^{-\alpha_2}}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)}\Big(\frac{s^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_\theta^{\alpha_2}\dot{\gamma})(t)\Big)dtds$$

$$= \frac{\lambda_2}{\Gamma(\alpha_1)\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)}\int_0^x\int_0^y (x-s)^{-\alpha_1}s^{\alpha_1-1}(y-t)^{-\alpha_2}(I_\theta^{\alpha_2}\dot{\gamma})(t)dtds$$

$$= \frac{\lambda_2}{\Gamma(\alpha_1)\Gamma(1-\alpha_1)}\int_0^x (x-s)^{-\alpha_1}s^{\alpha_1-1}ds\Big(\frac{1}{\Gamma(1-\alpha_2)}\int_0^y (y-t)^{-\alpha_2}(I_\theta^{\alpha_2}\dot{\gamma})(t)dt\Big)$$

$$= \frac{\lambda_2\mathcal{B}(\alpha_1, 1-\alpha_1)}{\Gamma(\alpha_1)\Gamma(1-\alpha_1)}\Big(I_\theta^{1-\alpha_2}(I_\theta^{\alpha_2}\dot{\gamma})(t)\Big)(y)$$

$$= \frac{\lambda_2}{\Gamma(1)}(I_\theta^1\dot{\gamma})(y) = \lambda_2(\gamma(y) - \gamma(0)) = \lambda_2\gamma(y)$$

and similarly $I_\theta^{1-\alpha}\Big[\frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_\theta^{\alpha_1}\dot{\varphi})(x)\Big](x,y) = \lambda_1\varphi(x)$. Thus,

$$(I_\theta^{1-\alpha}u_0)(x,y) = \lambda_2\gamma(y) + \lambda_1\varphi(x) + (I_\theta^1 f)(x,y). \tag{2.3}$$

By applying the operator D^2_{xy} on both sides of (2.3), we obtain

$$D^2_{xy}\Big[(I^{1-\alpha}_{\theta}u_0)(x,y)\Big] = D^2_{xy}\Big[\lambda_2\gamma(y) + \lambda_1\varphi(x) + (I^1_{\theta}f)(x,y)\Big]$$

and so $(D^{\alpha}_{\theta}u_0)(x,y) = f(x,y)$. By using (2.3), we get

$$(I^{1-\alpha}_{\theta}u_0)(x,0) = \lambda_2\gamma(0) + \lambda_1\varphi(x) + (I^1_{\theta}f)(x,0) = \lambda_1\varphi(x)$$

and $(I^{1-\alpha}_{\theta}u_0)(0,y) = \lambda_2\gamma(y) + \lambda_1\varphi(0) + (I^1_{\theta}f)(0,y) = \lambda_2\gamma(y)$. This completes the proof. □

Consider the Banach space $X = C(J_a \times J_b, \mathbb{R})$ endowed with the norm $\|u\| = \sup_{(x,y)\in J_a\times J_b}|u(t)|$. For $u \in X$, define the set of selections of F by

$$S_{F,u} := \{v \in L^1(J_a\times J_b,\mathbb{R}) :\ v(x,y) \in F(x,y,u(x,y)) \text{ for almost all } (x,y) \in J_a\times J_b\}.$$

It has been proved that $S_{F,u} \neq \emptyset$ for all $u \in C(J_a \times J_b, X)$ ([11]). We say that $u \in X$ is a solution for the boundary value problem (1.1)-(1.2) whenever it satisfies the boundary value conditions (1.2) and also there is a function $v \in L^1(J_a \times J_b, \mathbb{R})$ such that $v(x,y) \in F(x,y,u(x,y))$ for all $(x,y) \in J_a \times J_b$ and

$$u(x,y) = \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I^{\alpha_2}_{\theta}\dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I^{\alpha_1}_{\theta}\dot{\varphi})(x)$$
$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v(s,t)dtds$$

for almost all $(x,y) \in J_a \times J_b$. Define the multifunction $\mathcal{N} : X \to \mathcal{P}(X)$ by

$$\mathcal{N}(u) = \{h \in X : h(x,y)\frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I^{\alpha_2}_{\theta}\dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I^{\alpha_1}_{\theta}\dot{\varphi})(x)$$

$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v(s,t)dtds\} \text{ for all } (x,y) \in J_a\times J_b\}.$$

Here, we provide our main result.

