Novi Sad J. Math.
Vol. 46, No. 2, 2016, 163-180

GENERALIZED FUNCTIONS ON THE CLOSURE OF AN OPEN SET. APPLICATION TO UNIQUENESS OF SOME CHARACTERISTIC CAUCHY PROBLEM

Victor Dévoué[1], Jean-André Marti[2], Hans Vernaeve[3] and Jasson Vindas[4]

Abstract. To obtain existence and uniqueness when solving some nonlinear characteristic Cauchy problems, we define a special algebra $\mathcal{G}_{\mathcal{O}_M}\left(\overline{\Omega}\right)$ of generalized functions on the closure $\overline{\Omega}$ of an open set Ω in $\mathbb{R}^n$ constructed from the topological algebra $\mathcal{O}_M\left(\overline{\Omega}\right)$ of slowly increasing functions in $\overline{\Omega}$. Moreover other concepts are needed as slow scale elements and point values characterization of elements in $\mathcal{G}_{\mathcal{O}_M}\left(\overline{\Omega}\right)$.

AMS Mathematics Subject Classification (2010): 35A01; 35A02; 35D99; 35A25; 46F30; 46T30

Key words and phrases: tempered algebras of generalized functions; closure of open set; regularizing process; point values characterizations; nonlinear characteristic problems

1. Introduction

In many problems (as differential Cauchy ones with $f \in \mathrm{C}^1\left[0,+\infty\right[$ as initial data), we have to define some spaces or algebras on the closure of an open set Ω of $\mathbb{R}^n$.

In other cases the asymptotic analysis of a family of functions (as $e^{-\frac{x}{\varepsilon}}$) depending on a parameter (as ε) need the study in an algebra defined on the closure of an open set (as $\left[0,+\infty\right[$). However, the usual generalized functions (distributions, Colombeau-type algebras..) are a priori constructed on open set Ω in $\mathbb{R}^n$ for reasons principally linked to their sheaf structure (restriction operator, support, all ordered derivatives...). The starting point of our constructions is the algebra of smooth functions and we come back to the technics of continuous extension of such functions and their derivatives on the boundary of a closed subset of $\mathbb{R}^n$, following the definitions given in [3] and [4].

The space $\mathcal{O}_M\left(\mathbb{R}^n\right)$ of slowly increasing functions, endowed by the family of semi-norms $\left(p_{\varphi,\alpha}\right)_{(\varphi,\alpha)\in\mathcal{S}(\mathbb{R}^n)\times\mathbb{N}^n}$, becomes a topological algebra used in [5]

[1]Laboratoire CEREGMIA, Université des Antilles, Campus de Shoelcher, BP 7209, 97275 Schoelcher Cedex, Martinique, e-mail: devoue-vi@orange.fr

[2]Laboratoire CEREGMIA, Université des Antilles, Campus de Shoelcher, BP 7209, 97275 Schoelcher Cedex, Martinique, e-mail: jean.andre.marti@gmail.com

[3] Department of Mathematics, Ghent University, Building S22, Krijgslaan 281, B 9000 Gent, Belgium, e-mail: hvernaev@cage.UGent.be

[4]Department of Mathematics, Ghent University, Building S22, Krijgslaan 281, B 9000 Gent, Belgium, e-mail: jvindas@cage.Ugent.be

to define the generalized algebra $\mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^n)$ (which differs from $\mathcal{G}_\tau(\mathbb{R}^n)$). It is very useful to prove the uniqueness of some linear characteristic Cauchy problem studied in [2].

But in nonlinear cases, we cannot obtain the result without replacing $\mathbb{R}^n$ by a smaller closed set. When Ω is a convex open set in $\mathbb{R}^n$, we prove that $\mathcal{O}_M(\overline{\Omega})$, with the topology deducted from that of $\mathcal{O}_M(\mathbb{R}^n)$ by replacing $\mathcal{S}(\mathbb{R}^n)$ by $\mathcal{S}(\overline{\Omega})$, becomes also a locally convex algebra. Now, we define the generalized algebra $\mathcal{G}_{\mathcal{O}_M}(\overline{\Omega})$ as the quotient algebra $\mathcal{M}_{\mathcal{O}_M}(\overline{\Omega})/\mathcal{N}_{\mathcal{O}_M}(\overline{\Omega})$. When Ω is unbounded, it is given an alternative representation of $\mathcal{N}_{\mathcal{O}_M}(\overline{\Omega})$ leading to a point-value characterization ([8], [6]) of elements in $\mathcal{G}_{\mathcal{O}_M}(\overline{\Omega})$. There is the toolbox to obtain the uniqueness for nonlinear characteristic Cauchy problem involved above.

2. First extension of classical spaces

Definition 2.1. Following the C^∞-extension defined for example by H. Biagioni [4] for the closure $\overline{\Omega}$ of an open set Ω in $\mathbb{R}^n$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

$$\mathrm{C}^\infty(\overline{\Omega}) = \{f : \overline{\Omega} \to \mathbb{K} : f|_\Omega \in \mathrm{C}^\infty(\Omega); (\forall x \in \partial\Omega)\,(\forall \alpha \in \mathbb{N}^n)\,(\mathcal{D}^\alpha f(x) = \lim_{\Omega \ni y \to x} \mathcal{D}^\alpha f(y) < +\infty)\}$$ with $D^\alpha = \dfrac{\partial^{\alpha_1+\ldots+\alpha_n}}{\partial y_1^{\alpha_1}\ldots\partial y_n^{\alpha_n}}$ for $y = (y_1, \ldots, y_n) \in \Omega$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$.

Topology on $\mathrm{C}^\infty(\overline{\Omega})$

In a natural way the topology on $\mathrm{C}^\infty(\overline{\Omega})$ is the locally convex one defined by the family of seminorms $(p_{K,l})_{K \Subset \overline{\Omega}, l \in \mathbb{N}}$ where

$$\mathrm{C}^\infty(\overline{\Omega}) \ni f \to p_{K,l}(f) = \sup_{x \in K, |\alpha| \le l} |\mathcal{D}^\alpha f(x)|.$$

Definition 2.2. For the slowly increasing or rapidly decreasing functions on $\overline{\Omega}$, we define, in the same way

$$\mathcal{O}_M(\overline{\Omega}) = \{f \in \mathrm{C}^\infty(\overline{\Omega}), (\forall \alpha \in \mathbb{N}^n)\,(\exists p \in \mathbb{N})\,(\exists C > 0)\,(\forall x \in \overline{\Omega})\,(|\mathcal{D}^\alpha f(x)| \le C(1+|x|)^p)\},$$

$$\mathcal{S}(\overline{\Omega}) = \{\varphi \in \mathrm{C}^\infty(\overline{\Omega}), (\forall \alpha \in \mathbb{N}^n)\,(\forall q \in \mathbb{N})\,(\exists D > 0)\,(\forall x \in \overline{\Omega})\,(|\mathcal{D}^\alpha \varphi(x)| \le \frac{D}{(1+|x|)^q})\}.$$

Remark 2.3. All these spaces are in fact algebras, and $\mathcal{S}(\overline{\Omega})$ is an ideal of $\mathcal{O}_M(\overline{\Omega})$.

3. Topology on the algebra of slowly increasing functions

Let F be the closure of any open set in $\mathbb{R}^d$ (even $\mathbb{R}^d$ itself) and consider the function $\mathcal{O}_M(F) \to \mathbb{R}_+$

$$f \mapsto p_{\varphi,\alpha}(f) = \sup_{x \in F} |\varphi(x)\,\mathcal{D}^\alpha f(x)|$$

where $\varphi \in \mathcal{S}(F)$ and $\alpha \in \mathbb{N}^n$. We can see that $p_{\varphi,\alpha}$ is a semi-norm on $\mathcal{O}_M(F)$. Then, the family $\mathcal{P} = (p_{\varphi,\alpha})_{\varphi,\alpha\in\mathcal{S}(\overline{\Omega})\times\mathbb{N}^n}$ endows the algebra $\mathcal{O}_M(F)$ with a locally convex topology (a priori of vector space). We can refer to [5] about the continuity of the product in $\mathcal{O}_M(\mathbb{R}^d)$, but when $F = \overline{\Omega}$, for any open set Ω, the proof needs the following lemma.

Lemma 3.1. *Let $U \subseteq \mathbb{R}^d$ and f be a map $U \to \mathbb{C}$ which is rapidly decreasing in the sense that for each $m \in \mathbb{N}$, and $\langle x\rangle^m = (1+|x|)^m$*

$$\sup_{x\in U} \langle x\rangle^m |f(x)| < +\infty.$$

Then there exists $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $|f(x)| \leq \psi(x)$ for each $x \in U$.

