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ON A LORENTZIAN PARA-SASAKIAN MANIFOLD WITH RESPECT TO THE QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract. In this paper, we study certain curvature conditions satisfied by the conharmonic curvature tensor in a Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection.

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1. Introduction

In 1989, K. Matsumoto [17] introduced the notion of Lorentzian para-Sasakian manifolds. I. Mihai and R. Rosca [19] introduced the same notion independently and obtained several results. Lorentzian para-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [18]; U. C. De, K. Matsumoto and A. A. Shaikh [3] and many others such as ([20],[23]-[25]).

A linear connection $\bar{\nabla}$ in a Riemannian manifold M is said to be a quarter-symmetric connection [8] if the torsion tensor T of the connection $\bar{\nabla}$

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y]$$

satisfies

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

where η is a 1-form and ϕ is a $(1,1)$ tensor field. If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields of the manifold M, then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection. If we put $\phi X = X$ and $\phi Y = Y$, then the quarter-symmetric metric connection reduces

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to the semi-symmetric metric connection [7]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection. A quarter-symmetric metric connection has been studied by various authors ([1], [5], [9]-[11], [14]-[16]).

A relation between the quarter-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ in an n-dimensional Lorentzian para-Sasakian manifold M is given by [21]

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{1.1}$$

T. Takahashi [26] introduced the notion of local ϕ-symmetry on a Sasakian manifold and obtained a few interesting properties. U. C. De, A.A. Shaikh and S. Biswas [6] generalized the notion of ϕ-symmetric manifolds to ϕ-recurrent manifolds in the context of Sasakian manifolds. Venkatesha and C.S.Bagewadi [27] studied concircular ϕ-recurrent LP-Sasakian manifolds which generalize the notion of locally concircular ϕ-symmetric LP-Sasakian manifolds and obtained some interesting results. Recently, U. C. De and Pradip Manjhi have studied ϕ-Weyl semisymmetric and ϕ-projectively semisymmetric generalized Sasakian space forms and gave some illustrative examples [4].

Motivated by the above studies, in this paper we study certain curvature conditions satisfied by the conharmonic curvature tensor in a Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection.

The paper is organized as follows: In Section 2, we give a brief introduction of a Lorentzian para-Sasakian manifold. In Section 3, we deduce the relation between the curvature tensor of Lorentzian para-Sasakian manifolds with respect to the quarter-symmetric metric connection and the Levi-Civita connection. Sections 4 and 5 are devoted to study conharmonically flat and ϕ-conharmonically flat Lorentzian para-Sasakian manifolds with respect to the quarter-symmetric metric connection, respectively. Section 6 deals with the study of ϕ-conharmonically semi-symmetric η-Einstein Lorentzian para-Sasakian manifolds with respect to the quarter-symmetric metric connection. In Section 7, we study Lorentzian para-Sasakian manifolds satisfying the condition $\bar{C}(\xi, X) \cdot \bar{S} = 0$. A Lorentzian para-Sasakian manifold whose curvature tensor of manifold is covariant constant with respect to the quarter-symmetric metric connection and manifold if recurrent with a Levi-Civita connection is studied in Section 8.

2. Preliminaries

A differentiable manifold of dimension n is called a Lorentzian para-Sasakian manifold, if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \tag{2.1}$$

$$g(X,\xi) = \eta(X), \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.4}$$

$$\nabla_X \xi = \phi X, \tag{2.5}$$

where ∇ denotes the covariant differentiation with respect to the Lorentzian metric g.
If we put

$$\Phi(X, Y) = g(\phi X, Y) \tag{2.6}$$

for all vector fields X and Y, then the tensor field $\Phi(X, Y)$ is a symmetric $(0, 2)$ tensor field [17]. Also since the 1-form η is closed in an LP-Sasakian manifold, we have [3]

$$(\nabla_X \eta)(Y) = \Phi(X, Y), \quad \Phi(X, \xi) = 0 \tag{2.7}$$

for all vector fields X, $Y \in \chi(M)$.

Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q in a Lorentzian para-Sasakian manifold M with respect to the Levi-Civita connection satisfy the following equations [24]:

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.8}$$

$$R(\xi, X)Y = -R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.9}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.10}$$

$$R(\xi, X)\xi = -R(X, \xi)\xi = X + \eta(X)\xi, \tag{2.11}$$

$$S(X, \xi) = (n-1)\eta(X), \quad Q\xi = (n-1)\xi, \tag{2.12}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y) \tag{2.13}$$

for all vector fields $X, Y \in \chi(M)$.

