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ON NEARLY QUASI-EINSTEIN WARPED PRODUCTS

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Abstract. We study nearly quasi-Einstein warped product manifolds for arbitrary dimension $n \geq 3$. In the last section we also give an example of warped product on nearly quasi-Einstein manifold.

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1. Introduction

A Riemannian manifold (M^n, g), $(n > 2)$ is Einstein if its Ricci tensor S of type (0,2) is of the form $S = \alpha g$, where α is smooth function, which turns into $S = \frac{r}{n}g$, r being the scalar curvature of the manifold. Let (M^n, g), $(n > 2)$ be a Riemannian manifold and $U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\}$, then the manifold (M^n, g) is said to be quasi-Einstein manifold [1, 2] if on $U_S \subset M$, we have

$$S - \alpha g = \beta A \otimes A, \tag{1}$$

where A is a 1-form on U_S and α and β some functions on U_S. It is clear that the 1-form A as well as the function β are nonzero at every point on U_S. From the above definition, it follows that every Einstein manifold is quasi-Einstein. In particular, every Ricci-flat manifold (e.g., Schwarzchild spacetime) is quasi-Einstein.The scalars α, β are known as the associated scalars of the manifold. Also, the 1-form A is called the associated 1-form of the manifold defined by $g(X, \rho) = A(X)$ for any vector field X, ρ being a unit vector field, called the generator of the manifold. Such an n-dimensional quasi-Einstein manifold is denoted by $(QE)_n$.

In [3], De and Gazi introduced nearly quasi-Einstein manifold, denoted by $N(QE)_n$ and gave an example of a 4-dimensional Riemannian nearly quasi Einstein manifold, where the Ricci tensor S of type (0,2) which is not identically zero satisfies the condition

$$S(X, Y) = lg(X, Y) + mD(X, Y), \tag{2}$$

where l and m are non-zero scalars and D is a non-zero symmetric tensor of type (0,2).

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Also in [3], De and Gazi introduced the notion of a Riemannian manifold (M, g) of a nearly quasi-constant sectional curvature as a Riemannian manifold with the curvature tensor satisfies the condition

(3)
$$\begin{aligned} R(X,Y,Z,W) =& a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + b[g(Y,Z)D(X,W) \\ & - g(X,Z)D(Y,W) + g(X,W)D(Y,Z) - g(Y,W)D(X,Z)], \end{aligned}$$

where a, b are scalar functions with $b \neq 0$ and D is nonzero symmetric (0,2) tensor.

Let M be an m-dimensional, $m \geq 3$, Riemannian manifold and $p \in M$. Denote by $K(\pi)$ or $K(u \wedge v)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, where $\{u, v\}$ is an orthonormal basis of π. For any n-dimensional subspace $L \subseteq T_pM$, $2 \leq n \leq m$, its scalar curvature $\tau(L)$ is denoted in [4] by $\tau(L) = 2\Sigma_{1\leq i<j\leq n}K(e_i \wedge e_j)$, where $\{e_1, e_2, ...e_n\}$ is any orthonormal basis of L [5]. When $L = T_pM$, the scalar curvature $\tau(L)$ is just the scalar curvature $\tau(p)$ of M at p.

2. Warped product manifolds

The notion of warped product generalizes that of a surface of revolution. It was introduced in [6] for studying manifolds of negative curvature. Let (B, g_B) and (F, g_F) be two Riemannian manifolds and let f be a positive differentiable function on B. Consider the product manifold $B \times F$ with its projections $\pi : B \times F \to B$ and $\sigma : B \times F \to F$. The warped product $B \times_f F$ is the manifold $B \times F$ with the Riemannian structure such that $||X||^2 = ||\pi^*(X)||^2 + f^2(\pi(p))||\sigma^*(X)||^2$, for any vector field X on M. Thus we have

$$g = g_B + f^2 g_F \tag{4}$$

holds on M. The function f is called the warping function of the warped product [8].