Theorem 2.2. *Suppose that $\psi : [0,\infty) \to [0,\infty)$ is a nondecreasing upper semi-continuous map such that $\liminf_{t\to\infty}(t-\psi(t)) > 0$ and $\psi(t) < t$ for all $t > 0$, $F : J_a \times J_b \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is an integrable bounded multifunction such that $F(\cdot,\cdot,u) : J_a \times J_b \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for all $u \in \mathbb{R}$. Assume that there exits $m \in C(J_a \times J_b, [0,\infty))$ such that*

$$H_d(F(x,y,u) - F(x,y,u')) \leq \frac{1}{\Lambda_1}m(x,y)\psi(|u-u'|)$$

for all $(x,y) \in J_a\times J_b$ and $u,u' \in \mathbb{R}$, where $\Lambda_1 = \|m\|\Big\{\frac{a^{\alpha_1}b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\Big\}$. If the multifunction $\mathcal{N}$ has the approximate endpoint property, then the fractional partial differential inclusion problem (1.1)-(1.2) has a solution.

Proof. First, we prove that the multifunction $\mathcal{N}$ has at least one endpoint. Let $u \in X$. Since the multivalued map $(x, y) \mapsto F(x, y, u(x, y))$ is measurable and is closed-value, it has measurable selection and so $S_{F,u}$ is nonempty. Let $\{p_n\}_{n\geq 1}$ be a sequence in $\mathcal{N}(u)$ with $p_n \to p$. For each n, choose $v_n \in S_{F,u_n}$ such that

$$p_n(x,y) = \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_\theta^{\alpha_2}\dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_\theta^{\alpha_1}\dot{\varphi})(x)$$
$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v_n(s,t)dtds$$

for all $(x, y) \in J_a \times J_b$. Since the operator F is compact, the sequence $\{v_n\}_{n\geq 1}$ has a subsequence converging to some $v \in L^1(J_a \times J_b)$. We denote this subsequence again by $\{v_n\}_{n\geq 1}$. It is easy to see that $v \in S_{F,u}$ and

$$p(x,y) = \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_\theta^{\alpha_2}\dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_\theta^{\alpha_1}\dot{\varphi})(x)$$
$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v(s,t)dtds$$

for all $(x, y) \in J_a \times J_b$. This shows that $p \in \mathcal{N}(u)$ and so $\mathcal{N}(u)$ is closed. Note that, $\mathcal{N}(u)$ is bounded because F has a compact values. Now, we show that $H_d(\mathcal{N}(u), \mathcal{N}(w)) \leq \psi(\|u - w\|)$ for all $u, w \in X$. Let $u, w \in X$ and $h_1 \in \mathcal{N}(w)$. Choose $v_1 \in S_{F,w}$ such that

$$h_1(x,y) = \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_\theta^{\alpha_2}\dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_\theta^{\alpha_1}\dot{\varphi})(x)$$
$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v_1(s,t)dtds$$

for almost all $(x, y) \in J_a \times J_b$. By using the hypothesis, we have

$$H_d(F(x,y,u(x,y)) - F(x,y,w(x,y))) \leq \frac{1}{\Lambda_1}m(x,y)\psi(|u(x,y) - w(x,y)|)$$

and so we can choose $z \in F(x, y, u(x, y))$ such that

$$|v_1(x,y) - z| \leq \frac{1}{\Lambda_1}m(x,y)\psi(|u(x,y) - w(x,y)|).$$

Define the multivalued map $U : J_a \times J_b \to \mathcal{P}(\mathbb{R})$ by

$$U(x,y) = \{z \in \mathbb{R} : \ |v_1(x,y) - z| \leq \frac{1}{\Lambda_1}m(x,y)\psi(|u(x,y) - w(x,y)|).$$

Since v_1 and $\eta = \dfrac{1}{\Lambda_1}m\psi\Big(|u - w|\Big)$ are measurable, the multifunction $U(\cdot, \cdot) \cap F(\cdot, \cdot, u(\cdot, \cdot))$ is measurable. Hence, there exists $v_2(x, y) \in F(x, y, u(x, y))$ such that $|v_1(x,y)-v_2(x,y)| \leq \dfrac{1}{\Lambda_1} m(x,y)\psi(|u(x,y)-w(x,y)|)$. Now, consider the element $h_2 \in \mathcal{N}(u)$ defined by

$$h_2(x,y) = \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_\theta^{\alpha_2}\dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_\theta^{\alpha_1}\dot{\varphi})(x)$$