Definition 2.1. A Lorentzian para-Sasakian manifold M is said to be an η-Einstein manifold if its Ricci tensor S of type $(0, 2)$ satisfies

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.14}$$

where a and b are smooth functions on M. In particular, if $b = 0$, then an η-Einstein manifold is an Einstein manifold.

Contracting (2.14), we have

$$r = na - b. \tag{2.15}$$

On the other hand, putting $X = Y = \xi$ in (2.14) and using (2.1), (2.2) and (2.12), we also have

$$-(n-1) = -a + b. \tag{2.16}$$

Hence it follows from (2.15) and (2.16) that

$$a = \frac{r}{n-1} - 1, \qquad b = \frac{r}{n-1} - n.$$

So the Ricci tensor S of an η-Einstein Lorentzian para-Sasakian manifold is given by

$$S(X,Y) = (\frac{r}{n-1} - 1)g(X,Y) + (\frac{r}{n-1} - n)\eta(X)\eta(Y). \tag{2.17}$$

3. Curvature tensor of Lorentzian para-Sasakian manifolds with respect to the quarter-symmetric metric connection

Let R and $\bar{R}$, respectively, be the curvature tensors of the Levi-Civita connection ∇ and the quarter-symmetric metric connection $\bar{\nabla}$ in a Lorentzian para-Sasakian manifold M. Then we have [2]

$$\bar{R}(X,Y)Z = R(X,Y)Z + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi, \tag{3.1}$$

$$\bar{R}(\xi,Y)Z = -\bar{R}(Y,\xi)Z = -2\eta(Z)Y - 2\eta(X)\eta(Y)\xi, \tag{3.2}$$

$$\bar{R}(X,Y)\xi = 2\eta(Y)X - \eta(X)Y, \tag{3.3}$$

$$\bar{S}(Y,Z) = S(Y,Z) - g(Y,Z) - n\eta(Y)\eta(Z), \tag{3.4}$$

$$\bar{Q}Y = QY - Y - n\eta(Y)\xi, \quad \bar{Q}\xi = 2(n-1)\xi, \tag{3.5}$$

$$\bar{S}(Y,\xi) = 2(n-1)\eta(Y), \quad \bar{S}(\xi,\xi) = -2(n-1), \tag{3.6}$$

$$\bar{S}(\phi Y,\phi Z) = S(Y,Z) - g(Y,Z) - (n-2)\eta(Y)\eta(Z) \tag{3.7}$$

for all vector fields $X, Y, Z \in \chi(M)$.

4. Conharmonically flat Lorentzian para-Sasakian manifolds with respect to the quarter-symmetric metric connection

As a special subgroup of the conformal transformation group, Ishii [13] introduced the notion of conharmonic transformation under which a harmonic function transforms into a harmonic function. The conharmonic curvature tensor C of type $(1,3)$ in a Lorentzian para-Sasakian manifold M of dimension n is defined by ([12],[13])

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \tag{4.1}$$

for all vector fields $X, Y, Z \in \chi(M)$, which is invariant under conharmonic transformation.

Analogous to the equation (4.1), we define the conharmonic curvature tensor $\bar{C}$ in a Lorentzian para-Sasakian manifold M with respect to the quarter symmetric metric connection $\bar{\nabla}$ by

$$\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y], \tag{4.2}$$

where $\bar{R}$, $\bar{S}$ and $\bar{Q}$ are the Riemannian curvature tensor, the Ricci tensor and the Ricci operator with respect to the connection $\bar{\nabla}$, respectively. A manifold whose conharmonic curvature tensor vanishes at every point of the manifold is called conharmonically flat manifold.

Let us assume that the manifold M with respect to the quarter-symmetric metric connection is conharmonically flat, that is, $\bar{C} = 0$. Then from (4.2), we have

$$\bar{R}(X,Y)Z = \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y]. \tag{4.3}$$

By putting $Y = Z = \xi$ in (4.3) and then using (3.3), (3.6) and (2.2), we find

$$\bar{Q}X = -2X - 2n\eta(X)\xi. \tag{4.4}$$

Similarly we have

$$\bar{Q}Y = -2X - 2n\eta(Y)\xi. \tag{4.5}$$

Now from (4.3)-(4.5), we have

$$\bar{R}(X,Y)Z = \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y - 2g(Y,Z)X + 2g(X,Z)Y \tag{4.6}$$

$$-2ng(Y,Z)\eta(X)\xi + 2ng(X,Z)\eta(Y)\xi].$$

Putting $X = \xi$ and using (3.2), (3.6), (2.1) and (2.2), we get

$$\bar{S}(Y,Z) = -2(n-1)g(Y,Z) - 4(n-1)\eta(Y)\eta(Z)$$

which by using (3.4) becomes

$$S(Y,Z) = -(2n-3)g(Y,Z) - (3n-4)\eta(Y)\eta(Z). \tag{4.7}$$

Hence contracing (4.7), we obtain

$$r = -2(n-1)(n-2). \tag{4.8}$$

Thus we have the following theorem:

Theorem 4.1. *An n-dimensional conharmonically flat Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection is an η-Einstein manifold with the scalar curvature $r = -2(n-1)(n-2)$.*

5. ϕ-conharmonically flat Lorentzian para-Sasakian manifolds with respect to the quarter-symmetric metric connection

Definition 5.1 ([22]). An n-dimensional ($n > 3$) Lorentzian para-Sasakian manifold satisfying the condition

$$\phi^2 C(\phi X, \phi Y)\phi Z = 0 \tag{5.1}$$

is called ϕ-conharmonically flat.

Analogous to the equation (5.1), we define that an n-dimensional Lorentzian para-Sasakian manifold is ϕ-conharmonically flat with respect to the quarter-symmetric metric connection if it satisfies

$$\phi^2 \bar{C}(\phi X, \phi Y)\phi Z = 0 \tag{5.2}$$

for any vector fields $X, Y, Z \in \chi(M)$.

Assume that the manifold is ϕ-conharmonically flat with respect to the quarter-symmetric metric connection. Then from (5.2), we have

$$g(\bar{C}(\phi X, \phi Y)\phi Z, \phi W) = 0 \tag{5.3}$$

for any $X, Y, Z, W \in \chi(M)$. Using (4.2) in (5.3), we have

$$g(\bar{R}(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{(n-2)}[g(\phi Y, \phi Z)\bar{S}(\phi X, \phi W) \tag{5.4}$$

$$-g(\phi X, \phi Z)\bar{S}(\phi Y, \phi W) + g(\phi X, \phi W)\bar{S}(\phi Y, \phi Z) - g(\phi Y, \phi W)\bar{S}(\phi X, \phi Z)].$$

Now in view of (3.1) and (3.4), (5.4) becomes

$$g(R(\phi X,\phi Y)\phi Z,\phi W)=\frac{1}{(n-2)}[g(\phi Y,\phi Z)S(\phi X,\phi W) \tag{5.5}$$

$$-g(\phi X,\phi Z)S(\phi Y,\phi W)+g(\phi X,\phi W)S(\phi Y,\phi Z)-g(\phi Y,\phi W)S(\phi X,\phi Z)].$$

$$+\frac{1}{(n-2)}[-g(\phi Y,\phi Z)g(\phi X,\phi W)+g(\phi X,\phi Z)g(\phi Y,\phi W)$$

$$-g(\phi X,\phi W)g(\phi Y,\phi Z)+g(\phi Y,\phi W)g(\phi X,\phi Z)].$$

Let $\{e_1,e_2,.....,e_{n-1}\}$ be a local orthonormal basis of vector fields in M. Using that $\{\phi e_1,\phi e_2,.....,\phi e_{n-1},\xi\}$ is also a local orthonormal basis, if we put $X=W=e_i$ in (5.5) and sum up with respect to i, then

$$\sum_{i=1}^{n-1}g(R(\phi e_i,\phi Y)\phi Z,\phi e_i)=\frac{1}{(n-2)}\sum_{i=1}^{n-1}[g(\phi Y,\phi Z)S(\phi e_i,\phi e_i) \tag{5.6}$$

$$-g(\phi e_i,\phi Z)S(\phi Y,\phi e_i)+g(\phi e_i,\phi e_i)S(\phi Y,\phi Z)-g(\phi Y,\phi e_i)S(\phi e_i,\phi Z)].$$

$$+\frac{1}{(n-2)}\sum_{i=1}^{n-1}[-g(\phi Y,\phi Z)g(\phi e_i,\phi e_i)+g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)$$

$$-g(\phi e_i,\phi e_i)g(\phi Y,\phi Z)+g(\phi Y,\phi e_i)g(\phi e_i,\phi Z)].$$

It can be easily verified that [22]

$$\sum_{i=1}^{n-1}g(R(\phi e_i,\phi Y)\phi Z,\phi e_i)=S(\phi Y,\phi Z)+g(\phi Y,\phi Z), \tag{5.7}$$

$$\sum_{i=1}^{n-1}S(\phi e_i,\phi e_i)=r+n-1, \tag{5.8}$$

$$\sum_{i=1}^{n-1}g(\phi e_i,\phi Z)S(\phi Y,\phi e_i)=S(\phi Y,\phi Z), \tag{5.9}$$

$$\sum_{i=1}^{n-1}g(\phi e_i,\phi e_i)=n-1, \tag{5.10}$$

$$\sum_{i=1}^{n-1}g(\phi e_i,\phi Z)g(\phi Y,\phi e_i)=g(\phi Y,\phi Z). \tag{5.11}$$