Proof. Let $g(t) = \sup_{x\in U,|x|\geq t} |f(x)|$. Then, $g : [0,+\infty[\longrightarrow \mathbb{R}$ is a decreasing function (in particular $g \in \mathrm{L}^1_{Loc}(\mathbb{R})$) and is rapidly decreasing since for each $m \in \mathbb{N}$

$$g(t) \leq \sup_{x\in U,|x|\geq t} \frac{|x|^m}{t^m} |f(x)| \leq \frac{C}{t^m}.$$

Let $\Phi \in \mathcal{D}(\,[0,1]\,)$, $\Phi \geq 0$, $\int \Phi = 1$. Extend g as a constant function on $(-\infty, 0]$. Then $g * \Phi \in \mathrm{C}^\infty(\mathbb{R})$ and

$$(g * \Phi)(t) = \int_{t-1}^{t} g(s)\Phi(t-s)ds \geq g(t) \int \Phi = g(t).$$

Further $g * \Phi \in \mathcal{S}(\mathbb{R})$. Possibly increasing the values of $g * \Phi$, we find $h \in \mathcal{S}(\mathbb{R})$ with $h \geq g$ and h constant on a neighbourhood of 0. Hence

$$(x \mapsto \psi(x) = h(|x|)) \in \mathcal{S}(\mathbb{R}^d)$$

with $\psi(x) \geq g(|x|) \geq |f(x)|$ for each $x \in U$.

□

Theorem 3.2. $\mathcal{O}_M\left(\overline{\Omega}\right)$ *is a locally convex algebra.*

Proof. Let $\varphi \in \mathcal{S}\left(\overline{\Omega}\right)$. By Lemma 1 there exists $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\sqrt{\varphi} \leq \psi$ on $\overline{\Omega}$. Let $u, v \in \mathcal{O}_M\left(\overline{\Omega}\right)$. Then

$$p_{\varphi,0}(uv) = \sup_{\overline{\Omega}} |\varphi uv| \leq \sup_{\overline{\Omega}} |\psi u| \sup_{\overline{\Omega}} |\psi v| = p_{\psi,0}(u)\, p_{\psi,0}(v)$$

and similarly, by the Leibnitz rule, writing $\nu_{\varphi,m} = \max_{|\alpha|\leq m} p_{\varphi,\alpha}$,

$$\nu_{\varphi,m}(uv) \leq C_m \nu_{\psi,m}(u)\nu_{\psi,m}(v).$$

□

4. A tempered algebra on the closure of an open set

Definition 4.1. Let $\mathcal{G}_{\mathcal{O}_M}(\overline{\Omega})$ be the algebra $\mathcal{M}_{\mathcal{O}_M}\left(\overline{\Omega}\right)/\mathcal{N}_{\mathcal{O}_M}\left(\overline{\Omega}\right)$ where

$$\begin{aligned}\mathcal{M}_{\mathcal{O}_M}(\overline{\Omega}) = \{(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\overline{\Omega})^{(0,1]} : (\forall\varphi \in \mathcal{S}(\overline{\Omega}))\ (\forall\alpha \in \mathbb{N}^n)\\ (\exists M \in \mathbb{N})\ \ (\exists\varepsilon_0)\ \ (\forall\varepsilon < \varepsilon_0)\ \ (p_{\varphi,\alpha}(u_\varepsilon) \leq \varepsilon^{-M})\};\\ \mathcal{N}_{\mathcal{O}_M}(\overline{\Omega}) = \{(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\overline{\Omega})^{(0,1]} : (\forall\varphi \in \mathcal{S}(\overline{\Omega}))\ (\forall\alpha \in \mathbb{N}^n)\\ (\forall m \in \mathbb{N})\ \ (\exists\varepsilon_0)\ \ (\forall\varepsilon < \varepsilon_0)\ \ (p_{\varphi,\alpha}(u_\varepsilon) \leq \varepsilon^{m})\}.\end{aligned}$$

This definition is consistent. We can involve, for example, the framework of $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebra with $\mathcal{E} = \mathcal{O}_M(\overline{\Omega})$, $\mathcal{P} = (p_{\varphi,\alpha})$ and $\mathcal{C}$ generated by $(\varepsilon)_\varepsilon$.

Example 4.2. We deal with the characteristic Cauchy problem (P_g) for the transport equation formally written in characteristic coordinates

$$\frac{\partial u}{\partial t} = u;\quad u\,|_{\{x=0\}} = v.$$

However, we cannot prove the existence of a solution to (P_g) in $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$; indeed the regularized problem becomes

$$(P_\infty)\qquad \frac{\partial u_\varepsilon}{\partial t}(t,x) = u_\varepsilon(t,x)\,; u_\varepsilon(t,\varepsilon t) = v(t)$$

whose solution is $u_\varepsilon(t,x) = v(\frac{x}{\varepsilon})e^{-\frac{x}{\varepsilon}}e^t$ which clearly is not in $\mathcal{M}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$.

Example 4.3. Without changing asymptotic scale, we can estimate

$$\sup_{t,x} |\varphi(t,x)|\, v(\frac{x}{\varepsilon})e^{-\frac{x}{\varepsilon}}e^t$$

for $\varphi \in \mathcal{S}([0,T] \times [0,\infty[)$, $x \geq 0$ and $t \leq T$ in $\mathcal{G}_{Om}([0,T] \times [0,\infty[)$ with its polynomial $\frac{1}{\varepsilon}$-scale. Indeed, with $x \geq 0$ one has

$$\left(\frac{1}{e^{\frac{1}{\varepsilon}}}\right)^x \leq \varepsilon^x \leq 1$$

and the computation is easy. Then $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}([0,T]\times[0,\infty[)$ and $[u_\varepsilon]$ solve (P_g) in

$$\mathcal{G}_{\mathcal{O}_M}([0,T] \times [0,\infty[).$$

5. Point values characterization

In the following, we suppose that Ω is a convex open set.

5.1. A new definition of $\mathcal{G}_{\mathcal{O}_M}(\overline{\Omega})$

Theorem 5.1.

$$\begin{aligned}\mathcal{M}_{\mathcal{O}_M}(\overline{\Omega}) = \{(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\overline{\Omega})^{(0,1]} \mid (\forall\alpha\in\mathbb{N}^d)\ (\exists m\in\mathbb{N})\ (\exists p\in\mathbb{N})\\ (\exists\varepsilon_0)\ (\forall\varepsilon<\varepsilon_0)\ (\sup_{x\in\overline{\Omega}}(1+|x|)^{-p}|\partial^\alpha u_\varepsilon(x)| \le \varepsilon^{-m})\}.\\ \mathcal{N}_{\mathcal{O}_M}(\overline{\Omega}) = \{(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\overline{\Omega})^{(0,1]} \mid (\forall\alpha\in\mathbb{N}^d)\ (\forall m\in\mathbb{N})\ (\exists p\in\mathbb{N})\\ (\exists\varepsilon_0)\ (\forall\varepsilon<\varepsilon_0)\ (\sup_{x\in\overline{\Omega}}(1+|x|)^{-p}|\partial^\alpha u_\varepsilon(x)| \le \varepsilon^{m})\}.\end{aligned}$$

Proof. (Sketch) We follow the lines of proposition 5 in [5], which proves that $\mathcal{M}_{\mathcal{O}_M}(\mathbb{R}^n) = \mathcal{M}_\tau(\mathbb{R}^n)$, extended to the case when Ω is unbounded. But the ideal $\mathcal{N}_{\mathcal{O}_M}(\overline{\Omega})$ differs from the tempered one $\mathcal{N}_\tau(\overline{\Omega})$ and its characterization needs some other arguments. □

5.2. Zero derivative and slow scale elements

Definition 5.2. A subset $U \subseteq \mathbb{R}^d$ has the cone property if there exist $r>0$ and $c>0$ such that for each $x\in U$, there exists a rotation A such that $x + A\Gamma_{c,r} \subseteq U$, where $\Gamma_{c,r} = \{(x,y)\in\mathbb{R}\times\mathbb{R}^{d-1} : 0\le x\le r, |y|\le cx\}$.

This condition is used in Sobolev space theory [1, Ch. IV]. If $\overline{\Omega}$ is bounded and convex, then $\overline{\Omega}$ has the cone property: take any open ball $B(x_0,r)\subseteq\Omega$. Then for each $x\in\overline{\Omega}$, the cone at x with base $B(x_0,r)$ is contained in $\overline{\Omega}$. As Ω is bounded, this cone contains a cone $x + A\Gamma_{c,r}$ with c independent of x. However, if $\overline{\Omega}$ is unbounded and convex, this property may fail:

Example 5.3. Let $\overline{\Omega}\subseteq\mathbb{R}^3$ be the convex closure of the half lines $L_1 = \{(0,t,0) : t\ge 0\}$, $L_2 = \{(0,t,1) : t\ge 0\}$ and $L_3 = \{(1+t,t,0) : t\ge 0\}$. Then points on L_3 intersect $\overline{\Omega}$ in cones with smaller and smaller angles as $t\to+\infty$. Hence $\overline{\Omega}$ is the closure of an open convex set, but it does not have the cone property.

