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ON THE FAST GROWTH OF SOLUTIONS TO HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS

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Abstract. In this paper, we investigate the iterated order of solutions of higher order homogeneous and nonhomogeneous linear differential equations

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0$$

and

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z),$$

where $A_0(z) \not\equiv 0, A_1(z), \cdots, A_k(z) \not\equiv 0$ and $F(z) \not\equiv 0$ are entire functions of finite iterated $p-$order. We improve and extend some results of He, Zheng and Hu; Long and Zhu by using the concept of the iterated order and we obtain general estimates of the iterated convergence exponent and the iterated p-order of solutions for the above equations.

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1. Introduction

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory of meromorphic functions see [6, 9, 20], such as $T(r,f)$, $N(r,f)$, $m(r,f)$. For the definition of iterated order of meromorphic function, we use the same definition as in [12, 14]. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all $r \in (0, +\infty)$ sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$.

Definition 1.1. [12, 14] Let f be a meromorphic function. Then the iterated p-order $\rho_p(f)$ of f is defined by

$$\rho_p(f) := \limsup_{r\to+\infty} \frac{\log_p^+ T(r,f)}{\log r}, \ (p \geq 1 \text{ is an integer}).$$

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For $p=1$, this notation is called order and for $p=2$ hyper-order. If f is an entire function, then the iterated p-order $\rho_p(f)$ of f is defined by

$$\rho_p(f) := \limsup_{r\to+\infty} \frac{\log_p^+ T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log_{p+1}^+ M(r,f)}{\log r}, \ (p\geq 1 \text{ is an integer}),$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Definition 1.2. [12] The finiteness degree of the order of a meromorphic function f is defined by

$$i(f) := \begin{cases} 0, & \text{for } f \text{ rational}, \\ \min\{j\in\mathbb{N} : \rho_j(f)<\infty\}, & \text{for } f \text{ transcendental for which} \\ & \text{some } j\in\mathbb{N} \text{ with } \rho_j(f)<\infty \text{ exists}, \\ +\infty, & \text{for } f \text{ with } \rho_j(f)=+\infty,\ \forall j\in\mathbb{N}. \end{cases}$$

Definition 1.3. [12] Let $n(r,a)$ be the unintegrated counting function for the sequence of a-points of a meromorphic function $f(z)$. Then the iterated convergence exponent of a-points of $f(z)$ is defined by

$$\lambda_p(f,a) := \limsup_{r\to+\infty} \frac{\log_p^+ n(r,a)}{\log r}, \ (p\geq 1 \text{ is an integer}).$$

In the definition of the iterated convergence exponent, we may replace $n(r,a)$ with the integrated counting function $N(r,a)$, and we have

$$\lambda_p(f,a) := \limsup_{r\to+\infty} \frac{\log_p^+ N(r,a)}{\log r}, \ (p\geq 1 \text{ is an integer}),$$

where $N(r,a) = N(r,a,f) = N\left(r, \frac{1}{f-a}\right)$. If $a=0$, then the iterated convergence exponent of the zero-sequence is defined by

$$\lambda_p(f) := \limsup_{r\to+\infty} \frac{\log_p^+ N\left(r,\frac{1}{f}\right)}{\log r}, \ (p\geq 1 \text{ is an integer}),$$

where $N\left(r,\frac{1}{f}\right)$ is the integrated counting of zeros of $f(z)$ in $\{z : |z|\leq r\}$. Similarly, if $a=\infty$, then the iterated convergence exponent of the pole-sequence is defined by

$$\lambda_p\left(\frac{1}{f}\right) := \limsup_{r\to+\infty} \frac{\log_p^+ N(r,f)}{\log r}, \ (p\geq 1 \text{ is an integer}).$$

Definition 1.4. [12] The finiteness degree of the iterated convergence exponent is defined by

$$i_\lambda(f,a) := \begin{cases} 0, & \text{if } n(r,a) = O(\log r), \\ \min\{j\in\mathbb{N} : \lambda_j(f,a)<\infty\}, & \text{for some } j\in\mathbb{N} \\ & \text{with } \lambda_j(f,a)<\infty \text{ exists}, \\ +\infty, & \text{if } \lambda_j(f,a)=\infty \text{ for all } j\in\mathbb{N}. \end{cases}$$

Remark 1.5. If $a=0$, then we set $i_{\lambda}(f,a)=i_{\lambda}(f)$. If $a=\infty$, then we set $i_{\lambda}(f,a)=i_{\lambda}\left(\frac{1}{f}\right)$.

Definition 1.6. [11, 16] The iterated lower p-order $\mu_p(f)$ of a meromorphic function f is defined by

$$\mu_p(f):=\liminf_{r\to+\infty}\frac{\log_p^+ T(r,f)}{\log r},\quad (p\geq 1 \text{ is an integer}).$$

The Lebesgue linear measure of a set $E\subset[0,\infty)$ is $m(E)=\int_E dt$, and the logarithmic measure of a set $F\subset(1,+\infty)$ is $m_l(F)=\int_F\frac{dt}{t}$. The upper density of $E\subset[0,\infty)$ is given by

$$\overline{dens}(E)=\limsup_{r\to+\infty}\frac{m(E\cap[0,r])}{r}$$

and the upper logarithmic density of the set $F\subset(1,+\infty)$ is defined by

$$\overline{\log dens}(F)=\limsup_{r\longrightarrow+\infty}\frac{m_l(F\cap[1,r])}{\log r}.$$

Proposition 1.7. *[2] For all $H\subset(1,+\infty)$ the following statements hold:*
(i) If $m_l(H)=\infty$, then $m(H)=\infty$;
(ii) If $\overline{dens}(H)>0$, then $m(H)=\infty$;
(iii) If $\overline{\log dens}(H)>0$, then $m_l(H)=\infty$.

In [1], the author extended the results of Kwon [13], Chen and Yang [4] from second order to higher order linear differential equations by considering more general conditions to entire coefficients as follows.

