

# SEMILATTICES OF NIL-EXTENSIONS OF SIMPLE LEFT (RIGHT) $\pi$ -REGULAR ORDERED SEMIGROUPS

QingShun Zhu<sup>1</sup>

**Abstract.** The purpose of this paper is to describe the semilattice of nil-extensions of simple left (right)  $\pi$ -regular ordered semigroups. We will divide this discussion into two parts. In the first part we will give the characterizations of semilattice of nil-extensions of simple left (right)  $\pi$ -regular ordered semigroups. As applications, in the second part we characterize nil-extensions of simple completely  $\pi$ -regular ordered semigroups.

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## 1. Introduction

We are often interested in building more complex semigroups, lattices, ordered sets, and ordered or topological semigroups out of some of “simple” structure and this can be sometimes achieved by constructing the ideal extensions. The ideal extensions of semigroups -without order- have been first considered by A. H. Clifford (1950) [1] with exposition of the theory appearing in [2, 10]. Ideal extensions of ordered semigroups have been studied in [5]. For nil-extensions of simple ordered semigroups we refer to [3]. The aim of this paper is to study semilattice decompositions of ordered semigroups which are nil-extensions of simple left (right)  $\pi$ -regular ordered semigroups. As applications, we characterize completely  $\pi$ -regular ordered semigroups.

## 2. Preliminaries

Throughout this paper,  $Z^+$  will denote the set of all positive integers. An *ordered semigroup*  $(S, \cdot, \leq)$  is an ordered set  $(S, \leq)$  at the same time a semigroup  $(S, \cdot)$  such that: for any  $a, b, x \in S$ ,  $a \leq b$  implies  $ax \leq bx$  and  $xa \leq xb$ . For  $H \subseteq S$ , we denote  $(H) := \{t \in S | t \leq h \text{ for some } h \in H\}$ . For  $H = \{a\}$ , we write  $(a)$  instead of  $(\{a\})$  ( $a \in S$ ). A subsemigroup  $T$  of an ordered semigroup  $S$  is *completely regular* if it is regular, left regular and right regular [6]. Equivalently,  $a \in (a^2Ta^2)$  for any  $a \in T$  [11]. A subsemigroup  $T$  of  $S$  is *completely  $\pi$ -regular* if for every  $a \in T$ , there exists  $m \in Z^+$  such that  $a^m \in (a^{2m}Ta^{2m})$  [12].  $S$  is

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<sup>1</sup>Department of Statistics and Applied Mathematics, Institutes of Sciences Information Engineering University, ZhengZhou, 450001, P.R.China, e-mail: zqs65@126.com

called a *nil-extension* of an ordered semigroup  $K$  if: (i)  $K$  is an ideal of  $S$ ; and (ii) for every  $a \in S, a^n \in K$  for some  $n \in \mathbb{Z}^+$  [3].  $S$  is called *Archimedean*, if for any  $a, b \in S$  there exists  $n \in \mathbb{Z}^+$  such that  $a^n \in (SbS)$ .

An ordered semigroup  $S$  is called a *complete semilattice of subsemigroups*, if there exists a complete semilattice  $Y$  and a family  $\{S_\alpha | \alpha \in Y\}$  of subsemigroups of  $S$  such that

- (i)  $S_\alpha \cap S_\beta = \emptyset, \forall \alpha, \beta \in Y, \alpha \neq \beta$ .
- (ii)  $S = \bigcup \{S_\alpha | \alpha \in Y\}$ .
- (iii)  $S_\alpha S_\beta \subseteq S_{\alpha\beta}, \forall \alpha, \beta \in Y$ .
- (iv)  $S_\alpha \cap (S_\beta) \neq \emptyset$  implies  $\alpha \preceq \beta$ , where “ $\preceq$ ” is the order of the semilattice  $Y$  defined as follows:  $\preceq := \{(\alpha, \beta) | \alpha = \alpha\beta (= \beta\alpha)\}$  [7].