Theorem 5.4. *If $\overline{\Omega}$ has the cone property, then*

$$\begin{aligned}\mathcal{N}_{\mathcal{O}_M}(\overline{\Omega}) = \{(u_\varepsilon)_\varepsilon \in \mathcal{O}_M(\overline{\Omega})^{(0,1]} \mid (\forall m\in\mathbb{N})\ (\exists p\in\mathbb{N})\\ (\exists\varepsilon_0)\ \ (\forall\varepsilon<\varepsilon_0)\ \ \big(\sup_{x\in\overline{\Omega}}(1+|x|)^{-p}|u_\varepsilon(x)| \le \varepsilon^{m}\big)\}.\end{aligned}$$

Proof. We will in fact only assume a weaker property on $\overline{\Omega}$ than the cone property: we will only require that there exist $r>0$ and $M\in\mathbb{N}$ such that for each $x\in U$, there exists a rotation A such that $x + A\Gamma\subseteq\overline{\Omega}$, where Γ is the cusp $\{(x,y)\in\mathbb{R}\times\mathbb{R}^{d-1} : 0\le x\le r, |y|\le x^M\}$.

Let $(u_\varepsilon)_\varepsilon$ satisfy the estimates in the statement of the theorem. Let $x\in\overline{\Omega}$. Let A be such that $x + A\Gamma\subseteq\overline{\Omega}$. Let $\{e_1,\dots,e_n\}$ be the standard basis of $\mathbb{R}^d$. Let $e'_k = Ae_k$ (then e'_1 is along the symmetry axis of $A\Gamma$). As $x + A\Gamma\subseteq\overline{\Omega}$, the line segments $[x, x+\varepsilon^q e'_1]$ and $[x+\varepsilon^q e'_1, x+\varepsilon^q e'_1+\varepsilon^{Mq}e'_k]$ $(k=2,\dots,d)$ are contained in $\overline{\Omega}$, as soon as $\varepsilon\le r$, $q\ge 1$. Let $m\in\mathbb{N}$. Applying the Taylor argument from [7, Thm. 1.2.25] to these line segments, we find $p\in\mathbb{N}$ such that

$|\nabla u_\varepsilon(x)| \le e_1'^m \langle x\rangle^p$ and $|\nabla u_\varepsilon(x+\varepsilon^q e_1')| \le e_k'^m \langle x\rangle^p$, as soon as $\varepsilon \le \varepsilon_0$ and q sufficiently large. Then also

$$|\nabla u_\varepsilon(x)\cdot e_k'| \le \underbrace{|(\nabla u_\varepsilon(x+\varepsilon^q e_1') - \nabla u_\varepsilon(x))\cdot e_k'|}_{\le C\varepsilon^{q-N}\langle x\rangle^p} + C\varepsilon^m \langle x\rangle^p \le C'^m \langle x\rangle^p$$

as soon as $\varepsilon \le \varepsilon_0$ and q is sufficiently large. Hence $\|\nabla u_\varepsilon(x)\| \le C''^m\langle x\rangle^p$, with C'' independent of ε and x. Inductively, we obtain the bounds for the derivatives of any order. □

Remark 5.5. If one assumes a weaker kind of cone property where $r > 0$ depends on x, this characterization may fail (see the counterexample [7, 1.2.26]).

5.3. Point values characterization of elements of $\mathcal{G}_{\mathcal{O}_M}(\overline{\Omega})$

Definition 5.6. An element $\widetilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}^d$ is of slow scale if

$$(\forall n \in \mathbb{N})\ (\exists \varepsilon_0)\ (\forall \varepsilon < \varepsilon_0)\ \left(|x_\varepsilon| \le \varepsilon^{-1/n}\right).$$

We can consider $\widetilde{\overline{\Omega}} \subseteq \widetilde{\mathbb{R}}^d$ (containing those $\widetilde{x}$ having a representative $(x_\varepsilon)_\varepsilon \in \overline{\Omega}^{(0,1]}$).

Theorem 5.7. *Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^d)$ and let $\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\overline{\Omega}}$ be of slow scale. Then the point value $u(\tilde{x}) = [(u_\varepsilon(x_\varepsilon))_\varepsilon] \in \widetilde{\mathbb{C}}$ is well-defined.*

Proof. Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\overline{\Omega})$ be a representative of u. As in [7, Prop. 1.2.45], $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\overline{\Omega})$ implies that $(u_\varepsilon(x_\varepsilon))_\varepsilon \in \mathcal{M}_{\mathbb{R}}$, and $(u_\varepsilon(x_\varepsilon) - u_\varepsilon(x_\varepsilon'))_\varepsilon \in \mathcal{N}_{\mathbb{R}}$ if $(x_\varepsilon')_\varepsilon \in \widetilde{\overline{\Omega}}$ is another representative of $\tilde{x}$. The latter argument requires that $[x_\varepsilon, x_\varepsilon'] \subseteq \overline{\Omega}$, which is satisfied because $\overline{\Omega}$ is convex. It remains to be shown that the definition of the point value does not depend on the choice of representative of u. So let $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\overline{\Omega})$. Let $m \in \mathbb{N}$. Choose $p \in \mathbb{N}$ as in the statement of theorem 5.4. Then for sufficiently small ε,

$$|u_\varepsilon(x_\varepsilon)| \le \varepsilon^m(1+|x_\varepsilon|)^p \le \varepsilon^m(2|x_\varepsilon|)^p \le \varepsilon^m(2\varepsilon^{-1/p})^p = 2^p\varepsilon^{m-1}.$$

Since $m \in \mathbb{N}$ is arbitrary, $(u_\varepsilon(x_\varepsilon))_\varepsilon \in \mathcal{N}_{\mathbb{C}}$. □

Theorem 5.8. *Let $\overline{\Omega}$ have the cone property. Let $u \in \mathcal{G}_{\mathcal{O}_M}(\overline{\Omega})$. Then $u = 0$ iff $u(\tilde{x}) = 0$ for each slow scale point $\tilde{x} \in \widetilde{\overline{\Omega}}$.*

Proof. If $u = 0$, then clearly $u(\tilde{x}) = 0$ for each slow scale point in $\widetilde{\overline{\Omega}}$ (since the definition of point values does not depend on the representative of u). Conversely, let $u(\tilde{x}) = 0$ for each slow scale point $\tilde{x} \in \widetilde{\overline{\Omega}}$. We first show by contradiction that

(1) $\quad (\forall m \in \mathbb{N})\ (\exists n \in \mathbb{N})\ (\exists \varepsilon_0)\ (\forall \varepsilon < \varepsilon_0)\ \left(\sup_{x\in\overline{\Omega},\, |x|\le\varepsilon^{-1/n}} |u_\varepsilon(x)| \le \varepsilon^m\right).$

Assuming the contrary, we find $M \in \mathbb{N}$, a decreasing sequence $(\varepsilon_n)_n$ tending to 0 and $x_{\varepsilon_n} \in \overline{\Omega}$ with $|x_{\varepsilon_n}| \leq \varepsilon_n^{-1/n}$ and $|u_{\varepsilon_n}(x_{\varepsilon_n})| > \varepsilon_n^M$, for each n. Let x_ε be a fixed element $x_0 \in \overline{\Omega}$ if $\varepsilon \notin \{\varepsilon_n : n \in \mathbb{N}\}$. Then $\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\overline{\Omega}}$ is of slow scale and $(u_\varepsilon(x_\varepsilon))_\varepsilon \notin \mathcal{N}_\mathbb{R}$, contradicting $u(\tilde{x}) = 0$.

Now let $m \in \mathbb{N}$ arbitrary. Choose n as in equation (1). Since $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\overline{\Omega})$, there exists by theorem 5.1 some $N \in \mathbb{N}$ such that for small ε,

$$\sup_{x\in\overline{\Omega}}(1+|x|)^{-N}|u_\varepsilon(x)| \leq \varepsilon^{-N}.$$

Let $p = nm + nN + N$. Then, for small ε,

$$\begin{aligned}
&\sup_{x\in\overline{\Omega}}(1+|x|)^{-p}|u_\varepsilon(x)| = \\
&\qquad \max\Big(\sup_{\substack{|x|\leq\varepsilon^{-1/n}\\ x\in\overline{\Omega}}}(1+|x|)^{-p}|u_\varepsilon(x)|,\ \sup_{\substack{|x|\geq\varepsilon^{-1/n}\\ x\in\overline{\Omega}}}(1+|x|)^{-p}|u_\varepsilon(x)|\Big) \\
&\leq \max\Big(\sup_{x\in\overline{\Omega},\,|x|\leq\varepsilon^{-1/n}}|u_\varepsilon(x)|,\ \sup_{x\in\overline{\Omega}}(1+|x|)^{-N}|u_\varepsilon(x)|\sup_{x\in\overline{\Omega},\,|x|\geq\varepsilon^{-1/n}}(1+|x|)^{N-p}\Big) \\
&\qquad\qquad \leq \max\left(\varepsilon^m, \varepsilon^{-N}(2\varepsilon^{-1/n})^{N-p}\right) = 2\varepsilon^m.
\end{aligned}$$