Theorem 1.8. *[1] Let H be a set of complex numbers satisfying $\overline{dens}\{|z|:z\in H\}>0$, and let $A_0(z),\cdots,A_{k-1}(z)$ be entire functions such that for real constants α, β, μ, where $0\leq\beta<\alpha$ and $\mu>0$, we have*

$$|A_0(z)|\geq e^{\alpha|z|^{\mu}}$$

and

$$|A_j(z)|\leq e^{\beta|z|^{\mu}},\ j=1,\cdots,k-1$$

as $z\to\infty$ for $z\in H$. Then every solution $f\not\equiv 0$ of the equation

$$f^{(k)}+\sum_{j=1}^{k-1}A_jf^{(j)}+A_0f=0, \tag{1.1}$$

has infinite order and $\rho_2(f)\geq\mu$.

Theorem 1.9. *[1] Let H be a set of complex numbers satisfying* $\overline{dens}\{|z| : z \in H\} > 0$, *and let* $A_0(z), \cdots, A_{k-1}(z)$ *be entire functions with* $\max\{\rho(A_j) : j = 1, \cdots, k-1\} \leq \rho(A_0) = \rho < +\infty$ *such that for real constants* α, β $(0 \leq \beta < \alpha)$, *we have for any given* $\varepsilon > 0$

$$|A_0(z)| \geq e^{\alpha |z|^{\rho-\varepsilon}}$$

and

$$|A_j(z)| \leq e^{\beta |z|^{\rho-\varepsilon}}, \quad j = 1, \cdots, k-1$$

as $z \to \infty$ *for* $z \in H$. *Then every solution* $f \not\equiv 0$ *of equation* (1.1) *has infinite order and* $\rho_2(f) = \rho(A_0)$.

Very recently, Long and Zhu improved the previous results in [1, 4, 13, 18] by studying the growth of meromorphic solutions of higher-order linear differential equations (1.1) and

$$f^{(k)} + \sum_{j=1}^{k-1} A_j f^{(j)} + A_0 f = F, \tag{1.2}$$

where $A_0(z) \not\equiv 0$, $A_1(z), \cdots, A_{k-1}(z)$ and $F(z) \not\equiv 0$ are meromorphic functions. A precise estimation of the hyper-order of meromorphic solutions of the above equations has been given provided that there exists one dominant coefficient.

Theorem 1.10. *[15] Let E be a set of complex numbers satisfying* $m_l(\{|z| : z \in E\}) = \infty$, *and let* $A_j(z)$ $(j = 0, 1, \cdots, k-1)$, *be meromorphic functions. Suppose there exists an integer* s, $0 \leq s \leq k-1$, *satisfying*

$$\max\left\{\rho(A_j), \ j \neq s, \ \lambda\left(\frac{1}{A_s}\right)\right\} < \mu(A_s) \leq \rho(A_s) = \rho < \infty$$

and for some constants $0 \leq \beta < \alpha$, *we have, for all* $\varepsilon > 0$ *sufficiently small,*

$$|A_j(z)| \leq \exp\left(\beta |z|^{\rho-\varepsilon}\right), \ j \neq s,$$

$$|A_s(z)| \geq \exp\left(\alpha |z|^{\rho-\varepsilon}\right),$$

as $z \to \infty$ *for* $z \in E$. *Then every nontrivial meromorphic solution* f *whose poles are of uniformly bounded multiplicities of equation* (1.1) *satisfies* $\rho_2(f) = \rho(A_s)$.

For the case of non-homogeneous equation, they get the following result.

Theorem 1.11. *[15] Let E and $A_j(z)$ $(j = 0, 1, \cdots, k-1)$, be defined as in Theorem 1.10, and let $F(z) \not\equiv 0$ be a meromorphic function. Suppose there exists an integer s, $0 \le s \le k-1$, satisfying*

$$\max\left\{\rho(A_j),\ j \ne s,\ \lambda\left(\frac{1}{A_s}\right),\ \rho(F)\right\} < \mu(A_s) \le \rho(A_s) = \rho < \infty$$

and for some constants $0 \le \beta < \alpha$, we have, for all $\varepsilon > 0$ sufficiently small,

$$|A_j(z)| \le \exp\left(\beta |z|^{\rho-\varepsilon}\right),\ j \ne s,$$

$$|A_s(z)| \ge \exp\left(\alpha |z|^{\rho-\varepsilon}\right),$$

as $z \to \infty$ for $z \in E$. Then every meromorphic solution f of equation (1.2), whose poles are of uniformly bounded multiplicities, satisfies $\rho_2(f) = \rho(A_s)$.

In this paper, we consider for $k \ge 2$ the homogeneous and the nonhomogeneous linear differential equations

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.3}$$

$$A_k(z) f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z), \tag{1.4}$$

where $A_0(z) \not\equiv 0$, $A_1(z), \cdots, A_k(z) \not\equiv 0$ and $F(z) \not\equiv 0$ are entire functions of finite iterated $p-$order. It is well-known that if $A_k(z) \equiv 1$, then all solutions of the linear differential equation (1.3) and (1.4) are entire functions but when A_k is a nonconstant entire function, equation (1.3) or (1.4) can possess meromorphic solutions. For instance the equation

$$zf''' + 3f'' - 2e^{-2z} f' + \left((z-2)e^{-3z} + (3z-2)e^{-2z} + ze^{-z}\right) f = 0$$

has a meromorphic solution

$$f(z) = \frac{e^{e^{-z}}}{z}.$$

We also know that if some of coefficients $A_0(z)$, $A_1(z), \cdots,$ $A_{k-1}(z)$ are transcendental and $A_k(z) \equiv 1$, then equation (1.3) has at least one solution of infinite order. Recently several authors have investigated the properties of solutions of equations (1.3), (1.4) and obtained many results about their growth, see [5, 10, 11, 19]. Thus, there arise some interesting questions such as:

Question 1.1. What conditions on $A_0(z)$, $A_1(z), \cdots,$ $A_k(z)$ will guarantee that every solution $f \not\equiv 0$ of (1.3) and (1.4) is of infinite iterated order?