Since $B \times_f F$ is a warped product, then we have $\nabla_X Z = \nabla_Z X = (X ln f)Z$ for unit vector fields X, Z on B and F, respectively. Hence, we find $K(X \wedge Z) = g(\nabla_Z\nabla_X X - \nabla_X\nabla_Z X, Z) = (1/f)\{\nabla_X X_f - X^2 f\}$. If we chose a local orthonormal frame $e_1, ..., e_n$ such that $e_1, ..., e_{n_1}$ are tangent to B and $e_{n_1+1}, ..., e_n$ are tangent to F, then we have

$$\frac{\Delta f}{f} = \sum_{i=1}^{n} K(e_i \wedge e_j), \tag{5}$$

for each $s = n_1 + 1, ..., n$ [7]. We need the following two lemmas from [7], for later use:

Lemma 2.1. *Let $M = B \times_f F$ be a warped product, with Riemannian curvature tensor R_M. Given field X, Y, Z on B and U, V, W on F, then:*

(1) $R_M(X, Y)Z = R_B(X, Y)Z$,

(2) $R_M(V, X)Y = -(H^f(X, Y)/f)V$, where H^f is the Hessian of f,

(3) $R_M(X,Y)V = R_M(V,W)X = 0$,
(4) $R_M(X,V)W = -(g(V,W)/f)\nabla_X(grad\ f)$,
(5) $R_M(V,W)U = R_F(V,W)U + (||grad\ f||^2\ /\ f^2)\{g(V,U)W - g(W,U)V\}$.

Lemma 2.2. *Let $M = B \times_f F$ be a warped product, with Ricci tensor S_M. Given fields X, Y on B and V, W on F, then:*
(1) $S_M(X,Y) = S_B(X,Y) - \frac{d}{f}H^f(X,Y)$, *where* $d = dim\, F$
(2) $S_M(X,V) = 0$,
(3) $S_M(V,W) = S_F(V,W) - g(V,W)f^{\#}, f^{\#} = \frac{\Delta f}{f} + \frac{d-1}{f^2}||grad\ f||^2$, *where Δf is the Laplacian of f on B.*

Moreover, the scalar curvature τ_M of the manifold M satisfies the condition

$$\tau_M = \tau_B + \frac{\tau_F}{f^2} - 2d\frac{\Delta f}{f} - d(d-1)\frac{|\nabla f|^2}{f^2}, \tag{6}$$

where τ_B and τ_F are the scalar curvatures of B and F, respectively.

In [8], Gebarowski studied Einstein warped product manifolds and proved the following three theorems.

Theorem 2.1. *Let (M,g) be a warped product $I \times_f F$, $dim\, I = 1$, $dim\, F = n-1$ $(n \geq 3)$. Then (M,g) is an Einstein manifold if and only if F is Einstein with constant scalar curvature τ_F in the case $n = 3$ and f is given by one of the following formulae, for any real number b,*

$$f^2(t) = \left\{ \begin{array}{ll} \frac{4}{a}K\sinh^2\frac{\sqrt{a}(t+b)}{2}, & a > 0 \\ K(t+b)^2, & a = 0 \\ -\frac{4}{a}K\sin^2\frac{\sqrt{-a}(t+b)}{2}, & a < 0 \end{array} \right\}$$

for $K > 0$, $f^2(t) = b\ exp(at)$ $\quad (a \neq 0)$, for $K = 0$, $f^2(t) = -\frac{4}{a}K\cosh^2\frac{\sqrt{a}(t+b)}{2}$, $(a > 0)$, for $K < 0$, where a is the constant appearing after first integration of the equation $q^{''}e^q + 2K = 0$ and $K = \frac{\tau_F}{(n-1)(n-2)}$.

Theorem 2.2. *Let (M,g) be a warped product $B \times_f F$ of a complete connected r-dimensional $(1 < r < n)$ Riemannian manifold B and an $(n-r)$-dimensional Riemannian manifold F. If (M,g) is a space of constant sectional curvature $K > 0$, then B is a sphere of radius $\frac{1}{\sqrt{K}}$.*

Theorem 2.3. *Let (M,g) be a warped product $B \times_f F$ of a complete connected $n-1$-dimensional Riemannian manifold B and an one-dimensional Riemannian manifold F. If (M,g) is an Einstein manifold with scalar curvature $\tau_M > 0$ and the Hessian of f is proportional to the metric tensor g_B, then*
(1) (B, g_B) *is an $(n-1)$-dimensional sphere of radius* $\rho = (\frac{\tau_B}{(n-1)(n-2)})^{-\frac{1}{2}}$.
(2) (M,g) *is a space of constant sectional curvature* $K = \frac{\tau_M}{n(n-1)}$.