Hence $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\overline{\Omega})$ by theorem 5.4. □

6. Application: uniqueness for a nonlinear Cauchy problem

The Characteristic Cauchy problems for Partial Differential Equations with the data given on a locally or globally characteristic manifold are generally ill-posed in the classical context. In [2], are pointed out some important works on the question and described another method to solve it. To simplify, it is chosen the two-variables characteristic Cauchy problem for the transport equation (in basic form)

$$(P_c) \qquad \frac{\partial u}{\partial t} = F(.,.,u); \quad u|_\gamma = v$$

where γ of equation $x = 0$ is globally characteristic. For focusing only on the characteristic singularity, v and F are supposed to be regular enough. Clearly (P_c) is ill-posed but can be associated to a generalized problem

$$(P_g) \qquad \frac{\partial u}{\partial t} = \mathcal{F}(u); \quad \mathcal{R}(u) = v.$$

- (P_g) is well formulated in some convenient algebras of $(\mathcal{C},\mathcal{E},\mathcal{P})$-type (where $\mathcal{C}$ define the asymptotics and $(\mathcal{E},\mathcal{P})$a basic presheaf of topological algebra), with $u \in \mathcal{A}_\mathcal{C}(\mathcal{E},\mathcal{P})(\mathbb{R}^2)$ supposed to be "stable under F" and $v \in \mathcal{A}_\mathcal{C}(\mathcal{E},\mathcal{P})(\mathbb{R})$.

To obtain (P_g) from (P_c), two generalized mappings have to be defined:

$\mathcal{F}$ is a generalized mapping $\mathcal{F} : \mathcal{A}_{\mathcal{C}}(\mathcal{E},\mathcal{P})(\mathbb{R}^2) \longrightarrow \mathcal{A}_{\mathcal{C}}(\mathcal{E},\mathcal{P})(\mathbb{R}^2)$, associated to F and

$\mathcal{R} : \mathcal{A}_{\mathcal{C}}(\mathcal{E},\mathcal{P})(\mathbb{R}^2) \longrightarrow \mathcal{A}_{\mathcal{C}}(\mathcal{E},\mathcal{P})(\mathbb{R})$ is obtained by replacing $\{x = 0\}$ by a family

$$(\gamma_\varepsilon : x = l_\varepsilon(t))_\varepsilon$$

of non characteristic lines.

- If $\mathcal{T}$ is the usual topology of $\mathcal{E} = \mathrm{C}^\infty$, and $\mathcal{C} = [B_{reg}]$ *overgenerated* by a finite family of elements in relationship with the regularization of the problem, we know from previous works the existence in $\mathcal{A}_{[B_{reg}]}(\mathrm{C}^\infty,\mathcal{T})(\mathbb{R}^2) = \mathcal{A}(\mathbb{R}^2)$ (non uniqueness) of a solution to (P_g) depending a priori of the choice of the "decharacterizing" process ([2], Theorem 5).

- A better result is obtained when choosing the decharacterizing process in a tempered class $\mathcal{G}_\tau$. Then, the above solution (always non unique) depends only on this tempered class ([2], Theorem 6).

- It is possible to recover the uniqueness in the homogeneous case ([2], Theorem 13) when working in the new algebra

$$\mathcal{A}_{[(\varepsilon)_\varepsilon]}(\mathcal{O}_M, \mathcal{Q})(\mathbb{R}^2) = \mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^2)$$

  based on the space of slowly increasing smooth functions $\mathcal{O}_M(\mathbb{R}^2)$ endowed with its usual locally convex topology $\mathcal{Q}$. In that algebra it is impossible to obtain uniqueness for nonlinear case.

Now, we are focusing on the nonlinear case. We can prove that $\mathcal{F}$ can be defined as a mapping of $\mathcal{G}_{\mathcal{O}_M}([0,T] \times [0,\infty[)$ into itself and $\mathcal{R}$ as a mapping $\mathcal{G}_{\mathcal{O}_M}([0,T] \times [0,\infty[) \longrightarrow \mathcal{G}_{\mathcal{O}_M}([0,\infty[)$. Finally the uniqueness can be expected thanks to the tools and results detailed in the above Section 6.

Assume that

$$\exists M > 0, \forall n \in \mathbb{N}, \exists \mu_n > 0, \sup_{(t,x,z)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R},|\alpha|\leq n} |D^\alpha F(t,x,z)| = M_n \leq \mu_n M.$$

Lemma 6.1. *Let $\mathcal{L}_{\mathcal{O}_M}(\mathbb{R}_+)$ be the subset in $\mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$ of families $(g_\varepsilon)_\varepsilon$ such that $g'_\varepsilon > 0$, and $[g_\varepsilon^{-1}] \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$ preserves slow scale points, $\lim_{\varepsilon\to 0,\mathcal{D}'(\mathbb{R})} g_\varepsilon = 0$. Assume that $V \in \mathcal{O}_M(\mathbb{R}_+)$, $(L_\varepsilon)_\varepsilon \in \mathcal{L}_{\mathcal{O}_M}(\mathbb{R}_+)$. Take $U = [U_\varepsilon] \in \mathcal{G}(\mathbb{R}_+^2)$ such that, for any $(t,x) \in \mathbb{R}_+^2$,*

$$U_\varepsilon(t,x) = V(L_\varepsilon^{-1}(x)) + \int_{L_\varepsilon^{-1}(x)}^t F(\tau,x,U_\varepsilon(\tau,x))\,\mathrm{d}\tau.$$

Then U is solution to

$$(P_g^*) : \quad \frac{\partial U}{\partial t} = \mathcal{F}(U); \quad \mathcal{R}(U) = V.$$

and, for any $(t,x) \in \mathbb{R}_+^2$,

$$|U_\varepsilon(t,x)| \leq M_1 e^{M_1\left|t-L_\varepsilon^{-1}(x)\right|} \left| \int_{L_\varepsilon^{-1}(x)}^t r_\varepsilon(t,x)\mathrm{d}\tau \right| + r_\varepsilon(t,x).$$

where

$$m_\varepsilon(t,x) = \int_{L_\varepsilon^{-1}(x)}^t |F(\tau,x,0)|\,\mathrm{d}\tau, r_\varepsilon(t,x) = \left(\left|V(L_\varepsilon^{-1}(x))\right| + |m_\varepsilon(t,x)|\right).$$

Proof. For $(t,x) \in \mathbb{R}_+^2$, we have

$$F(t,x,U_\varepsilon(t,x)) - F(t,x,0) = \frac{\partial F}{\partial z}(t,x,\theta_\varepsilon(t,x))U_\varepsilon(t,x)$$

where $\theta_\varepsilon(\tau,x) = \theta(\tau,x)U_\varepsilon(\tau,x)$ and $0 < \theta(\tau,x) < 1$. Then

$$(E1)\ \ U_\varepsilon(t,x) = V(L_\varepsilon^{-1}(x)) + \int_{L_\varepsilon^{-1}(x)}^t (\frac{\partial F}{\partial z}(\tau,x,\theta_\varepsilon(\tau,x))U_\varepsilon(\tau,x) + F(\tau,x,0))\mathrm{d}\tau.$$

Assume that $L_\varepsilon^{-1}(x) \leq t$. According to $(E1)$ we have
$(E2)$

$$|U_\varepsilon(t,x)| \leq \left|V(L_\varepsilon^{-1}(x))\right| + M_1 \int_{L_\varepsilon^{-1}(x)}^t |U_\varepsilon(\tau,x)|\ \mathrm{d}\tau + \int_{L_\varepsilon^{-1}(x)}^t |F(\tau,x,0)|\,\mathrm{d}\tau.$$

Define

$$H_\varepsilon(t,x) = \int_{L_\varepsilon^{-1}(x)}^t |U_\varepsilon(\tau,x)|\ \mathrm{d}\tau$$

and observe that

$$(E3)\qquad \frac{\partial H_\varepsilon}{\partial t}(t,x) = |U_\varepsilon(t,x)|.$$

That means you can write

$$\frac{\partial H_\varepsilon}{\partial t}(t,x) \leq M_1 H_\varepsilon(t,x) + \left|V(L_\varepsilon^{-1}(x))\right| + m_\varepsilon(t,x).$$

and multiplying that by an integrating factor

$$e^{-M_1(t-L_\varepsilon^{-1}(x))}\frac{\partial H_\varepsilon}{\partial t}(t,x) - e^{-M_1(t-L_\varepsilon^{-1}(x))}M_1 H_\varepsilon(t,x)$$
$$\leq e^{-M_1(t-L_\varepsilon^{-1}(x))}\left(\left|V(L_\varepsilon^{-1}(x))\right| + m_\varepsilon(t,x)\right)$$

which means
$(E4)$

$$\frac{d}{dt}\left(e^{-M_1(t-L_\varepsilon^{-1}(x))}H_\varepsilon(t,x)\right) \leq e^{-M_1(t-L_\varepsilon^{-1}(x))}\left(\left|V(L_\varepsilon^{-1}(x))\right| + m_\varepsilon(t,x)\right).$$