Question 1.2. Can we replace the meromorphic coefficients of equations (1.1) and (1.2) in Theorem 1.10 and Theorem 1.11 by entire functions for equations (1.3) and (1.4)?

The main purpose of this paper is to consider the above questions. The remainder of the paper is organized as follows. In Section 2, we shall show our main results which improve and extend many results in the above-mentioned papers, and thus the above questions are answered. Section 3 is for some lemmas. The last section is for the proofs of our main results.

2. Main results

For equation (1.3), our first result is an extension of Theorem 1.9 and Theorem 1.10.

Theorem 2.1. *Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ (or $m_l(\{|z| : z \in H\}) = \infty$) and let $A_j(z)$ $(j = 0, 1, \cdots, k)$, be entire functions such that $A_k \not\equiv 0$. Suppose there exists an integer s, $0 \leq s \leq k$ such that $i(A_s) = p$, $0 < p < +\infty$, and satisfying*

$$\max\{\rho_p(A_j),\ j \neq s,\ j = 0, 1, \cdots, k\} < \mu_p(A_s) \leq \rho_p(A_s) < +\infty$$

($p \geq 1$ is an integer) and for some constants $0 \leq \beta < \alpha$, we have, for all $\varepsilon > 0$ sufficiently small,

$$|A_j(z)| \leq \exp_p\left(\beta|z|^{\rho_p(A_s)-\varepsilon}\right),\ j \neq s, j = 0, 1, \cdots, k,$$

$$|A_s(z)| \geq \exp_p\left(\alpha|z|^{\rho_p(A_s)-\varepsilon}\right),$$

as $z \to \infty$ for $z \in H$. Then every transcendental meromorphic solution f of equation (1.3) with $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$ satisfies $i(f) = p + 1$ and $\rho_{p+1}(f) = \rho_p(A_s)$.

When $A_k(z) \equiv 1$, we obtain the following corollary for entire solutions.

Corollary 2.2. *Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ (or $m_l(\{|z| : z \in H\}) = \infty$) and let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$, be entire functions. Suppose there exists an integer s, $0 \leq s \leq k-1$ such that $i(A_s) = p$, $0 < p < +\infty$, and satisfying*

$$\max\{\rho_p(A_j),\ j \neq s,\ j = 0, 1, \cdots, k-1\} < \mu_p(A_s) \leq \rho_p(A_s) < +\infty$$

($p \geq 1$ is an integer) and for some constants $0 \leq \beta < \alpha$, we have, for all $\varepsilon > 0$ sufficiently small,

$$|A_j(z)| \leq \exp_p\left(\beta|z|^{\rho_p(A_s)-\varepsilon}\right),\ j \neq s, j = 0, 1, \cdots, k-1,$$

$$|A_s(z)| \geq \exp_p\left(\alpha|z|^{\rho_p(A_s)-\varepsilon}\right),$$

as $z \to \infty$ for $z \in H$. Then every transcendental solution f of equation (1.1) satisfies $i(f) = p + 1$ and $\rho_{p+1}(f) = \rho_p(A_s)$.

Corollary 2.3. *Let $A_j(z)$ $(j = 0, 1, \cdots, k)$, H satisfy all of the hypotheses of Theorem 2.1, and let $\varphi(z)$ be a transcendental meromorphic function satisfying $i(\varphi) < p+1$ or $\rho_{p+1}(\varphi) < \rho_p(A_s)$. Then every transcendental meromorphic solution $f(z)\,(\not\equiv 0)$ with $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$ of equation (1.3) satisfies $i_{\overline{\lambda}}(f-\varphi) = i_\lambda(f-\varphi) = p+1$ and $\overline{\lambda}_{p+1}(f-\varphi) = \lambda_{p+1}(f-\varphi) = \rho_{p+1}(f-\varphi) = \rho_p(A_s)$.*

Considering nonhomogeneous linear differential equation (1.4), we obtain an extension of Theorem 1.11.

Theorem 2.4. *Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ (or $m_l(\{|z| : z \in H\}) = \infty$) and let $A_j(z)$ $(j = 0, 1, \cdots, k)$, $F(z) \not\equiv 0$ be entire functions such that $A_k \not\equiv 0$. Suppose there exists an integer s, $0 \le s \le k$ such that $i(A_s) = p$, $0 < p < +\infty$, and satisfying*

$$\max\{\rho_p(A_j),\ j \ne s,\ j = 0, 1, \cdots, k,\ \rho_p(F)\} < \mu_p(A_s) \le \rho_p(A_s) < +\infty,$$

($p \ge 1$ is an integer) and for some constants $0 \le \beta < \alpha$, we have, for all $\varepsilon > 0$ sufficiently small,

$$|A_j(z)| \le \exp_p\left(\beta|z|^{\rho_p(A_s)-\varepsilon}\right),\ j \ne s, j = 0, 1, \cdots, k,$$

$$|A_s(z)| \ge \exp_p\left(\alpha|z|^{\rho_p(A_s)-\varepsilon}\right),$$

as $z \to \infty$ for $z \in H$. Then every transcendental meromorphic solution f of equation (1.4) with $\lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$ satisfies $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho_p(A_s)$.

When $A_k(z) \equiv 1$, we obtain the following corollary for entire solutions.