**Lemma 2.1.** *Let an ordered semigroup  $S$  be a complete semilattice  $Y$  of subsemigroups  $S_\alpha (\alpha \in Y)$ . Then  $S$  is left  $\pi$ -regular (right  $\pi$ -regular, completely  $\pi$ -regular) if and only if  $S_\alpha$  is left  $\pi$ -regular (right  $\pi$ -regular, completely  $\pi$ -regular) for all  $\alpha (\alpha \in Y)$*

*Proof.* We deal with the left  $\pi$ -regularity only. The proof is similar for other  $\pi$ -regularity. Let  $S$  be left  $\pi$ -regular, and let  $\alpha \in Y$  be an arbitrary element. We prove that  $S_\alpha$  is left  $\pi$ -regular. As  $S$  is left  $\pi$ -regular, for every  $a \in S_\alpha$ , there are elements  $\beta \in Y$  and  $x \in S_\beta$  such that  $a \leq xa^2 \leq x^2a^3 = (x^2a)a^2$ . By (iv) of the definition of complete semilattice of subsemigroups, we have  $\alpha \leq \beta\alpha = \alpha\beta$  and so  $\alpha = \alpha\beta$ . Thus  $x^2a \in S_\alpha$ . From this result it follows that  $S_\alpha$  is left  $\pi$ -regular. The converse statement is obvious.  $\square$

Let  $S$  be an ordered semigroup. A subsemigroup  $F$  of  $S$  is called a *filter* of  $S$  if (1)  $a, b \in S$  such that  $ab \in F$  implies  $a \in F$  or  $b \in F$  and (2) if  $a \in F$  and  $b \in S$  such that  $b \geq a$ , then  $b \in F$ . Denote by  $\mathcal{N}$  the relation on  $S$  defined by  $\mathcal{N} := \{(x, y) | N(x) = N(y)\}$  where  $N(a)$  denotes the filter of  $S$  generated by  $a (a \in S)$ . The relation  $\mathcal{N}$  is the least complete semilattice congruence on  $S$  [8].

### 3. Main results

Now, we consider ordered semigroups which are nil-extensions of simple left (right)  $\pi$ -regular ordered semigroups. We denote by  $LReg(S), Intra(S)$  the set of all left regular and intra-regular elements of an ordered semigroup  $S$ , respectively.

**Lemma 3.1** ([3]). *Let  $S$  be an Archimedean ordered semigroup. If  $Intra(S) \neq \emptyset$ , then*

- (i)  $S$  has a kernel  $K(S)$ , and  $K(S) = (SaS], Intra(S) \subseteq K(S)$  for every  $a \in Intra(S)$ .
- (ii)  $S$  is a nil-extension of the simple ordered semigroup  $K(S)$ .

**Theorem 3.2.** *Let  $S$  be an ordered semigroup. Then the following conditions are equivalent:*

- (i)  $S$  is a nil-extension of a simple left (right)  $\pi$ -regular ordered semigroup;

- (ii)  $(\forall a, b \in S)(\exists n \in Z^+)a^n \in (Sb^mSa^{2n})(a^n \in (a^{2n}Sb^mS))$  for every  $m \in Z^+$ .
- (iii)  $(\forall a, b \in S)(\exists n \in Z^+)a^n \in (SbSa^{2n})(a^n \in (a^{2n}SbS))$ .
- (iv)  $S$  is Archimedean left (right)  $\pi$ -regular.

*Proof.* (i) $\Rightarrow$ (ii) Let  $S$  be a nil-extension of a simple left  $\pi$ -regular ordered semigroup  $K$ . Assume  $a \in S$ . Then  $a^k \in K$ , for some  $k \in Z^+$ . Since  $K$  is left  $\pi$ -regular, for  $a^k$  there exist  $r \in Z^+$  and  $x \in K$  such that  $(a^k)^r \leq x(a^k)^{2r}$ , i.e.,  $a^n \leq xa^{2n}$ , where  $n = kr \in Z^+$ . For every  $b \in S$ , since  $K$  is an ideal of  $S$ , we have  $a^kb^m \in KS \subseteq K$  for every  $m \in Z^+$ . But  $K$  is simple, and so for  $a^n, a^kb^m \in K$  there exist  $u, v \in K$  such that  $a^n \leq ua^kb^mv$ . Now  $a^n \leq xa^{2n} \leq x(xa^{2n})a^n \leq x^2ua^kb^mva^{2n} = (x^2ua^k)b^mva^{2n}$ , which shows that  $a^n \in (Sb^mSa^{2n})$ .

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) The implications follow immediately.