$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v_2(s,t)dtds$$

for all $(x,y) \in J_a \times J_b$. Put $\sup_{(x,y)\in J_a\times J_b}|m(x,y)| = \|m\|$. Then, we have

$$\begin{aligned}
|h_1(x,y)-h_2(x,y)| \leq \Big| & \frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_\theta^{\alpha_2}\dot{\gamma})(y) + \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_\theta^{\alpha_1}\dot{\varphi})(x) \\
&+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v_1(s,t)dtds \\
&-\frac{\lambda_2 x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_\theta^{\alpha_2}\dot{\gamma})(y) - \frac{\lambda_1 y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_\theta^{\alpha_1}\dot{\varphi})(x) \\
&-\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v_2(s,t)dtds\Big| \\
\leq & \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}|v_1(s,t)-v_2(s,t)|dtds \\
\leq & \frac{1}{\Lambda_1}\|m\|\psi(\|u-w\|)\Big\{\frac{x^{\alpha_1}y^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\Big\} \\
= & \frac{\Lambda_1}{\Lambda_1}\psi(\|u-w\|) = \psi(\|u-w\|)
\end{aligned}$$

and so $\|h_1-h_2\| = \sup_{(x,y)\in J_a\times J_b}|h_1(x,y)-h_2(x,y)| = \psi(\|u-w\|)$. Hence, $H_d(\mathcal{N}(u),\mathcal{N}(w)) \leq \psi(\|u-w\|)$ for all $u,w \in X$. Since the multifunction $\mathcal{N}$ has approximate endpoint property according to Theorem 1.1, there exists $u^* \in X$ such that $\mathcal{N}(u^*) = \{u^*\}$. It is easy to check that u^* is a solution for the fractional partial differential inclusion problem (1.1)-(1.2). □

For illustration of our main result, we give the following example.

Example 2.3. Consider the fractional partial differential inclusion

$$D_\theta^\alpha u(x,y) \in \left[0, \frac{0.03xy|\sin u(x,y)|}{1+|\sin u(x,y)|}\right]$$

with boundary value conditions $(I_\theta^{1-\alpha}u)(x,0) = 0.1(e^x-1)$ and $(I_\theta^{1-\alpha}u)(0,y) = 0.01y^2$, where $(x,y) \in [0,1]\times[0,1]$. Let $\alpha = (\alpha_1,\alpha_2)$ with $\alpha_1,\alpha_2 \in (0,1]$, $\lambda_1 = 0.1$ and $\lambda_2 = 0.01$. Define the multifunction $F : [0,1]\times[0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ by $F(x,y,z) = \left[0, \dfrac{0.03xy|\sin z(t)|}{1+|\sin z(t)|}\right]$. If $m : [0,1]\times[0,1] \to [0,\infty)$ is defined by

$m(x,y) = \frac{3}{100}xy$, then $\|m\| = \frac{3}{100}$. Consider the map $\psi(t) = \frac{t}{2}$. It is clear that ψ is nondecreasing, upper semi-continuous on $(0,1]$, $\liminf_{t\to\infty}(t-\psi(t)) > 0$ and $\psi(t) < t$ for all $t > 0$. Since $\Gamma(\alpha_i + 1) < \frac{1}{2}$, we get

$$\Lambda_1 = \|m\|\Big\{\frac{a^{\alpha_1}b^{\alpha_2}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\Big\} = \frac{3}{100}\Big(\frac{1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\Big) < 0.12.$$

One can easily check that

$$H_d(F(x,y,u_1) - F(x,y,u_2)) \leq \frac{1}{\Lambda_1}m(x,y)\psi\Big(|u_1 - u_2|\Big).$$

Put $X = C_{\mathbb{R}}([0,1]\times[0,1])$. Define $\mathcal{N} : X \to \mathcal{P}(X)$ by

$$\mathcal{N}(u) = \{h \in X : \text{ there exists } v \in S_{F,u} \text{ such that}$$

$$h(x,y) = w(x,y) \text{ for all } (x,y) \in [0,1]\times[0,1]\},$$

where

$$w(x,y) = \frac{0.01x^{\alpha_1-1}}{\Gamma(\alpha_1)}(I_\theta^{\alpha_2}\dot{\gamma})(y) + \frac{0.1y^{\alpha_2-1}}{\Gamma(\alpha_2)}(I_\theta^{\alpha_1}\dot{\varphi})(x)$$
$$+\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v(s,t)dtds.$$

Since $\sup_{u\in\mathcal{N}(0)}\|u\| = 0$, $\inf_{u\in X}\sup_{s\in\mathcal{N}(u)}\|u-s\| = 0$ and so $\mathcal{N}$ has the approximate endpoint property. Now by using Theorem 2.2, we conclude that the above fractional partial differential inclusion problem has a solution.

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