Corollary 2.5. *Let H be a set of complex numbers satisfying $\overline{\log dens}\{|z| : z \in H\} > 0$ (or $m_l(\{|z| : z \in H\}) = \infty$) and let $A_j(z)$ $(j = 0, 1, \cdots, k-1)$, $F(z) \not\equiv 0$ be entire functions. Suppose there exists an integer s, $0 \le s \le k-1$ such that $i(A_s) = p$, $0 < p < +\infty$, and satisfying*

$$\max\{\rho_p(A_j),\ j \ne s,\ j = 0, 1, \cdots, k-1,\ \rho_p(F)\} < \mu_p(A_s) \le \rho_p(A_s) < +\infty,$$

($p \ge 1$ is an integer) and for some constants $0 \le \beta < \alpha$, we have, for all $\varepsilon > 0$ sufficiently small,

$$|A_j(z)| \le \exp_p\left(\beta|z|^{\rho_p(A_s)-\varepsilon}\right),\ j \ne s, j = 0, 1, \cdots, k-1,$$

$$|A_s(z)| \ge \exp_p\left(\alpha|z|^{\rho_p(A_s)-\varepsilon}\right),$$

as $z \to \infty$ for $z \in H$. Then every transcendental solution f of equation (1.2) satisfies $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho_p(A_s)$.

Corollary 2.6. *Let $A_j(z)$ $(j=0,1,\cdots,k)$, $F(z)$, H satisfy all of the hypotheses of Theorem 2.4, and let $\varphi(z)$ be a transcendental meromorphic function satisfying $i(\varphi)<p+1$ or $\rho_{p+1}(\varphi)<\rho_p(A_s)$. Then every transcendental meromorphic solution $f(z)$ with $\lambda_p\left(\frac{1}{f}\right)<\mu_p(f)$ of equation (1.4) satisfies $i_{\overline{\lambda}}(f-\varphi)=i_{\lambda}(f-\varphi)=p+1$ and $\overline{\lambda}_{p+1}(f-\varphi)=\lambda_{p+1}(f-\varphi)=\rho_{p+1}(f-\varphi)=\rho_p(A_s)$.*

3. Preliminary lemmas

Our proofs depend mainly upon the following lemmas.

Lemma 3.1. *[7] Let f be a transcendental meromorphic function in the plane, and let $\mu>1$ be a given constant. Then there exist a set $E_1\subset(1,+\infty)$ that has a finite logarithmic measure, and a constant $B>0$ depending only on μ and (m,n) $(m,n\in\{0,1,\cdots,k\})$ $m<n$ such that for all z with $|z|=r\notin[0,1]\cup E_1$, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right|\leq B\left(\frac{T(\mu r,f)}{r}(\log^{\mu}r)\log T(\mu r,f)\right)^{n-m}.$$

By using similar reasoning as in the proof of Lemma 3.5 in [17], we easily obtain the following lemma when $\rho_p(f)=\rho_p(g)=+\infty$.

Lemma 3.2. *Let $f(z)=\frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$, $d(z)$ are entire functions of finite iterated order satisfying $\mu_p(g)=\mu_p(f)=\mu\leq\rho_p(f)=\rho_p(g)\leq+\infty$, $0<p<+\infty$, $i(d)<p$ or $\rho_p(d)<\mu$. Let z be a point with $|z|=r$ at which $|g(z)|=M(r,g)$ and $\nu_g(r)$ denote be the central index of g. Then the estimation*

$$\frac{f^{(n)}(z)}{f(z)}=\left(\frac{\nu_g(r)}{z}\right)^n(1+o(1)),\ n\geq 1,$$

holds for all $|z|=r$ outside a set E_2 of r of finite logarithmic measure.

Lemma 3.3. *[3] Let $g(z)$ be an entire function of finite iterated order satisfying $i(g)=p+1$, $\rho_{p+1}(g)=\rho$, $i_{\mu}(g)=q+1$, $\mu_{q+1}(g)=\mu$, $0<p,q<\infty$, and let $\nu_g(r)$ be the central index of g. Then we have*

$$\limsup_{r\to+\infty}\frac{\log_{p+1}\nu_g(r)}{\log r}=\rho,\ \liminf_{r\to+\infty}\frac{\log_{q+1}\nu_g(r)}{\log r}=\mu.$$

Lemma 3.4. *[8] Let $\varphi:[0,+\infty)\to\mathbb{R}$ and $\psi:[0,+\infty)\to\mathbb{R}$ be monotone nondecreasing functions such that $\varphi(r)\leq\psi(r)$ for all $r\notin(E_3\cup[0,1])$, where E_3 is a set of finite logarithmic measure. Let $\alpha>1$ be a given constant. Then there exists an $r_1=r_1(\alpha)>0$ such that $\varphi(r)\leq\psi(\alpha r)$ for all $r>r_1$.*

Lemma 3.5. *[10] Let $p\geq 1$ be an integer and let $f(z)=\frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$, $d(z)$ are entire functions satisfying $\mu_p(g)=\mu_p(f)=\mu\leq$*

$\rho_p(f) = \rho_p(g) \leq +\infty$, $0 < p < +\infty$, $\rho_p(d) = \lambda_p\left(\frac{1}{f}\right) = \beta < \mu$. *Then there exists a set* E_4 *of finite logarithmic measure such that for all* $|z| = r \notin E_4$ *and* $|g(z)| = M(r,g)$ *and for* r *sufficiently large, we have*

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq r^{2s}, \quad (s \text{ is an integer}).$$

Lemma 3.6. *[5] Let* $p \geq 1$ *be an integer, and let* $f(z)$ *be an entire function such that* $i(f) = p$, $\rho_p(f) = \rho < +\infty$. *Then, there exists a set* $E_5 \subset (1,+\infty)$ *of* r *of finite linear measure such that for any given* $\varepsilon > 0$, *we have*

$$\exp\left\{-\exp_{p-1}\left\{r^{\rho+\varepsilon}\right\}\right\} \leq |f(z)| \leq \exp_p\left\{r^{\rho+\varepsilon}\right\} \quad (r \notin E_5).$$

Lemma 3.7. *[11] Let* $A_j(z)$ $(j = 0,1,\cdots,k)$, $F(z)\,(\not\equiv 0)$ *be meromorphic functions and let* $f(z)$ *be a meromorphic solution of* (1.4) *satisfying one of the following conditions:*
(i) $\max\{i(F) = p,\ i(A_j)\,(j = 0,1,\cdots,k)\} < i(f) = p+1\,(0 < p < +\infty)$,
(ii) $b = \max\{\rho_{p+1}(F),\ \rho_{p+1}(A_j)\,(j = 0,1,\cdots,k)\} < \rho_{p+1}(f)$, *then* $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f)$.