Since $H_\varepsilon(L_\varepsilon^{-1}(x),x)=0$, we can integrate both sides of $(E4)$ from $L_\varepsilon^{-1}(x)$ to t, we get

$$\begin{aligned} e^{-M_1(t-L_\varepsilon^{-1}(x))}H_\varepsilon(t,x) &\leq \int_{L_\varepsilon^{-1}(x)}^{t} e^{-M_1(\tau-L_\varepsilon^{-1}(x))}\left(\left|V(L_\varepsilon^{-1}(x))\right| + m_\varepsilon(\tau,x)\right)\mathrm{d}\tau \\ &\leq \int_{L_\varepsilon^{-1}(x)}^{t} \left(\left|V(L_\varepsilon^{-1}(x))\right| + m_\varepsilon(\tau,x)\right)\mathrm{d}\tau. \end{aligned}$$

So

$$(E5)\qquad H_\varepsilon(t,x) \leq e^{M_1(t-L_\varepsilon^{-1}(x))}\left(\int_{L_\varepsilon^{-1}(x)}^{t} \left(\left|V(L_\varepsilon^{-1}(x))\right| + m_\varepsilon(\tau,x)\right)\mathrm{d}\tau\right).$$

Substituting $(E5)$ into $(E2)$, you obtain

$$\begin{aligned} |U_\varepsilon(t,x)| \leq M_1 e^{M_1(t-L_\varepsilon^{-1}(x))}&\left(\int_{L_\varepsilon^{-1}(x)}^{t} \left(\left|V(L_\varepsilon^{-1}(x))\right| + m_\varepsilon(\tau,x)\right)\mathrm{d}\tau\right) \\ &+ \left(\left|V(L_\varepsilon^{-1}(x))\right| + m_\varepsilon(t,x)\right). \end{aligned}$$

Assume that $t < L_\varepsilon^{-1}(x)$. According to $(E1)$ we have

$$|U_\varepsilon(t,x)| \leq \left|V(L_\varepsilon^{-1}(x))\right| + M_1\int_t^{L_\varepsilon^{-1}(x)} |U_\varepsilon(\tau,x)|\,\mathrm{d}\tau + \int_t^{L_\varepsilon^{-1}(x)} |F(\tau,x,0)|\,\mathrm{d}\tau,$$

that is

$$\begin{aligned} |U_\varepsilon(t,x)| &\leq \left|V(L_\varepsilon^{-1}(x))\right| - M_1\int_{L_\varepsilon^{-1}(x)}^{t} |U_\varepsilon(\tau,x)|\,\mathrm{d}\tau - \int_{L_\varepsilon^{-1}(x)}^{t} |F(\tau,x,0)|\,\mathrm{d}\tau \\ (E6)\qquad &\leq \left|V(L_\varepsilon^{-1}(x))\right| - M_1\int_{L_\varepsilon^{-1}(x)}^{t} |U_\varepsilon(\tau,x)|\,\mathrm{d}\tau + \left|\int_{L_\varepsilon^{-1}(x)}^{t} |F(\tau,x,0)|\,\mathrm{d}\tau\right| \end{aligned}$$

According to $E3$, that means you can write

$$\frac{\partial H_\varepsilon}{\partial t}(t,x) \leq -M_1 H_\varepsilon(t,x) + \left|V(L_\varepsilon^{-1}(x))\right| + |m_\varepsilon(t,x)|.$$

and multiplying that by an integrating factor

$$\begin{aligned} &e^{M_1(t-L_\varepsilon^{-1}(x))}\frac{\partial H_\varepsilon}{\partial t}(t,x) + e^{M_1(t-L_\varepsilon^{-1}(x))}M_1 H_\varepsilon(t,x) \\ &\leq e^{M_1(t-L_\varepsilon^{-1}(x))}\left(\left|V(L_\varepsilon^{-1}(x))\right| + |m_\varepsilon(t,x)|\right) \end{aligned}$$

which means

$$(E7)\quad \frac{\partial}{\partial t}\left(e^{M_1(t-L_\varepsilon^{-1}(x))}H_\varepsilon(t,x)\right) \leq e^{M_1(t-L_\varepsilon^{-1}(x))}\left(\left|V(L_\varepsilon^{-1}(x))\right| + |m_\varepsilon(t,x)|\right).$$

Since $H_\varepsilon(L_\varepsilon^{-1}(x),x)=0$, we can integrate both sides of $(E7)$ from t to $L_\varepsilon^{-1}(x)$, we get

$$\begin{aligned}-e^{M_1(t-L_\varepsilon^{-1}(x))}H_\varepsilon(t,x) &\le \int_t^{L_\varepsilon^{-1}(x)} e^{M_1(\tau-L_\varepsilon^{-1}(x))}\left(\left|V(L_\varepsilon^{-1}(x))\right|+|m_\varepsilon(\tau,x)|\right)\mathrm{d}\tau\\ &\le \int_t^{L_\varepsilon^{-1}(x)}\left(\left|V(L_\varepsilon^{-1}(x))\right|+|m_\varepsilon(\tau,x)|\right)\mathrm{d}\tau.\end{aligned}$$

So

$$(E8)\quad -H_\varepsilon(t,x)\le e^{-M_1(t-L_\varepsilon^{-1}(x))}\left(\int_t^{L_\varepsilon^{-1}(x)}\left(\left|V(L_\varepsilon^{-1}(x))\right|+|m_\varepsilon(\tau,x)|\right)\mathrm{d}\tau\right).$$

Substituting $(E8)$ into $(E6)$, we obtain

$$\begin{aligned}&|U_\varepsilon(t,x)|\\ &\le M_1e^{-M_1(t-L_\varepsilon^{-1}(x))}\int_t^{L_\varepsilon^{-1}(x)}\left(\left|V(L_\varepsilon^{-1}(x))\right|+|m_\varepsilon(\tau,x)|\right)\mathrm{d}\tau\\ &\qquad\qquad+\left(\left|V(L_\varepsilon^{-1}(x))\right|+|m_\varepsilon(t,x)|\right)\\ &\le M_1e^{-M_1(t-L_\varepsilon^{-1}(x))}\left|\int_{L_\varepsilon^{-1}(x)}^t\left(\left|V(L_\varepsilon^{-1}(x))\right|+|m_\varepsilon(\tau,x)|\right)\mathrm{d}\tau\right|\\ &\qquad\qquad+\left(\left|V(L_\varepsilon^{-1}(x))\right|+|m_\varepsilon(t,x)|\right)\end{aligned}$$

So, in the both cases we have

$(E9)$

$$\begin{aligned}|U_\varepsilon(t,x)|\le M_1e^{M_1\left|t-L_\varepsilon^{-1}(x)\right|}&\left|\int_{L_\varepsilon^{-1}(x)}^t\left(\left|V(L_\varepsilon^{-1}(x))\right|+|m_\varepsilon(\tau,x)|\right)\mathrm{d}\tau\right|\\ &+\left(\left|V(L_\varepsilon^{-1}(x))\right|+|m_\varepsilon(t,x)|\right).\end{aligned}$$

Put

$$r_\varepsilon(t,x)=\left(\left|V(L_\varepsilon^{-1}(x))\right|+|m_\varepsilon(t,x)|\right),$$

then, we have in the both case

$$|U_\varepsilon(t,x)|\le M_1e^{M_1\left|t-L_\varepsilon^{-1}(x)\right|}\left|\int_{L_\varepsilon^{-1}(x)}^t r_\varepsilon(\tau,x)\mathrm{d}\tau\right|+r_\varepsilon(t,x).$$

□

Lemma 6.2. *Assume that $V\in\mathcal{O}_M(\mathbb{R}_+)$, $(L_\varepsilon)_\varepsilon\in\mathcal{L}_{\mathcal{O}_M}(\mathbb{R}_+)$. Let $[U_\varepsilon]\in\mathcal{G}\left(\mathbb{R}_+^2\right)$ be the solution to $\left(P_g^*\right)$ define in Lemma 6.1. Let $S=[S_\varepsilon]\in\mathcal{G}\left(\mathbb{R}_+^2\right)$ be another solution to $\left(P_g^*\right)$. For $(t,x)\in\mathbb{R}_+^2$ we have*

$$\begin{cases}\dfrac{\partial}{\partial t}\left(S_\varepsilon(t,x)\right)=F(t,x,S_\varepsilon(t,x)+I_\varepsilon(t,x)\\ \qquad S_\varepsilon\left(t,l_\varepsilon(t)\right)=V\left(t\right)+J_\varepsilon\left(t\right).\end{cases}$$

with $(J_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}_+), (I_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}_+^2)$. *Take* $W_\varepsilon = (S_\varepsilon - U_\varepsilon)$. *Then, for any* $(t,x) \in \mathbb{R}_+^2$, *we have*

$$|W_\varepsilon(t,x)| \leq M_1 e^{M_1|t-L_\varepsilon^{-1}(x)|} \int_t^{L_\varepsilon^{-1}(x)} \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + |\sigma_\varepsilon(\tau,x)|\right) \mathrm{d}\tau + \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + |\sigma_\varepsilon(t,x)|\right)$$

where, for all ε,

$$\sigma_\varepsilon(t,x) = \int_{L_\varepsilon^{-1}(x)}^t I_\varepsilon(\tau,x) d\tau.$$