(iv) $\Rightarrow$ (i) Let  $S$  be an Archimedean left  $\pi$ -regular. Clearly,  $S$  is intra- $\pi$ -regular and  $LReg(S) \subseteq Intra(S)$ , and so  $Intra(S) \neq \emptyset$ . Assume  $a \in Intra(S)$ . By Lemma 3.1, we conclude that  $S$  is a nil-extension of simple ordered subsemigroups  $K(S)$ . Let  $a \in K(S)$ . Since  $S$  is left  $\pi$ -regular, for  $a \in S$  there exist  $m \in Z^+$  and  $u \in S$  such that  $a^m \leq ua^{2m}$ . From  $a^m \leq ua^m(ua^{2m}) = (ua^m)u a^{2m}$ , in which  $ua^m u \in SK(S)S \subseteq K(S)$ , we obtain  $a^m \in (K(S)a^{2m}]_{K(S)}$ , i.e.,  $a^m \in LReg(K(S))$ . Thus  $K(S)$  is left  $\pi$ -regular. □

**Lemma 3.3.** ([12, Theorem 2.1]) *Let  $S$  be an ordered semigroup. Then the following conditions are equivalent*

- (i)  $S$  is completely  $\pi$ -regular.  
 (ii) For any  $a \in S$ , there exists  $m \in Z^+$  such that  $a^m \in (a^{2m}Sa^{2m})$ .  
 (iii)  $S$  is left and right  $\pi$ -regular.  
 (vi) every left (right) ideal of  $S$  is left and right  $\pi$ -regular.

**Corollary 3.4.** *Let  $S$  be an ordered semigroup. Then the following conditions are equivalent:*

- (i)  $S$  is a nil-extension of a simple completely  $\pi$ -regular ordered semigroup;  
 (ii)  $(\forall a, b \in S)(\exists n \in Z^+)a^n \in (a^{2n}Sb^mSa^{2n})$  for every  $m \in Z^+$ ;  
 (iii)  $(\forall a, b \in S)(\exists n \in Z^+)a^n \in (a^{2n}SbSa^{2n})$ ;  
 (iv)  $S$  is an Archimedean completely  $\pi$ -regular ordered semigroup.

*Proof.* The proof of this corollary is similar to Theorem 3.2, by Theorem 3.2 and Lemma 3.3. □

Let  $S$  be an ordered semigroup. We say that  $S$  has the  $P$ -property if for  $a, b \in S$ ,  $b \in I(a)$  implies  $b^k \in I(a^2)$  for some  $k \in Z^+$  [9]. If  $S$  is an Archimedean ordered semigroup, then for each pair  $a, b \in S$  one can find  $k \in Z^+$  such that  $b^k \in I(a^2)$ . Hence, every Archimedean ordered semigroup has the  $P$ -property. In [9, Theorem 2.8], the authors prove that: An ordered semigroup has the  $P$ -property if and only if it is a complete semilattice of Archimedean ordered semigroups, by [4, Theorem 1.7], we have the following lemma.

**Lemma 3.5.** *Let  $S$  be an ordered semigroup. Then the following conditions are equivalent:*

- (i)  $S$  is a complete semilattice of Archimedean ordered semigroups;
- (ii)  $S$  has the  $P$ -property;
- (iii)  $S$  is a semilattice of Archimedean ordered semigroups;
- (iv)  $(\forall a, b \in S)(\exists n \in \mathbb{Z}^+)(ab)^n \in (Sa^2S)$ ;
- (v)  $\mathcal{N}$  is the greatest semilattice congruence on  $S$  such that each of its congruence classes is an Archimedean ordered subsemigroup.

**Theorem 3.6.** *Let  $S$  be an ordered semigroup. Then the following conditions are equivalent:*

- (i)  $S$  is a complete semilattice of nil-extensions of simple left (right)  $\pi$ -regular ordered semigroups;
- (ii)  $S$  is left (right)  $\pi$ -regular and each  $\mathcal{I}$ -class of  $S$  containing a left (right) regular element is a subsemigroup;
- (iii)  $(\forall a, b \in S)(\exists n \in \mathbb{Z}^+)(ab)^n \in (S(ba)^n(ab)^nS(ab)^{2n})$ ;  
 $((\forall a, b \in S)(\exists n \in \mathbb{Z}^+)(ab)^n \in ((ab)^{2n}S(ba)^n(ab)^nS))$ ;
- (iv)  $(\forall a, b \in S)(\exists n \in \mathbb{Z}^+)(ab)^n \in (Sa^2S(ab)^{2n})((ab)^n \in ((ab)^{2n}Sa^2S))$ ;
- (v)  $S$  is left (right)  $\pi$ -regular and has the  $P$ -property.
- (vi)  $\mathcal{N}$  is the unique complete semilattice congruence on  $S$  such that each of its congruence classes is a nil-extension of a simple left (right)  $\pi$ -regular ordered semigroup.