By the virtue of (5.7)-(5.11), equation (5.6) can be written as

$$S(\phi Y,\phi Z)+g(\phi Y,\phi Z)=\frac{1}{(n-2)}[(r+n-1)g(\phi Y,\phi Z) \qquad (5.12)$$

$$+(n-3)S(\phi Y,\phi Z)]-2g(\phi Y,\phi Z),$$

from which it follows that

$$S(\phi Y,\phi Z)=(r-2n+5)g(\phi Y,\phi Z). \qquad (5.13)$$

By using (2.3) and (2.13) in (5.13), we get

$$S(Y,Z)=(r-2n+5)g(Y,Z)+(r-3n+6)\eta(Y)\eta(Z). \qquad (5.14)$$

Hence contracting (5.14), we obtain

$$r=\frac{2(n-1)(n-3)}{n-2}. \qquad (5.15)$$

Thus we have the following theorem:

Theorem 5.2. *Let M be an n-dimensional $(n>3)$, ϕ-conharmonically flat Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection. Then M is an η-Einstein manifold with the scalar curvature $r=\frac{2(n-1)(n-3)}{n-2}$.*

6. ϕ-conharmonically semi-symmetric η-Einstein Lorentzian para-Sasakian manifolds with respect to the quarter-symmetric metric connection

Definition 6.1. An η-Einstein Lorentzian para-Sasakian manifold (M^n,g), $n>1$ is said to be ϕ-conharmonically semisymmetric with respect to the quarter-symmetric metric connection if

$$\bar{C}(X,Y)\cdot\phi=0$$

on M for all $X,Y\in\chi(M)$.

Let M be an n-dimensional ϕ-conharmonically semi-symmetric η-Einstein Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection. Therefore $\bar{C}(X,Y)\cdot\phi=0$ becomes

$$(\bar{C}(X,Y)\cdot\phi)Z=\bar{C}(X,Y)\phi Z-\phi\bar{C}(X,Y)Z=0 \qquad (6.1)$$

for any vector fields $X,Y,Z\in\chi(M)$. From (4.2), we have

$$\bar{C}(X,Y)\phi Z=\bar{R}(X,Y)\phi Z-\frac{1}{(n-2)}[\bar{S}(Y,\phi Z)X-\bar{S}(X,\phi Z)Y \qquad (6.2)$$

$$+g(Y,\phi Z)\bar{Q}X-g(X,\phi Z)\bar{Q}Y].$$

By using (3.1), (3.4) and (3.5), the last equation takes the form

$$\bar{C}(X,Y)\phi Z = R(X,Y)\phi Z + \eta(X)g(Y,\phi Z)\xi - \eta(Y)g(X,\phi Z)\xi$$
$$-\frac{1}{(n-2)}[S(Y,\phi Z)X - g(Y,\phi Z)X - S(X,\phi Z)Y + g(X,\phi Z)Y$$
$$+g(Y,\phi Z)(QX - 2X - n\eta(X)\xi) - g(X,\phi Z)(QY - 2Y - n\eta(Y)\xi)]. \tag{6.3}$$

Also we have

$$\phi\bar{C}(X,Y)Z = \phi R(X,Y)Z + \eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X$$
$$-\frac{1}{(n-2)}[S(Y,Z)\phi X - S(X,Z)\phi Y - g(Y,Z)\phi X + g(X,Z)\phi Y$$
$$-n\eta(Y)\eta(Z)\phi X + n\eta(X)\eta(Z)\phi Y + g(Y,Z)(\phi QX - \phi X) - g(X,Z)(\phi QY - \phi Y)]. \tag{6.4}$$