Proof. For any $(t,x) \in \mathbb{R}_+^2$, we have

$$S_\varepsilon(t,x) = V(L_\varepsilon^{-1}(x)) + J_\varepsilon(L_\varepsilon^{-1}(x)) + \int_{L_\varepsilon^{-1}(x)}^t F(\tau,x,S_\varepsilon(\tau,x))\,\mathrm{d}\tau + \int_{L_\varepsilon^{-1}(x)}^t I_\varepsilon(\tau,x)\,\mathrm{d}\tau.$$

Take $(t,x) \in [0,T] \times [0,\infty[$. When putting $W_\varepsilon = (S_\varepsilon - U_\varepsilon)$ we get

$$W_\varepsilon(t,x) = J_\varepsilon(L_\varepsilon^{-1}(x)) + \sigma_\varepsilon(t,x) + \int_{L_\varepsilon^{-1}(x)}^t \left(F(\tau,x,S_\varepsilon(\tau,x)) - F(\tau,x,U_\varepsilon(\tau,x))\right)\,\mathrm{d}\tau.$$

Moreover we have

$$F(t,x,S_\varepsilon(t,x)) - F(t,x,U_\varepsilon(t,x)) = W_\varepsilon(t,x) \int_0^1 \frac{dF_\eta}{dz}(t,x,U_\varepsilon(t,x) + \theta W_\varepsilon((t,x))\,\mathrm{d}\theta.$$

Assume that $L_\varepsilon^{-1}(x) \leq t$. We have

$$\left|\int_{L_\varepsilon^{-1}(x)}^t W_\varepsilon(\tau,x) \left(\int_0^1 \frac{\partial F}{\partial z}(\tau,x,U_\varepsilon(\tau,x) + \theta W_\varepsilon((\tau,x))\,\mathrm{d}\theta\right)\,\mathrm{d}\tau.\right| \leq \int_{L_\varepsilon^{-1}(x)}^t |W_\varepsilon(\tau,x)|\, M_1\,\mathrm{d}\tau.$$

We deduce

$$(E10) \quad |W_\varepsilon(t,x)| \leq M_1 \int_{L_\varepsilon^{-1}(x)}^t |W_\varepsilon(\tau,x)|\,\mathrm{d}\tau + \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + |\sigma_\varepsilon(t,x)|\right).$$

According to $E3$, that means you can write

$$\frac{dH_\varepsilon}{dt}(t,x) \leq M_1 H_\varepsilon(t,x) + \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + |\sigma_\varepsilon(t,x)|\right)$$

and multiplying that by an integrating factor

$$\begin{aligned} & e^{-M_1(t-L_\varepsilon^{-1}(x))}\frac{dH_\varepsilon}{dt}(t,x) - e^{-M_1(t-L_\varepsilon^{-1}(x))}M_1 H_\varepsilon(t,x) \\ & \leq e^{-M_1(t-L_\varepsilon^{-1}(x))}\left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + \left|\sigma_\varepsilon\left(t,x\right)\right|\right) \end{aligned}$$

which means
$(E11)$

$$\frac{d}{dt}\left(e^{-M_1(t-L_\varepsilon^{-1}(x))}H_\varepsilon(t,x)\right) \leq e^{-M_1(t-L_\varepsilon^{-1}(x))}\left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + \left|\sigma_\varepsilon\left(t,x\right)\right|\right).$$

Since $H_\varepsilon(L_\varepsilon^{-1}(x),x) = 0$, we can integrate both sides of $(E11)$ from $L_\varepsilon^{-1}(x)$ to t, we get

$$\begin{aligned} e^{-M_1(t-L_\varepsilon^{-1}(x))}H_\varepsilon(t,x) & \leq \int_{L_\varepsilon^{-1}(x)}^t e^{-M_1(\tau-L_\varepsilon^{-1}(x))}\left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + \left|\sigma_\varepsilon\left(\tau,x\right)\right|\right)\mathrm{d}\tau \\ & \leq \int_{L_\varepsilon^{-1}(x)}^t \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + \left|\sigma_\varepsilon\left(\tau,x\right)\right|\right)\mathrm{d}\tau. \end{aligned}$$

So

$$(E12) \qquad H_\varepsilon(t,x) \leq e^{M_1(t-L_\varepsilon^{-1}(x))}\int_{L_\varepsilon^{-1}(x)}^t \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + \left|\sigma_\varepsilon\left(\tau,x\right)\right|\right)\mathrm{d}\tau.$$

Substituting $(E12)$ into $(E10)$, you obtain

$$\begin{aligned} |W_\varepsilon(t,x)| \leq M_1 e^{M_1(t-L_\varepsilon^{-1}(x))} & \int_{L_\varepsilon^{-1}(x)}^t \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + \left|\sigma_\varepsilon\left(\tau,x\right)\right|\right)\mathrm{d}\tau \\ & + \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + \left|\sigma_\varepsilon\left(t,x\right)\right|\right). \end{aligned}$$

Assume that $t < L_\varepsilon^{-1}(x)$, we have

$$\begin{aligned} & \left|\int_{L_\varepsilon^{-1}(x)}^t W_\varepsilon(\tau,x)\left(\int_0^1 \frac{\partial F}{\partial z}(\tau,x,U_\varepsilon(\tau,x)+\theta W_\varepsilon((\tau,x))\,\mathrm{d}\theta\right)\mathrm{d}\tau.\right| \\ & \leq \int_t^{L_\varepsilon^{-1}(x)} \left|W_\varepsilon(\tau,x)\right| M_1\,\mathrm{d}\tau. \end{aligned}$$

We deduce

$$(E13) \quad |W_\varepsilon(t,x)| \leq -M_1\int_{L_\varepsilon^{-1}(x)}^t \left|W_\varepsilon(\tau,x)\right|\,\mathrm{d}\tau + \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + \left|\sigma_\varepsilon\left(t,x\right)\right|\right).$$

According to $E3$, that means you can write

$$\frac{dH_\varepsilon}{dt}(t,x) \leq \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + \left|\sigma_\varepsilon\left(t,x\right)\right|\right) - M_1 H_\varepsilon(t,x)$$

and multiplying that by an integrating factor

$$\begin{aligned} & e^{M_1(t-L_\varepsilon^{-1}(x))}\frac{dH_\varepsilon}{dt}(t,x)+e^{M_1(t-L_\varepsilon^{-1}(x))}M_1H_\varepsilon(t,x) \\ & \leq e^{M_1(t-L_\varepsilon^{-1}(x))}\left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right|+\left|\sigma_\varepsilon(t,x)\right|\right) \end{aligned}$$

which means
$(E14)$

$$\frac{d}{dt}\left(e^{M_1(t-L_\varepsilon^{-1}(x))}H_\varepsilon(t,x)\right)\leq e^{M_1(t-L_\varepsilon^{-1}(x))}\left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right|+\left|\sigma_\varepsilon(t,x)\right|\right).$$

Since $H_\varepsilon(L_\varepsilon^{-1}(x),x)=0$, we can integrate both sides of $(E14)$ from t to $L_\varepsilon^{-1}(x)$, we get

$$\begin{aligned} -e^{M_1(t-L_\varepsilon^{-1}(x))}H_\varepsilon(t,x) & \leq \int_t^{L_\varepsilon^{-1}(x)} e^{M_1(\tau-L_\varepsilon^{-1}(x))}\left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right|+\left|\sigma_\varepsilon(\tau,x)\right|\right)\mathrm{d}\tau \\ & \leq \int_t^{L_\varepsilon^{-1}(x)}\left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right|+\left|\sigma_\varepsilon(\tau,x)\right|\right)\mathrm{d}\tau. \end{aligned}$$

So

$$(E15)\qquad -H_\varepsilon(t,x)\leq e^{-M_1(t-L_\varepsilon^{-1}(x))}\int_t^{L_\varepsilon^{-1}(x)}\left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right|+\left|\sigma_\varepsilon(\tau,x)\right|\right)\mathrm{d}\tau.$$

Substituting $(E15)$ into $(E13)$, you obtain

$$\begin{aligned} |W_\varepsilon(t,x)| \leq M_1 e^{-M_1(t-L_\varepsilon^{-1}(x))} & \int_t^{L_\varepsilon^{-1}(x)}\left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right|+\left|\sigma_\varepsilon(\tau,x)\right|\right)\mathrm{d}\tau \\ & +\left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right|+\left|\sigma_\varepsilon(t,x)\right|\right). \end{aligned}$$