*Proof.* (i) $\Rightarrow$ (ii) Let  $S$  be a complete semilattice  $Y$  of subsemigroups  $S_\alpha, \alpha \in Y$  which are nil-extensions of simple left  $\pi$ -regular ordered semigroups  $K_\alpha$ . By Lemma 2.1 and Theorem 3.2,  $S$  is left  $\pi$ -regular. Let  $T$  be a  $\mathcal{I}$ -class of  $S$  containing a left regular element  $a$ , and let  $a \in S_\alpha$ , for some  $\alpha \in Y$ . Then  $a \leq xa^2$ , for some  $x \in S$ , whence  $a \leq (xa)^na$ , for each  $n \in \mathbb{Z}^+$ . It is easy to verify that  $xa \in S_\alpha$ , so  $(xa)^m \in K_\alpha$ , for some  $m \in \mathbb{Z}^+$ . Now,  $a \leq (xa)^ma \in K_\alpha S_\alpha \subseteq K_\alpha$ . Thus,  $a \in K_\alpha$ . Since  $K_\alpha$  is simple, then every element of  $K_\alpha$  is  $\mathcal{I}$ -related with  $a$  in  $S$ , so  $K_\alpha \subseteq T$ . Further, assume  $b \in T$ . Then  $(a, b) \in \mathcal{I}$ , so  $b \in S_\alpha$ , and since  $b \leq uav$ , for some  $u, v \in S^1$ , then  $b \leq uxa^2v \leq u(xa)^2av = (uxax)a(av)$ . It is not hard to check that  $uxax, av \in S_\alpha$ , so  $b \in S_\alpha K_\alpha S_\alpha \subseteq K_\alpha$ , whence  $T \subseteq K_\alpha$ . Therefore,  $T = K_\alpha$ , so it is a subsemigroup of  $S_\alpha$ .

(ii) $\Rightarrow$ (iii) Let  $a, b \in S$ . Since  $S$  is left  $\pi$ -regular, then  $(ab)^n \leq x(ab)^{2n} \leq x^2(ab)^{3n}$ , for some  $n \in \mathbb{Z}^+, x \in S$ , whence  $(ab)^n \in (S(ba)^{n+1}S)$ , and clearly,  $(ba)^{n+1} \in (S(ab)^nS)$ , whence  $((ba)^{n+1}, (ab)^n) \in \mathcal{I}$ , i.e.,  $(ba)^{n+1} \in T$ , where  $T$  is the  $\mathcal{I}$ -class of  $(ab)^n$ . Similarly,  $(ab)^{n+1} \in T$ . By the hypothesis,  $T$  is a subsemigroup of  $S$ , so  $(ba)^{n+1}(ab)^{n+1} \in T$ , i.e.,  $((ba)^{n+1}(ab)^{n+1}, (ab)^n) \in \mathcal{I}$ . Therefore,  $(ab)^n \in (S^1(ba)^{n+1}(ab)^{n+1}S^1) \subseteq (S(ba)^n(ab)^nS)$ , so

$$(ab)^n \leq x^2(ab)^{3n} \in (S(S(ba)^n(ab)^nS)(ab)^{2n}) \subseteq (S(ba)^n(ab)^nS(ab)^{2n}).$$

(iii) $\Rightarrow$ (iv) and (vi) $\Rightarrow$ (i) This follows immediately.

(iv) $\Rightarrow$ (v) Clearly,  $S$  is left  $\pi$ -regular, and by Lemma 3.5, a simple argument shows that the statement holds.

(v) $\Rightarrow$ (vi) By Lemma 3.5, Lemma 2.1 and Theorem 3.2,  $\mathcal{N}$  is the unique complete semilattice congruence on  $S$  such that each of its congruence classes is a nil-extension of a simple left  $\pi$ -regular ordered semigroup.  $\square$

By Lemma 3.3 and Theorem 3.6, we give some characterizations of complete semilattices of simple complete  $\pi$ -regular ordered semigroups.