Motivated by the above study by Gebarowski, in the present paper our aim is to generalize Theorems 2.1, 2.2 and 2.3 for nearly quasi-Einstein manifolds. Also in the last section we give an example of warped product on nearly quasi-Einstein manifold.

3. Nearly quasi-Einstein warped products

In this section, we consider nearly quasi-Einstein warped product manifolds and prove some results concerning these type manifolds.

Theorem 3.1. *Let (M, g) be a warped product $I \times_f F$, $dim\, I = 1$, $dim\, F = n-1$ $(n \geq 3)$. If (M, g) is nearly quasi-Einstein manifold with associated scalars l, m, then F is a nearly quasi-Einstein manifold.*

Proof. Let us consider $(dt)^2$ to be the metric on I. Taking $f = \exp\{\frac{q}{2}\}$ and making use of Lemma 2.2, we can write

$$S_M(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -\frac{n-1}{4}[2q^{''} + (q^{'})^2] \tag{7}$$

and

$$S_M(V, W) = S_F(V, W) - \frac{1}{4}e^q[2q^{''} + (n-1)(q^{'})^2]g_F(V, W), \tag{8}$$

for all vector fields V, W on F.

Since M is nearly quasi-Einstein, from (2) we have

$$S_M(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = lg(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) + mD(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}), \tag{9}$$

and

$$S_M(V, W) = lg(V, W) + mD(V, W). \tag{10}$$

On the other hand, using (5), the equations (9) and (10) reduce to

$$S_M(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = l + mD(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) \tag{11}$$

and

$$S_M(V, W) = le^q g_F(V, W) + mD_F(V, W). \tag{12}$$

Comparing the right hand side of the equations (7) and (11) we get

$$l + mD(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -\frac{n-1}{4}[2q^{''} + (q^{'})^2]. \tag{13}$$

Similarly, comparing the right hand sides of (8) and (12) we obtain

$$S_F(V, W) = \frac{1}{4}e^q[2q^{''} + (n-1)(q^{'})^2 + 4l]g_F(V, W) + mD_F(V, W). \tag{14}$$

which implies that F is a nearly quasi-Einstein manifold. This completes the proof of the theorem. □

Theorem 3.2. *Let (M, g) be a warped product $B \times_f F$ of a complete connected r-dimensional $(1 < r < n)$ Riemannian manifold B and an $(n-r)$-dimensional Riemannian manifold F.*

If (M, g) is a space of nearly quasi-constant sectional curvature, the Hessian of f is proportional to the metric tensor g_B, then B is a nearly quasi-Einstein manifold.

Proof. Assume that M is a space of nearly quasi-constant sectional curvature. Then from equation (3), we can write

(15)
$$\begin{aligned} R(X,Y,Z,W) =& a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + b[g(Y,Z)D(X,W) \\ & - g(X,Z)D(Y,W) + g(X,W)D(Y,Z) - g(Y,W)D(X,Z)], \end{aligned}$$

for all vector fields X, Y, Z, W on B.

In view of Lemma 2.1 and by using (4) in equation (15) and then after a contraction over X and W (we put $X = W = e_i$), we get

$$S_B(Y,Z) = [a(r-1) + bD_B(e_i, e_i)]g_B(Y,Z) + brD_B(Y,Z), \tag{16}$$

which shows us B is a nearly quasi-Einstein manifold. Contracting from (16) over Y and Z, we can write

$$\tau_B = ar(r-1) + 2rbD_B(e_i, e_i). \tag{17}$$

Since M is a space of nearly quasi-constant sectional curvature, in view of (5) and (15) we get

$$\frac{\Delta f}{f} = \frac{ar + brD_B(e_i, e_i)}{2}. \tag{18}$$

On the other hand, since the Hesssian of f is proportional to the metric tensor g_B, it can be written as follows

$$H^f(X,Y) = \frac{\Delta f}{r} g_B(X,Y). \tag{19}$$

Then by use of (17) and (18) in (19) we obtain $H^f + Kfg_B(X,Y) = 0$, where $K = \frac{r(3-r)bD_B(e_i,e_i) - \tau_B}{2r(r-1)}$ holds on B. So by Obata's theorem [9], B is isometric to the sphere of radius $\frac{1}{\sqrt{K}}$ in the $(r+1)$-dimensional Euclidean space. This gives us that B is a nearly quasi-Einstein manifold. Since $b \neq 0$ and also $r \neq 0$, therefore B is a nearly quasi-Einstein manifold of dimension $n \geq 2$. □