So, in the both cases we have

$$\begin{aligned} |W_\varepsilon(t,x)| \leq M_1 e^{M_1\left|t-L_\varepsilon^{-1}(x)\right|} & \left|\int_{l_\varepsilon^{-1}(x)}^{t}\left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right|+\left|\sigma_\varepsilon(\tau,x)\right|\right)\mathrm{d}\tau\right| \\ & +\left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right|+\left|\sigma_\varepsilon(t,x)\right|\right). \end{aligned}$$

□

Definition 6.3. The generalized function $[u_\varepsilon]$ is solution to Problem $\left(P_g^{**}\right)$ if there are $U=[U_\varepsilon]\in\mathcal{G}\left(\mathbb{R}_+^2\right)$, $V\in\mathcal{O}_M\left(\mathbb{R}_+\right)$, $\left(L_\varepsilon\right)_\varepsilon\in\mathcal{L}_{\mathcal{O}_M}\left(\mathbb{R}_+\right)$ such that

(1) U is solution to $\left(P_g^*\right)$

(2) $\left\{\begin{array}{c} u_\varepsilon=\left.U_\varepsilon\right|_{[0,T]\times[0,\infty[}\,;l_\varepsilon=\left.L_\varepsilon\right|_{[0,T]\times[0,\infty[}\,; \\ u_\varepsilon\left(t,l_\varepsilon(t)\right)=\left.V\right|_{[0,T]}(t)=v(t); \end{array}\right.$

(3) $\{[u_\varepsilon]\in\mathcal{G}_{\mathcal{O}_M}([0,T]\times[0,\infty[)$.

Moreover, for $(t,x)\in\mathbb{R}_+^2$, we have

$$U_\varepsilon(t,x)=V(L_\varepsilon^{-1}(x))+\int_{L_\varepsilon^{-1}(x)}^{t}F(\tau,x,U_\varepsilon(\tau,x))\,\mathrm{d}\tau.$$

Take $m_\varepsilon(t,x) = \int_{l_\varepsilon^{-1}(x)}^{t} |F(\tau,x,0)|\,\mathrm{d}\tau$, assume that $(m_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}\left((\mathbb{R}_+)^2\right)$.

Theorem 6.4. *Suppose that $(l_\varepsilon)_\varepsilon$ is taken in $\mathcal{L}_{\mathcal{O}_M}([0,T])$. Then, if $v \in \mathcal{O}_M([0,T])$, the generalized function $u = [u_\varepsilon]_{\mathcal{G}_{\mathcal{O}_M}([0,T]\times[0,\infty[)}$, where u_ε is defined in Definition 6.3, depends only on $l = [l_\varepsilon]_{\mathcal{G}_{\mathcal{O}_M}([0,T])}$.*

Moreover u is the unique solution to $\left(P_g^{**}\right)$ in $\mathcal{G}_{\mathcal{O}_M}([0,T]\times[0,\infty[)$.

Proof. The first step is to prove the existence, and it is not possible to do that in $\mathcal{G}_{\mathcal{O}_M}(\mathbb{R}^2)$ if $F \neq 0$ ([2], Remark 3).

Let $\mathcal{L}_{\mathcal{O}_M}(\mathbb{R}_+)$ be the subset in $\mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$ of families $(g_\varepsilon)_\varepsilon$ such that $g'_\varepsilon > 0$,and $\left[g_\varepsilon^{-1}\right] \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$ preserves slow scale points, $\lim_{\varepsilon\to 0,\mathcal{D}'(\mathbb{R})} g_\varepsilon = 0$.

From Lemma (6.1), we have for any $(t,x) \in \mathbb{R}_+^2$,

$$|U_\varepsilon(t,x)| \leq M_1 e^{M_1\left|t-L_\varepsilon^{-1}(x)\right|}\left|\int_{L_\varepsilon^{-1}(x)}^{t} r_\varepsilon(\tau,x)\mathrm{d}\tau\right| + r_\varepsilon(t,x).$$

where

$$m_\varepsilon(t,x) = \int_{L_\varepsilon^{-1}(x)}^{t} |F(\tau,x,0)|\,\mathrm{d}\tau,\ r_\varepsilon(t,x) = \left(\left|V(L_\varepsilon^{-1}(x))\right| + |m_\varepsilon(t,x)|\right).$$

As $(L_\varepsilon)_\varepsilon \in \mathcal{L}_{\mathcal{O}_M}([0,T])$, we know that $\left(L_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$ moreover $V \in \mathcal{O}_M(\mathbb{R})$ so $\left(\left|V \circ L_\varepsilon^{-1}\right|\right)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$. We have also $(m_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}\left((\mathbb{R}_+)^2\right)$. Thus

$$(r_\varepsilon)_\varepsilon = \left(\left|V \circ L_\varepsilon^{-1}\right| + |m_\varepsilon|\right)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}\left(\mathbb{R}_+^2\right).$$

Put

$$a_\varepsilon(t,x) = \left|\int_{L_\varepsilon^{-1}(x)}^{t} r_\varepsilon(\tau,x)\mathrm{d}\tau\right|$$

then $(a_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}\left((\mathbb{R}_+)^2\right)$. Thus $\exists m \in \mathbb{N}$, $\exists p \in \mathbb{N}$, $\exists \varepsilon_0$, $\forall \varepsilon < \varepsilon_0$,

$$\sup_{(t,x)\in[0,T]\times[0,\infty[} (1+|t|+|x|)^{-p} r_\varepsilon(t,x) \leq \sup_{(t,x)\in(\mathbb{R}_+)^2} (1+|t|+|x|)^{-p} r_\varepsilon(t,x) \leq \varepsilon^{-m}$$

and : $\exists m \in \mathbb{N}$, $\exists p \in \mathbb{N}$, $\exists \varepsilon_0$, $\forall \varepsilon < \varepsilon_0$,

$$\sup_{(t,x)\in[0,T]\times[0,\infty[} (1+|t|+|x|)^{-p} a_\varepsilon(t,x) \leq \sup_{(t,x)\in(\mathbb{R}_+)^2} (1+|t|+|x|)^{-p} a_\varepsilon(t,x) \leq \varepsilon^{-m}.$$

Take $u_\varepsilon = U_\varepsilon|_{[0,T]\times[0,\infty[}$. So, we have

$$\sup_{(t,x)\in[0,T]\times[0,\infty[} (1+|t|+|x|)^{-p}\,|u_\varepsilon(t,x)|$$
$$\leq M_1\big(\sup_{(t,x)\in[0,T]\times[0,\infty[} e^{M_1\left|t-L_\varepsilon^{-1}(x)\right|}\big)\varepsilon^{-m} + \varepsilon^{-m}.$$

Take $l_\varepsilon = L_\varepsilon|_{[0,T]}$. Then, for $(l_\varepsilon)_\varepsilon \in \mathcal{L}_{\mathcal{O}_M}([0,T])$, we have

$$\sup_{(t,x)\in[0,T]\times[0,\infty[} e^{M_1\left|t-L_\varepsilon^{-1}(x)\right|} = \sup_{(t,x)\in[0,T]\times[0,\infty[} e^{M_1\left|t-l_\varepsilon^{-1}(x)\right|} \leq e^{M_1 T}.$$

So: $\exists k \in \mathbb{N}$, $\exists p \in \mathbb{N}$, $\exists \varepsilon_1$, $\forall \varepsilon < \varepsilon_1$,

$$\sup_{(t,x)\in[0,T]\times[0,\infty[} (1+|t|+|x|)^{-p}\,|u_\varepsilon(t,x)| \leq \varepsilon^{-k}.$$

According to Theorem 5.4,

$$(u_\varepsilon)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}([0,T]\times[0,\infty[)$$

and the class of $(u_{\varepsilon,})_\varepsilon$ in $\mathcal{G}_{\mathcal{O}_M}([0,T][\times[0,\infty[)$ is a solution to problem $\left(P_g^{**}\right)$.

Uniqueness.

Let $s = [s_\varepsilon] \in \mathcal{G}_{\mathcal{O}_M}([0,T]\times[0,\infty[)$ be another solution to $\left(P_g^{**}\right)$. That is to say there are $[S_\varepsilon] \in \mathcal{G}\left(\mathbb{R}_+^2\right)$, $V \in \mathcal{O}_M(\mathbb{R}_+)$, $(L_\varepsilon)_\varepsilon \in \mathcal{L}_{\mathcal{O}_M}(\mathbb{R}_+)$, $(J_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}_+), (I_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}_+^2\right)$. such that

$$(1)\begin{cases} \dfrac{\partial S_\varepsilon}{\partial t}(t,x) = F(t,x,U_\varepsilon(t,x)) + I_\varepsilon(t,x)\,; \\ S_\varepsilon(t,L_\varepsilon(t)) = V(t) + J_\varepsilon(t)\,; \end{cases}$$

$$(2)\begin{cases} s_\varepsilon = S_\varepsilon|_{[0,T]\times[0,\infty[}\,; l_\varepsilon = L_\varepsilon|_{[0,T]}\,; v = V|_{[0,T]}\,; \\ s_\varepsilon(t,l_\varepsilon(t)) = V|_{[0,T]}(t) + J_\varepsilon|_{[0,T]}(t) = v(t) + J_\varepsilon|_{[0,T]}(t)\,; \end{cases}$$

$$(3)\,\{[s_\varepsilon] \in \mathcal{G}_{\mathcal{O}_M}([0,T]\times[0,\infty[).$$

Take $W_\varepsilon = (S_\varepsilon - U_\varepsilon)$.