**Corollary 3.7.** *Let  $S$  be an ordered semigroup. Then the following conditions are equivalent:*

- (i)  $S$  is a complete semilattice of nil-extensions of simple complete  $\pi$ -regular ordered semigroups;
- (ii)  $S$  is a complete  $\pi$ -regular and a complete semilattice of Archimedean ordered semigroups;
- (iii)  $S$  is completely  $\pi$ -regular and has the  $P$ -property;
- (iv)  $(\forall a, b \in S)(\exists n \in Z^+)(ab)^n \in ((ab)^{2n}Sa^mS(ab)^{2n})$  for every  $m \in Z^+$ ;
- (v)  $(\forall a, b \in S)(\exists n \in Z^+)(ab)^n \in ((ab)^{2n}Sa^2S(ab)^{2n})$ ;
- (vi)  $S$  is left  $\pi$ -regular and a complete semilattice of Archimedean right  $\pi$ -regular ordered semigroups;
- (vii)  $S$  is right  $\pi$ -regular and a complete semilattice of Archimedean left  $\pi$ -regular ordered semigroups;
- (viii)  $\mathcal{N}$  is the unique complete semilattice congruence on  $S$  such that each of its congruence classes is a nil-extension of a simple complete  $\pi$ -regular ordered semigroup.

*Proof.* (i) $\Leftrightarrow$ (iii), (ii) $\Leftrightarrow$ (iii) This follows by Lemma 3.5 and Theorem 3.6.

(i) $\Rightarrow$ (iv) Let  $S$  be a complete semilattice  $Y$  of ordered subsemigroups  $S_\alpha$  ( $\alpha \in Y$ ) which are nil-extensions of simple completely  $\pi$ -regular ordered semigroups. Let  $a \in S_\alpha, b \in S_\beta$  for some  $\alpha, \beta \in Y$ . We have  $ab, a^mb \in S_{\alpha\beta}$  for every  $m \in Z^+$ , so there exists  $n \in Z^+$  such that

$$(ab)^n \in ((ab)^{2n}S_{\alpha\beta}a^mbS_{\alpha\beta}(ab)^{2n}) \subseteq ((ab)^{2n}Sa^mS(ab)^{2n})$$

by Corollary 3.4.

(iv) $\Rightarrow$ (v), (viii) $\Rightarrow$ (i) These are obvious.

(v) $\Rightarrow$ (vi) It is clear that  $S$  is left and right  $\pi$ -regular and has the  $P$ -property. By Lemma 3.5,  $S$  is a complete semilattice  $Y$  of Archimedean ordered semigroups  $S_\alpha, \alpha \in Y$ . By Lemma 2.1,  $S_\alpha$  is right  $\pi$ -regular. Hence  $S$  is left  $\pi$ -regular and a complete semilattice of Archimedean right  $\pi$ -regular ordered semigroups.

(vi) $\Rightarrow$ (vii) Let  $S$  be left  $\pi$ -regular and a complete semilattice of Archimedean right  $\pi$ -regular ordered semigroups. By Theorem 3.2 and Theorem 3.6,  $S$  is right  $\pi$ -regular and has the  $P$ -property. Now, from  $S$  is left  $\pi$ -regular and has the  $P$ -property, by Theorem 3.6 and Theorem 3.2, we get that  $S$  is a complete semilattice of Archimedean left  $\pi$ -regular ordered semigroups.

(vii) $\Rightarrow$ (viii) Let  $S$  be right  $\pi$ -regular and a complete semilattice  $Y$  of Archimedean left  $\pi$ -regular ordered semigroups  $S_\alpha, \alpha \in Y$ . By Theorem 3.2,

$S_\alpha$  is a nil-extension of a simple left  $\pi$ -regular ordered semigroup, so by Theorem 3.6, we have that  $\mathcal{N}$  is the unique complete semilattice congruence on  $S$  such that each of its congruence classes  $(x)_\mathcal{N}$  ( $x \in S$ ) is a nil-extension of a simple left  $\pi$ -regular ordered semigroup. From this by Theorem 3.2 it follows that  $(x)_\mathcal{N}$  is an Archimedean left  $\pi$ -regular ordered subsemigroup. Since  $S$  is right  $\pi$ -regular, by Lemma 2.1,  $(x)_\mathcal{N}$  is right  $\pi$ -regular, so  $(x)_\mathcal{N}$  is completely  $\pi$ -regular by Lemma 3.3. By Corollary 3.4, we obtain that  $(x)_\mathcal{N}$  is a nil-extension of a simple complete  $\pi$ -regular ordered semigroup.  $\square$

For an ordered semigroup  $S$ ,  $\sigma$  a semilattice congruence on  $S$ , we denote by “ $\preceq$ ” the order on the semigroup  $S/\sigma = \{(x)_\sigma | x \in S\}$  defined by:

$$(x)_\sigma \preceq (y)_\sigma \Leftrightarrow (x)_\sigma = (xy)_\sigma$$

$(S/\sigma, \cdot, \preceq)$  is an ordered semigroup.  $S$  is called a *chain* of ordered semigroups if there exists a semilattice congruence  $\sigma$  on  $S$  such that  $(x)_\sigma$  is an ordered subsemigroup of  $S$  for every  $x \in S$  and  $(S/\sigma, \preceq)$  is a chain.