Theorem 3.3. *Let (M, g) be a warped product $B \times_f F$ of a complete connected $n-1$-dimensional Riemannian manifold B and one-dimensional Riemannian manifold I. If (M, g) is a nearly quasi-Einstein manifold with constant associated scalars l, m and the Hessian of f is proportional to the metric tensor g_B, then (B, g_B) is an $(n-1)$-dimensional sphere of radius $\varrho = \frac{n-1}{\sqrt{\tau_B + l}}$.*

Proof. Assume that M is a warped product manifold. Then by use of Lemma 2.2 we can write

$$S_B(X,Y) = S_M(X,Y) + \frac{1}{f}H^f(X,Y) \tag{20}$$

for any vector fields X, Y on B. On the other hand, since M is a nearly quasi-Einstein manifold we have

$$S_M(X,Y) = lg(X,Y) + mD(X,Y). \tag{21}$$

In view of (4) and (21) the equation (20) can be written as

$$S_B(X,Y) = lg_B(X,Y) + mD_B(X,Y) + \frac{1}{f}H^f(X,Y). \tag{22}$$

By a contraction from the above equation over X, Y, we find

$$\tau_B = l(n-1) + mD_B(e_i,e_i) + \frac{\Delta f}{f}. \tag{23}$$

On the other hand, we know from the equation (21) that

$$\tau_M = ln + mD_B(e_i,e_i). \tag{24}$$

By use of (24) in (23) we get $\tau_B = \tau_M - l + \frac{\Delta f}{f}$. In view of Lemma 2.2 we also know that

$$-\frac{\tau_M}{n} = \frac{\Delta f}{f}. \tag{25}$$

The last two equations give us $\tau_B = \frac{n-1}{n}\tau_M - l$. On the other hand, since the Hessian of f is proportional to the metric tensor g_B, it can be written as follows $H^f(X,Y) = \frac{\Delta f}{n-1}g_B(X,Y)$. As the consequence of the equation (25) we have $\frac{\Delta f}{n-1} = -\frac{1}{n(n-1)}\tau_M f$, which implies that

$$H^f(X,Y) + \frac{\tau_B + l}{(n-1)^2} f g_B(X,Y) = 0.$$

So, B is isometric to the $(n-1)$-dimensional sphere of radius $\frac{n-1}{\sqrt{\tau_B + l}}$. Hence the Theorem is proved. □

4. Example of warped product on nearly quasi-Einstein manifold

In [3], De and Gazi established the 4-dimensional example of nearly quasi-Einstein manifold. Let (M_4, g) be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = (dx^4)^2 + (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M_4. Then they have shown that it is nearly quasi-Einstein manifold with nonzero and nonconstant scalar curvature.

To define warped product on $N(QE)_4$, we consider the warping function $f : \mathbf{R} \longrightarrow (0, \infty)$ by $f(x^4) = \sqrt{(x^4)^{\frac{4}{3}}}$, here we observe that $f = \sqrt{(x^4)^{\frac{4}{3}}} > 0$ and is a smooth function. The line element defined on $\mathbf{R} \times \mathbf{R^3}$ which is of the form $I \times_f F$, where $I = \mathbf{R}$ is the base and $F = \mathbf{R^3}$ is the fibre.

Therefore the metric $ds_M^2 = ds_B^2 + f^2 ds_F^2$ that is

$$ds^2 = g_{ij} dx^i dx^j = (dx^4)^2 + (x^4)^{\frac{4}{3}} [(dx^1)^2 + (dx^2)^2 + (dx^3)^2],$$

is the example of Riemannian warped product on $N(QE)_4$.

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