By using (6.3) and (6.4), (6.1) takes the form

$$g(Y,\phi Z)X - g(X,\phi Z)Y + \eta(X)g(Y,\phi Z)\xi - \eta(Y)g(X,\phi Z)\xi - g(Y,Z)\phi X$$
$$+g(X,Z)\phi Y - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X$$
$$-\frac{1}{(n-2)}[S(Y,\phi Z)X - g(Y,\phi Z)X - S(X,\phi Z)Y + g(X,\phi Z)Y$$
$$+g(Y,\phi Z)(QX - 2X - n\eta(X)\xi) - g(X,\phi Z)(QY - 2Y - n\eta(Y)\xi)]$$
$$+\frac{1}{(n-2)}[S(Y,Z)\phi X - S(X,Z)\phi Y - g(Y,Z)\phi X + g(X,Z)\phi Y$$
$$-n\eta(Y)\eta(Z)\phi X + n\eta(X)\eta(Z)\phi Y + g(Y,Z)(\phi QX - \phi X) - g(X,Z)(\phi QY - \phi Y)] = 0. \tag{6.5}$$

Taking $Y = \xi$ and then using (2.1), (2.2), (2.12) and (3.6), (6.5) reduces to

$$S(X,\phi Z)\xi + (2n-4)g(X,\phi Z)\xi + n\eta(Z)\phi X = 0$$

which in view of (2.17) becomes

$$(\frac{r}{n-1} + 2n - 5)g(X,\phi Z)\xi + n\eta(Z)\phi X = 0.$$

Now considering Z to be orthogonal to ξ, then $\eta(Z) = 0$ and $g(X,\phi Z) \neq 0$, which implies that

$$r = -(n-1)(2n-5).$$

Thus we can state the following theorem:

Theorem 6.2. *For an n-dimensional ϕ-conharmonically semi-symmetric η-Einstein Lorentzian para-Sasakian manifold with respect to the quarter-symmetric metric connection, the scalar curvature is $r = -(n-1)(2n-5)$.*

7. Lorentzian para-Sasakian manifolds satisfying $\bar{C}(\xi, X) \cdot \bar{S} = 0$

Let us consider a Lorentzian para-Sasakian manifold satisfying $\bar{C}(\xi, X)\cdot\bar{S} = 0$. Then we have

$$\bar{S}(\bar{C}(\xi, X)Y, Z) + \bar{S}(Y, \bar{C}(\xi, X)Z) = 0. \tag{7.1}$$

In view of (4.2), we have

$$\bar{C}(\xi, X)Y = -\frac{1}{(n-2)}[-\eta(Y)X + (2n-4)\eta(X)\eta(Y)\xi + S(X,Y)\xi \\ -\eta(Y)QX + (2n-3)g(X,Y)\xi]. \tag{7.2}$$

Making use of (7.2), (7.1) takes the form

$$\bar{S}(\bar{C}(\xi, X)Y, Z) = -\frac{1}{(n-2)}[\eta(Y)g(X,Z) + 2(n-1)(2n-3)g(X,Y)\eta(Z) \\ +(4(n-1)(n-2) + n(n-1) + n)\eta(X)\eta(Y)\eta(Z) \\ +2(n-1)\eta(Z)S(X,Y) - \eta(Y)S(QX,Z)]. \tag{7.3}$$

Similarly we have

$$\bar{S}(Y, \bar{C}(\xi, X)Z) = -\frac{1}{(n-2)}[\eta(Z)g(X,Y) + 2(n-1)(2n-3)g(X,Z)\eta(Y) \\ +(4(n-1)(n-2) + n(n-1) + n)\eta(X)\eta(Y)\eta(Z) \\ +2(n-1)\eta(Y)S(X,Z) - \eta(Z)S(QX,Y)]. \tag{7.4}$$

Using (7.3) and (7.4) in (7.1), we have

$$2(4(n-1)(n-2) + n(n-1) + n)\eta(X)\eta(Y)\eta(Z) + \eta(Y)g(X,Z) \\ +\eta(Z)g(X,Y) + 2(n-1)\eta(Z)S(X,Y) + 2(n-1)\eta(Y)S(X,Z) \\ +2(n-1)(2n-3)g(X,Y)\eta(Z) + 2(n-1)(2n-3)g(X,Z)\eta(Y) \\ -\eta(Y)S(QX,Z) - \eta(Z)S(QX,Y) = 0. \tag{7.5}$$