From Lemma (6.2), for any $(t,x)\in\mathbb{R}_+^2$, we have

$$|W_\varepsilon(t,x)| \leq M_1 e^{M_1\left|t-L_\varepsilon^{-1}(x)\right|}\left|\int_{L_\varepsilon^{-1}(x)}^t \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + |\sigma_\varepsilon(\tau,x)|\right)\mathrm{d}\tau\right| + \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + |\sigma_\varepsilon(t,x)|\right).$$

where, for all ε,

$$\sigma_\varepsilon(t,x) = \int_{L_\varepsilon^{-1}(x)}^t I_\varepsilon(\tau,x)d\tau.$$

As $(L_\varepsilon)_\varepsilon \in \mathcal{L}_{\mathcal{O}_M}(\mathbb{R}_+)$, we know that $\left(L_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$ moreover $V \in \mathcal{O}_M(\mathbb{R}_+)$ so $V\circ L_\varepsilon^{-1} \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$.
Furthermore, as $(J_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}_+)$, $\left(L_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{M}_{\mathcal{O}_M}(\mathbb{R}_+)$ and they preserve slow scale points, we have that $\left(J_\varepsilon\circ L_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}_+)$.

We set, for all ε,

$$\sigma_\varepsilon(t,x) = \int_{l_\varepsilon^{-1}(x)}^t I_\varepsilon(\tau,x)d\tau.$$

We have to check that

$$(|\sigma_\varepsilon|)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}_+^2\right).$$

Let $[(t_\varepsilon, x_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}_+^2}$ be a slow scale point. Then $[(x_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}_+}$ is a slow scale point and $[(y_\varepsilon)_\varepsilon] = \left[\left(L_\varepsilon^{-1}(x_\varepsilon)\right)_\varepsilon\right]$ is also a slow scale point. We have

$$\forall \varepsilon, \exists c_\varepsilon \in [y_\varepsilon, t_\varepsilon], |\sigma_\varepsilon(t_\varepsilon, x_\varepsilon)| = \left| \int_{y_\varepsilon}^{t_\varepsilon} I_\varepsilon(\tau, x_\varepsilon) d\tau \right| = |t_\varepsilon - y_\varepsilon| \, I_\varepsilon(c_\varepsilon, x_\varepsilon)$$

but as $|c_\varepsilon| \leq \max(|y_\varepsilon|, |t_\varepsilon|)$, $[(c_\varepsilon)_\varepsilon]$ is also a slow scale point. But then $[(c_\varepsilon, x_\varepsilon)_\varepsilon]$ is a slow scale point of $\widetilde{\mathbb{R}_+^2}$ so that $(I_\varepsilon(c_\varepsilon, x_\varepsilon))_\varepsilon \in \mathcal{N}_{\mathbb{R}_+}$ and finally $(|\sigma_\varepsilon(t_\varepsilon, x_\varepsilon)|)_\varepsilon \in \mathcal{N}_{\mathbb{R}_+}$, thus $(|\sigma_\varepsilon|)_\varepsilon$ lies in $\mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}_+^2\right)$.

We have $(|\sigma_\varepsilon|)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}_+^2\right)$ and $\left(V \circ L_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}_+\right)$, thus

$$\left((t,x) \mapsto \left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + |\sigma_\varepsilon(t,x)|\right)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}_+^2\right).$$

Put

$$b_\varepsilon(t,x) = \left| \int_{l_\varepsilon^{-1}(x)}^{t} \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + |\sigma_\varepsilon(\tau,x)| \right) \mathrm{d}\tau \right|,$$

then $(b_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}_+^2\right)$.

So, for each $\forall m \in \mathbb{N}, \exists p \in \mathbb{N}, \exists \varepsilon_0, \forall \varepsilon < \varepsilon_0$,

$$\sup_{(t,x)\in\mathbb{R}_+^2} (1+|t|+|x|)^{-p} \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + |\sigma_\varepsilon(t,x)| \right) \leq \varepsilon^m$$

and

$$\sup_{(t,x)\in\mathbb{R}_+^2} (1+|t|+|x|)^{-p} \left|(b_\varepsilon(t,x))\right| \leq \varepsilon^m.$$

Thus

$$\sup_{(t,x)\in[0,T]\times[0,\infty[} (1+|t|+|x|)^{-p} \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + |\sigma_\varepsilon(t,x)| \right) \leq \varepsilon^m$$

and

$$\sup_{(t,x)\in[0,T]\times[0,\infty[} (1+|t|+|x|)^{-p} \left|(b_\varepsilon(t,x))\right| \leq \varepsilon^m.$$

Consequently $(\forall m \in \mathbb{N})\,(\exists \varepsilon_0)\,(\forall \varepsilon < \varepsilon_0)$

$$\begin{aligned}
&\sup_{(t,x)\in[0,T]\times[0,\infty[} (1+|t|+|x|)^{-p} \left|w_\varepsilon(t,x)\right| \\
&\leq \sup_{(t,x)\in[0,T]\times[0,\infty[} (1+|t|+|x|)^{-p} \left(\left|J_\varepsilon(L_\varepsilon^{-1}(x))\right| + |\sigma_\varepsilon(t,x)| \right) \\
&+ M_1 (\sup_{(t,x)\in[0,T]\times[0,\infty[} e^{M_1\left|t-L_\varepsilon^{-1}(x)\right|}) \sup_{(t,x)\in[0,T]\times[0,\infty[} (1+|t|+|x|)^{-p} \left|(b_\varepsilon(t,x))\right| \\
&\leq \varepsilon^m + M_1 (\sup_{(t,x)\in[0,T]\times[0,\infty[} e^{M_1\left|t-L_\varepsilon^{-1}(x)\right|}) \varepsilon^m.
\end{aligned}$$

Take $l_\varepsilon = L_\varepsilon|_{[0,T]}$. Then, for $(l_\varepsilon)_\varepsilon \in \mathcal{L}_{\mathcal{O}_M}([0,T])$, we have

$$\sup_{(t,x)\in[0,T]\times[0,\infty[} e^{M_1\left|t-L_\varepsilon^{-1}(x)\right|} = \sup_{(t,x)\in[0,T]\times[0,\infty[} e^{M_1\left|t-l_\varepsilon^{-1}(x)\right|} \leq e^{M_1 T}.$$

So $\forall k \in \mathbb{N}$, $\exists p \in \mathbb{N}$, $\exists \varepsilon_1$, $\forall \varepsilon < \varepsilon_1$,

$$\sup_{(t,x)\in[0,T]\times[0,\infty[} (1 + |t| + |x|)^{-p} |w_\varepsilon(t,x)| \leq \varepsilon^k.$$

According to Theorem 5.4, we deduce that

$$(w_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{O}_M}([0,T] \times [0,\infty[).$$

Thus the solution is unique in $\mathcal{G}_{\mathcal{O}_M}([0,T] \times [0,\infty[)$. □

References

[1] Adams, R., Sobolev spaces. New York: Academic Press, 1975.

[2] Allaud, E., Dévoué, V., Delcroix, A., Marti, J.-A., Vernaeve, H., On the Well-posedeness in Some Generalized Characteristic Cauchy Problems. Mathematical Modelling of Natural Phenomena 11 (2) (2016), 89-99.

[3] Aragona,J., Colombeau generalized functions on quasi-regular sets. Publ. Math. Debrecen 68 (3-4) (2006), 371-399.

[4] Biagioni, H., A Nonlinear Theory of Generalized Functions. Lecture Notes in Math. 1421, Springer-Verlag, 1990.

[5] Delcroix, A., A new approach to temperate generalized Colombeau functions. Publ. Inst. Math. Beograd Novi Sad, 84(98) (2008), 109-121.

[6] Hasler, M., Marti, J.-A., Towards point-value characterizations in multi-parameter algebras. Novi Sad J. Math. 41 (1) (2011), 21-31.

[7] Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R., Geometric Theory of Generalized Functions with Applications to General Relativity. Dordrecht: Kluwer Academic Publ., 2001.

[8] Vernaeve, H., Pointwise characterizations in generalized function algebras. Monatsh. Math. 158 (2009), 195-213.

Received by the editors July 5, 2016
First published online August 13, 2016