Lemma 3.8. *Let* $f(z) = \frac{g(z)}{d(z)}$ *be a meromorphic function, where* $g(z)$, $d(z)$ *are entire functions. If* $0 \leq \rho_p(d) < \mu_p(f)$, *then* $\mu_p(g) = \mu_p(f)$ *and* $\rho_p(g) = \rho_p(f)$. *Moreover, if* $\rho_p(f) = +\infty$, *then* $\rho_{p+1}(g) = \rho_{p+1}(f)$.

Proof. We divided into the following three cases.
Case 1. $\rho_p(f) < +\infty$. By definition of the iterated order, there exists an increasing sequence $\{r_n\}$, $(r_n \to +\infty)$ and a positive integer n_0 such that for all $n > n_0$ and for any given $\varepsilon \in \left(0, \frac{\rho_p(f)-\rho_p(d)}{2}\right)$ (as $0 \leq \rho_p(d) < \mu_p(f) \leq \rho_p(f)$),

$$T(r_n, f) \geq \exp_{p-1}\left\{r_n^{\rho_p(f)-\varepsilon}\right\}, \tag{3.1}$$

and

$$T(r_n, d) \leq \exp_{p-1}\left\{r_n^{\rho_p(d)+\varepsilon}\right\}. \tag{3.2}$$

Since $T(r,f) \leq T(r,g) + T(r,d) + O(1)$, we get, for all sufficiently large n,

$$\exp_{p-1}\left\{r_n^{\rho_p(f)-\varepsilon}\right\} \leq T(r_n, g) + \exp_{p-1}\left\{r_n^{\rho_p(d)+\varepsilon}\right\} + O(1). \tag{3.3}$$

Since $\varepsilon \in \left(0, \frac{\rho_p(f)-\rho_p(d)}{2}\right)$, then (3.3) becomes

$$(1 - o(1))\exp_{p-1}\left\{r_n^{\rho_p(f)-\varepsilon}\right\} \leq T(r_n, g) + O(1),$$

for all sufficiently large n. Hence

$$\rho_p(f) \leq \rho_p(g).$$

On the other hand, since $T(r,g) \leq T(r,f) + T(r,d)$, and $\rho_p(d) < \rho_p(f)$, so we obtain

$$\rho_p(g) \leq \rho_p(f).$$

Therefore, we get $\rho_p(g) = \rho_p(f)$. By using the similar way above and the definition of iterated lower p-order $\mu_p(f)$ and $\mu_p(g)$, we can prove

$$\mu_p(g) = \mu_p(f).$$

Case 2. $\rho_p(f) = +\infty$. Suppose on the contrary to the assertion that $\mu_p(g) < \mu_p(f)$. We aim for a contradiction. By the definition of iterated lower p-order, there exists an increasing sequence $\{r_n\}$, $(r_n \to +\infty)$ and a positive integer n_0 such that for all $n > n_0$ and for any given $\varepsilon > 0$

$$T(r_n, g) < \exp_{p-1}\left\{r_n^{\mu_p(g)+\varepsilon}\right\}, \; T(r_n, d) \leq \exp_{p-1}\left\{r_n^{\mu_p(d)+\varepsilon}\right\}.$$

Since $T(r_n, f) \leq T(r_n, g) + T(r_n, d) + O(1)$, we get, for all sufficiently large n,

$$T(r_n, f) \leq \exp_{p-1}\left\{r_n^{\mu_p(g)+\varepsilon}\right\} + \exp_{p-1}\left\{r_n^{\mu_p(d)+\varepsilon}\right\} + O(1),$$

hence $\mu_p(f) \leq \max\{\mu_p(g), \mu_p(d)\}$. This is a contradiction with our assumption.

Case 3. $\mu_p(f) < +\infty$ and $\rho_p(f) = +\infty$. By using the similar way in proving Cases 1 and 2, we can prove Case 3.

Finally, we will prove $\rho_{p+1}(g) = \rho_{p+1}(f)$. Suppose that $\rho_p(f) = +\infty$. Then there exists an increasing sequence $\{r_n\}$, $(r_n \to +\infty)$, such that

$$\rho_{p+1}(f) = \lim_{n\to\infty} \frac{\log_{p+1}^{+} T(r_n, f)}{\log r_n}.$$

Combining $\rho_p(d) < \mu_p(f)$ and the definitions of the iterated p-order and the iterated lower p-order, we get

$$\lim_{n\to\infty} \frac{T(r_n, d)}{T(r_n, f)} = 0.$$

Then, there exists a positive integer N, such that for $n > N$

$$T(r_n, f) \leq 2T(r_n, g) + O(1).$$

Thus, $\rho_{p+1}(f) \leq \rho_{p+1}(g)$. By using a similar argument as in proving Case 1, since $T(r,g) \leq T(r,f) + T(r,d)$, then there exists a positive integer N, such that for $n > N$

$$T(r_n, g) \leq 2T(r_n, f).$$

Hence, $\rho_{p+1}(g) \leq \rho_{p+1}(f)$. Therefore $\rho_{p+1}(f) = \rho_{p+1}(g)$. □

Remark 3.9. Lemma 3.8 was obtained for $p = 1$ by Long and Zhu in [15].

4. Proof of Theorems and Corollaries

Proof of Theorem 2.1.