Let λ be the eigenvalue of the endomorphism Q corresponding to an eigenvector X. Then

$$QX = \lambda X. \tag{7.6}$$

By using (7.6), (7.5) takes the form

$$2(4(n-1)(n-2) + n(n-1) + n)\eta(X)\eta(Y)\eta(Z) + \eta(Y)g(X,Z) \\ +\eta(Z)g(X,Y) + 2(n-1)\lambda\eta(Z)g(X,Y) + 2(n-1)\lambda\eta(Y)g(X,Z) \\ +2(n-1)(2n-3)g(X,Y)\eta(Z) + 2(n-1)(2n-3)g(X,Z)\eta(Y) \tag{7.7}$$

$$-\lambda^2\eta(Y)g(X,Z) - \lambda^2\eta(Z)g(X,Y) = 0$$

which after putting $Z = \xi$ reduces to

$$[\lambda^2 - 2(n-1)\lambda - 2(n-1)(2n-3) - 1]g(X,Y) \tag{7.8}$$

$$-[\lambda^2-2(n-1)\lambda-2(n-1)(2n-3)-1+2(4(n-1)(n-2)+n(n-1)+n)]\eta(X)\eta(Y) = 0.$$

By replacing $Y = \xi$ in (7.8), we get

$$[\lambda^2-2(n-1)\lambda-2(n-1)(2n-3)-1+4(n-1)(n-2)+n(n-1)+n]\eta(X) = 0.$$

This gives

$$\lambda^2 - 2(n-1)\lambda + (n-1)^2 = 0, \quad \eta(X) \neq 0.$$

Hence we can state the following:

Theorem 7.1. *If an n-dimensional Lorentzian para-Sasakian manifold satisfies $\bar{C}(\xi, X)\cdot\bar{S} = 0$, then the non-zero eigenvalues of the symmetric endomorphism Q of the tangent space corresponding to S are congruent, such as $(n-1)$.*

8. A Lorentzian para-Sasakian manifold whose curvature tensor of manifold is covariant constant with respect to the quarter-symmetric metric connection and M is recurrent with respect to the Levi-Civita connection

Definition 8.1 ([2])**.** A Lorentzian para-Sasakian manifold with respect to the Levi-Civita connection is called the recurrent, if its curvature tensor R satisfies the condition

$$(\nabla_W R)(X,Y)Z = A(W)R(X,Y)Z, \tag{8.1}$$

where A is the 1-form

Analogous to the equation (8.1), a Lorentzian para-Sasakian manifold with respect to the quarter symmetric metric connection $\bar{\nabla}$ is called the recurrent, if its curvature tensor $\bar{R}$ satisfies the condition

$$(\bar{\nabla}_W \bar{R})(X,Y)Z = A(W)\bar{R}(X,Y)Z, \tag{8.2}$$

where $\bar{R}$ is the curvature tensor with respect to the connection $\bar{\nabla}$.

Theorem 8.2. *If an n-dimensional Lorentzian para-Sasakian manifold whose curvature tensor of manifold is covariant constant with respect to the quarter-symmetric metric connection and the manifold is recurrrent with respect to the Levi-Civita connection and the associated 1-form A is equal to the associated 1-form η, then the scalar curvatre r vanishes by providing the trace of ϕ is zero.*

Proof. From (1.1), (2.6) and (2.8), we have

$$(\bar{\nabla}_W R)(X,Y)Z = \bar{\nabla}_W R(X,Y)Z - R(\bar{\nabla}_W X,Y)Z - R(X,\bar{\nabla}_W Y)Z \tag{8.3}$$

$$-R(X,Y)\bar{\nabla}_W Z = (\nabla_W R)(X,Y)Z + 2\eta(W)[g(Y,\phi Z)X - g(X,\phi Z)Y].$$

Suppose that $(\bar{\nabla}_W R)(X,Y)Z = 0$, then from (8.3) it follows that

$$(\nabla_W R)(X,Y)Z + 2\eta(W)[g(Y,\phi Z)X - g(X,\phi Z)Y] = 0 \tag{8.4}$$

which after applying (8.1) becomes

$$A(W)R(X,Y)Z + 2\eta(W)[g(Y,\phi Z)X - g(X,\phi Z)Y] = 0. \tag{8.5}$$

Now contracting X in (8.5), we get

$$A(W)S(Y,Z) + 2(n-1)g(Y,\phi Z)\eta(W) = 0. \tag{8.6}$$

Suppose the associated 1-form A is equal to the associated 1-form η, then from the last equation, we get

$$S(Y,Z) = -2(n-1)g(Y,\phi Z), \qquad \eta(W) \neq 0. \tag{8.7}$$

Hence contracting (8.7), we get

$$r = -2(n-1)\psi, \quad \text{where} \quad \psi = trace\phi \tag{8.8}$$

which completes the proof of the theorem. □

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