Proof. By (1.3), we have

$$|A_s| \leq \left|\frac{f}{f^{(s)}}\right| \left(|A_0| + \sum_{\substack{j=1 \\ j\neq s}}^{k} |A_j| \left|\frac{f^{(j)}}{f}\right| \right). \tag{4.1}$$

Using Lemma 3.1, there exists a set $E_1 \subset (0, +\infty)$ with $m(E_1) < \infty$ and a constant $B > 0$, such that for all z satisfying $|z| = r \notin E_1$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B \left[T(2r, f)\right]^{k+1}, \ j = 1, 2, \cdots, k, \ j \neq s. \tag{4.2}$$

By Lemma 3.5, there exists a set E_4 of finite logarithmic measure such that for all $|z| = r \notin E_4$ and $|g(z)| = M(r, g)$ and for r sufficiently large

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq r^{2s}, \tag{4.3}$$

where $g(z)$ is an entire function satisfying $\mu_p(g) = \mu_p(f) = \mu \leq \rho_p(f) = \rho_p(g) \leq +\infty$, $0 < p < +\infty$. By the hypotheses of Theorem 2.1, there exists a set H with $\overline{\log dens}\{|z| : z \in H\} > 0$ (or $m_l(\{|z| : z \in H\}) = \infty$) such that for all $z \in H$ as $z \to \infty$, we have

$$|A_j(z)| \leq \exp_p\left(\beta |z|^{\rho_p(A_s)-\varepsilon}\right), \ j \neq s, j = 0, 1, \cdots, k \tag{4.4}$$

$$|A_s(z)| \geq \exp_p\left(\alpha |z|^{\rho_p(A_s)-\varepsilon}\right). \tag{4.5}$$

Set $H_1 = \{|z| : z \in H\} \setminus (E_1 \cup E_4)$, so $m_l(H_1) = \infty$. It follows from (4.1), (4.2), (4.3), (4.4) and (4.5) that for all z satisfying $|z| = r \in H_1$ and $|g(z)| = M(r, g)$,

$$\exp_p\left(\alpha |z|^{\rho_p(A_s)-\varepsilon}\right) \leq kBr^{2s}\left(T(2r, f)\right)^{k+1} \exp_p\left(\beta |z|^{\rho_p(A_s)-\varepsilon}\right),$$

using $0 \leq \beta < \alpha$, we obtain

$$\exp\left((1 - o(1)) \exp_{p-1}\left(\alpha |z|^{\rho_p(A_s)-\varepsilon}\right)\right) \leq kBr^{2s}\left(T(2r, f)\right)^{k+1}. \tag{4.6}$$

It follows from (4.6) and Lemma 3.4 that

$$\rho_p(A_s) \leq \rho_{p+1}(f).$$

On the other hand, by the hypotheses of Theorem 2.1, for sufficiently large r, we have

$$|A_j(z)| \leq \exp_p\left(r^{\rho_p(A_s)+\varepsilon}\right), \ j \neq s, \ j = 0, 1, \cdots, k \tag{4.7}$$

and by Lemma 3.6, for any given $\varepsilon > 0$, there exists a set $E_5 \subset (1, +\infty)$ of finite linear measure, such that for all z satisfying $|z| = r \notin E_5$ we obtain

$$|A_k(z)| \geq \exp\left\{-\exp_{p-1}\left(r^{\rho_p(A_k)+\varepsilon}\right)\right\} \geq \exp\left\{-\exp_{p-1}\left(r^{\rho_p(A_s)+\varepsilon}\right)\right\}. \tag{4.8}$$

It follows by (1.3) that

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \frac{1}{|A_k(z)|}\left(\sum_{j=1}^{k-1}|A_j(z)|\left|\frac{f^{(j)}(z)}{f(z)}\right| + |A_0(z)|\right). \tag{4.9}$$

By Hadamard factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions of finite iterated order satisfying $\mu_p(g) = \mu_p(f) \leq \rho_p(f) = \rho_p(g) \leq +\infty$, $0 < p < +\infty$, $i(d) < p$ or $\rho_p(d) = \lambda_p(d) = \lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$. By Lemma 3.2, there exists a set E_2 of finite logarithmic measure such that for all z satisfying $|z| = r \notin E_2$ at which $|g(z)| = M(r, g)$ we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1 + o(1)), \ j = 1, \cdots, k. \tag{4.10}$$

By substituting (4.7), (4.8) and (4.10) into (4.9), we obtain

$$\begin{aligned} &\left|\frac{\nu_g(r)}{z}\right|^k |1 + o(1)| \\ &\leq \frac{1}{\exp\left\{-\exp_{p-1}\left(r^{\rho_p(A_s)+\varepsilon}\right)\right\}}\left\{\sum_{j=1}^{k-1}\left|\frac{\nu_g(r)}{z}\right|^j |1 + o(1)| + 1\right\}\exp_p\left(r^{\rho_p(A_s)+\varepsilon}\right) \\ &= \left\{\sum_{j=1}^{k-1}\left|\frac{\nu_g(r)}{z}\right|^j |1 + o(1)| + 1\right\}\exp\left\{2\exp_{p-1}\left(r^{\rho_p(A_s)+\varepsilon}\right)\right\}. \end{aligned}$$

Hence

$$|\nu_g(r)|\,|1 + o(1)| \leq kr^k\,|1 + o(1)|\exp\left\{2\exp_{p-1}\left(r^{\rho_p(A_s)+\varepsilon}\right)\right\}, \tag{4.11}$$

for all z satisfying $|z| = r \notin ([0, 1] \cup E_2 \cup E_5)$ and $|g(z)| = M(r, g)$, $r \to +\infty$. By (4.11) and Lemma 3.4, we get

$$\limsup_{r\to+\infty}\frac{\log_{p+1}\nu_g(r)}{\log r} \leq \rho_p(A_s) + \varepsilon. \tag{4.12}$$

Since $\varepsilon > 0$ is arbitrary, by (4.12) and Lemma 3.3, we obtain

$$\rho_p(A_s) \geq \rho_{p+1}(g)$$

since $\rho_p(d) < \mu_p(f)$, so by Lemma 3.8 we have $\rho_{p+1}(g) = \rho_{p+1}(f)$. This and the fact that $\rho_p(A_s) \leq \rho_{p+1}(f)$ yield $\rho_{p+1}(f) = \rho_p(A_s)$ and $i(f) = p + 1$. Theorem 2.1 is thus proved. □

Proof of Corollary 2.3.

Proof. Setting $h = f - \varphi$, where φ is such that $i(\varphi) < p+1$ or $\rho_{p+1}(\varphi) < \rho_p(A_s)$. By Theorem 2.1, we have $i(f) = p+1$ and $\rho_{p+1}(f) = \rho_p(A_s)$. Using the properties of iterated order, we get $\rho_{p+1}(h) = \rho_{p+1}(f) = \rho_p(A_s)$. By substituting $f = h + \varphi$ into (1.3), we obtain

$$A_k(z)\,h^{(k)} + A_{k-1}(z)\,h^{(k-1)} + \cdots + A_1(z)\,h' + A_0(z)\,h$$

$$= -\left(A_k(z)\,\varphi^{(k)} + A_{k-1}(z)\,\varphi^{(k-1)} + \cdots + A_1(z)\,\varphi' + A_0(z)\,\varphi\right). \tag{4.13}$$

Set $K(z) = A_k(z)\,\varphi^{(k)} + A_{k-1}(z)\,\varphi^{(k-1)} + \cdots + A_1(z)\,\varphi' + A_0(z)\,\varphi$. If $i(\varphi) < p+1$ or $\rho_{p+1}(\varphi) < \rho_p(A_s)$, then by Theorem 2.1, we deduce that φ is not a solution of equation (1.3), implying that $K(z) \not\equiv 0$, and in this case we have $\rho_{p+1}(K) \leq \rho_{p+1}(\varphi) < \rho_p(A_s) = \rho_{p+1}(f)$, so $\max\{\rho_{p+1}(K),\ \rho_{p+1}(A_j)\,(j = 0, 1, \cdots, k)\} < \rho_{p+1}(f) = \rho_p(A_s)$ and by Lemma 3.7, we obtain $i_{\overline{\lambda}}(f - \varphi) = i_\lambda(f - \varphi) = p+1$ and $\overline{\lambda}_{p+1}(f - \varphi) = \lambda_{p+1}(f - \varphi) = \rho_{p+1}(f - \varphi) = \rho_p(A_s)$. □

Proof of Theorem 2.4.

Proof. By (1.4), we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq \frac{1}{|A_k(z)|}\left(\sum_{j=1}^{k-1}|A_j(z)|\left|\frac{f^{(j)}(z)}{f(z)}\right| + |A_0(z)| + \left|\frac{F(z)}{f(z)}\right|\right). \tag{4.14}$$

By Hadamard factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions of finite iterated order satisfying $\mu_p(g) = \mu_p(f) \leq \rho_p(f) = \rho_p(g) \leq +\infty$, $0 < p < +\infty$, $i(d) < p$ or $\rho_p(d) = \lambda_p(d) = \lambda_p\left(\frac{1}{f}\right) < \mu_p(f)$. By Lemma 3.2, there exists a set E_2 of finite logarithmic measure such that for all z satisfying $|z| = r \notin E_2$ at which $|g(z)| = M(r, g)$ we have (4.10). By the hypotheses of Theorem 2.4 and Lemma 3.6, for any given $\varepsilon > 0$, there exists a set $E_5 \subset (1, +\infty)$ of a finite linear measure, such that for all z satisfying $|z| = r \notin E_5$

$$|A_k(z)| \geq \exp\left\{-\exp_{p-1}\left(r^{\rho_p(A_k)+\varepsilon}\right)\right\} \geq \exp\left\{-\exp_{p-1}\left(r^{\rho_p(A_s)+\varepsilon}\right)\right\}. \tag{4.15}$$

On the other hand, for sufficiently large r, we have

$$|F(z)| \leq \exp_p\left(r^{\rho_p(A_s)+\varepsilon}\right),\ |A_j(z)| \leq \exp_p\left(r^{\rho_p(A_s)+\varepsilon}\right),\quad j \neq s,\ j = 0, 1, \cdots, k. \tag{4.16}$$

So, for any given ε $(0 < 2\varepsilon < \mu_p(g) - \rho_p(d))$ and sufficiently large r, we obtain

$$\left|\frac{F(z)}{f(z)}\right| = \frac{|F(z)\,d(z)|}{|g(z)|} = \frac{|F(z)|\,|d(z)|}{M(r,g)}$$

$$\leq \frac{\exp_p\left(r^{\rho_p(A_s)+\varepsilon}\right)\exp_p\left(r^{\rho_p(d)+\varepsilon}\right)}{\exp_p\left(r^{\mu_p(g)-\varepsilon}\right)} \leq \exp_p\left(r^{\rho_p(A_s)+\varepsilon}\right). \tag{4.17}$$

By substituting (4.10), (4.15), (4.16) and (4.17) into (4.14), for z satisfying $|z| = r \notin ([0,1] \cup E_2 \cup E_5)$, $r \to +\infty$ and $|g(z)| = M(r,g)$, we have

$$\left|\frac{\nu_g(r)}{z}\right|^k |1+o(1)|$$

$$\leq \frac{1}{\exp\left\{-\exp_{p-1}\left(r^{\rho_p(A_s)+\varepsilon}\right)\right\}} \left\{\sum_{j=1}^{k-1}\left|\frac{\nu_g(r)}{z}\right|^j |1+o(1)| + 2\right\} \exp_p\left(r^{\rho_p(A_s)+\varepsilon}\right)$$

$$= \left\{\sum_{j=1}^{k-1}\left|\frac{\nu_g(r)}{z}\right|^j |1+o(1)| + 2\right\} \exp\left\{2\exp_{p-1}\left(r^{\rho_p(A_s)+\varepsilon}\right)\right\}.$$

Hence

$$|\nu_g(r)|\,|1+o(1)| \leq (k+1)\, r^k\, |1+o(1)| \exp\left\{2\exp_{p-1}\left(r^{\rho_p(A_s)+\varepsilon}\right)\right\}, \tag{4.18}$$

for all z satisfying $|z| = r \notin ([0,1] \cup E_2 \cup E_5)$ and $|g(z)| = M(r,g)$, $r \to +\infty$. By (4.18) and Lemma 3.4, we get

$$\limsup_{r\to+\infty} \frac{\log_{p+1}\nu_g(r)}{\log r} \leq \rho_p(A_s) + \varepsilon. \tag{4.19}$$

Since ε $(0 < 2\varepsilon < \mu_p(g) - \rho_p(d))$ is arbitrary, by (4.19) and Lemma 3.3, we obtain

$$\rho_p(A_s) \geq \rho_{p+1}(g)$$

since $\rho_p(d) < \mu_p(f)$, so by Lemma 3.8 we have $\rho_{p+1}(g) = \rho_{p+1}(f)$, hence $\rho_{p+1}(f) \leq \rho_p(A_s)$. On the other hand, by (1.4), we have

$$|A_s| \leq \left|\frac{f}{f^{(s)}}\right| \left(\left|\frac{F(z)}{f(z)}\right| + |A_0| + \sum_{\substack{j=1\\ j\neq s}}^{k} |A_j| \left|\frac{f^{(j)}}{f}\right|\right). \tag{4.20}$$

Using Lemma 3.1, there exists a set $E_1 \subset (0,+\infty)$ with $m(E_1) < +\infty$ and a constant $B > 0$, such that for all z satisfying $|z| = r \notin E_1$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\,[T(2r,f)]^{k+1},\ j = 1,2,\cdots,k,\ j\neq s. \tag{4.21}$$

By Lemma 3.5, there exists a set E_4 of finite logarithmic measure such that for all $|z| = r \notin E_4$ and $|g(z)| = M(r,g)$ and for r sufficiently large

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leq r^{2s}. \tag{4.22}$$

By the hypotheses of Theorem 2.4, there exists a set H with $\overline{\log dens}\{|z| : z \in H\} > 0$ (or $m_l(\{|z| : z \in H\}) = \infty$) such that for all $z \in H$, we have that

(4.4) and (4.5) hold. Set $H_1 = \{|z| : z \in H\} \setminus (E_1 \cup E_4)$, so $m_l(H_1) = \infty$. By substituting (4.4), (4.5), (4.17), (4.21) and (4.22) into (4.20), for z satisfying $|z| = r \in H_1$, $r \to +\infty$ and $|g(z)| = M(r, g)$, we have

$$\exp_p\left(\alpha|z|^{\rho_p(A_s)-\varepsilon}\right) \leq (k+1)\, Br^{2s}\,(T(2r, f))^{k+1} \exp_p\left(\beta|z|^{\rho_p(A_s)-\varepsilon}\right),$$

using $0 \leq \beta < \alpha$, we obtain

$$\exp\left((1 - o(1)) \exp_{p-1}\left(\alpha|z|^{\rho_p(A_s)-\varepsilon}\right)\right) \leq (k+1)\, Br^{2s}\,(T(2r, f))^{k+1}. \tag{4.23}$$

It follows from (4.23) and Lemma 3.4 that

$$\rho_p(A_s) \leq \rho_{p+1}(f).$$

This and the fact that $\rho_p(A_s) \geq \rho_{p+1}(f)$ yield $\rho_p(A_s) = \rho_{p+1}(f)$ and $i(f) = p+1$. Since $\max\{\rho_{p+1}(F),\ \rho_{p+1}(A_j)\,(j = 0, 1, \cdots, k)\} < \rho_{p+1}(f) = \rho_p(A_s)$, then $\max\{i(F),\ i(A_j)\,(j = 0, 1, \cdots, k)\} < i(f) = p+1\ (0 < p < +\infty)$, by Lemma 3.7, we obtain $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f) = \rho_p(A_s)$. Theorem 2.4 is thus proved. □

Proof of Corollary 2.6.

Proof. Setting $h = f - \varphi$ such that $i(\varphi) < p+1$ or $\rho_{p+1}(\varphi) < \rho_p(A_s)$. By Theorem 2.4, we have $i(f) = p+1$ and $\rho_{p+1}(f) = \rho_p(A_s)$. Using the properties of iterated order, we get $\rho_{p+1}(h) = \rho_{p+1}(f) = \rho_p(A_s)$. By substituting $f = h + \varphi$ into (1.4), we get

$$A_k(z)\, h^{(k)} + A_{k-1}(z)\, h^{(k-1)} + \cdots + A_1(z)\, h' + A_0(z)\, h$$

$$= F(z) - \left(A_k(z)\, \varphi^{(k)} + A_{k-1}(z)\, \varphi^{(k-1)} + \cdots + A_1(z)\, \varphi' + A_0(z)\, \varphi\right). \tag{4.24}$$

Set $G(z) = F(z) - \left(A_k(z)\, \varphi^{(k)} + A_{k-1}(z)\, \varphi^{(k-1)} + \cdots + A_1(z)\, \varphi' + A_0(z)\, \varphi\right)$. If $i(\varphi) < p+1$ or $\rho_{p+1}(\varphi) < \rho_p(A_s)$, then by Theorem 2.4, we deduce that φ is not a solution of equation (1.4), implying that $G(z) \not\equiv 0$, and in this case we have $\rho_{p+1}(G) \leq \rho_{p+1}(\varphi) < \rho_p(A_s) = \rho_{p+1}(f)$, so $\max\{\rho_{p+1}(G),\ \rho_{p+1}(A_j)\,(j = 0, 1, \cdots, k)\} < \rho_{p+1}(f) = \rho_p(A_s)$ and by Lemma 3.7, we obtain $i_{\overline{\lambda}}(f - \varphi) = i_\lambda(f - \varphi) = p+1$ and $\overline{\lambda}_{p+1}(f - \varphi) = \lambda_{p+1}(f - \varphi) = \rho_{p+1}(f - \varphi) = \rho_p(A_s)$. □

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