Further, we will consider chains of nil-extensions of simple left (*right*)  $\pi$ -regular semigroups.

**Theorem 3.8.** *Let  $S$  be an ordered semigroup. Then the following conditions are equivalent:*

(i)  *$S$  is a chain of nil-extensions of simple left (*right*)  $\pi$ -regular ordered semigroups;*

(ii)  *$(\forall a, b \in S)(\exists n \in \mathbb{Z}^+)a^n \in (Sa^mb^rSa^{2n})$  or  $b^n \in (Sa^mb^rSb^{2n})$  for every  $m, r \in \mathbb{Z}^+$ ;*

*$(\forall a, b \in S)(\exists n \in \mathbb{Z}^+)a^n \in (a^{2n}Sa^mb^rS)$  or  $b^n \in (b^{2n}Sa^mb^rS)$  for every  $m, r \in \mathbb{Z}^+$ ;*

(iii)  *$(\forall a, b \in S)(\exists n \in \mathbb{Z}^+)a^n \in (SabSa^{2n})$  or  $b^n \in (SabSb^{2n})$ ;*

*$((\forall a, b \in S)(\exists n \in \mathbb{Z}^+)a^n \in (a^{2n}SabS)$  or  $b^n \in (b^{2n}SabS)$ ).*

*Proof.* (i) $\Rightarrow$ (ii) Let  $\sigma$  be a semilattice congruence of  $S$  such that  $(x)_\sigma$  is a nil-extension of a simple left  $\pi$ -regular ordered semigroup  $K_x$  of  $S$  for every  $x \in S$  and  $(S/\sigma, \preceq)$  is a chain. Let  $a, b \in S$ . For  $(a)_\sigma, (b)_\sigma$ , we have  $(a)_\sigma \preceq (b)_\sigma$  or  $(b)_\sigma \preceq (a)_\sigma$ . If  $(a)_\sigma \preceq (b)_\sigma$ , then  $a, ab \in (a)_\sigma$ , so for every  $m, r \in \mathbb{Z}^+$ , we have  $a, a^mb^r \in (a)_\sigma$ . By Theorem 3.2, there exists  $n \in \mathbb{Z}^+$  such that  $a^n \in ((a)_\sigma a^mb^r (a)_\sigma a^{2n}) \subseteq (Sa^mb^rSa^{2n})$ . If  $(b)_\sigma \preceq (a)_\sigma$ , in a similar way, we obtain  $b^n \in (Sa^mb^rSb^{2n})$ .

(ii) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (i) It is clear that  $S$  is Archimedean left  $\pi$ -regular. Let  $a, b \in S$ , for  $a^2, ab$ , there exists  $n \in \mathbb{Z}^+$  such that  $(ab)^n \in (Sa^2S)$ . From this follows by Lemma 3.5 that  $S$  has the  $P$ -property. In view of Theorem 3.6,  $\mathcal{N}$  is the unique complete semilattice congruence on  $S$  such that  $(x)_\mathcal{N}$  is a nil-extension of a simple left  $\pi$ -regular ordered semigroup for every  $x \in S$ . Let  $(a)_\mathcal{N}, (b)_\mathcal{N} \in S/\mathcal{N}$ . By hypothesis, there exists  $n \in \mathbb{Z}^+$  such that  $a^n \in (SabSa^{2n}) \subseteq (SabS)$  or

$b^n \in (SabSb^{2n}) \subseteq (SabS]$ . If  $a^n \in (SabS]$ , then there exist  $u, v \in S$  such that  $a^n \leq uabv$ . Then  $N(a) \ni a^n \leq uabv$ , so we have  $uabv \in N(a)$ , it implies  $ab \in N(a)$ , we get  $N(ab) \subseteq N(a)$ . If  $b^m \in (SabS]$ , in a similar way, we have  $N(ab) \subseteq N(b)$ . On the other hand,  $ab \in N(ab)$ , we have  $a, b \in N(ab)$ . Therefore,  $N(a) \subseteq N(ab)$  and  $N(b) \subseteq N(ab)$ . Thus, we have  $N(ab) = N(a)$  or  $N(ab) = N(b)$  i.e.  $(a)_{\mathcal{N}} = (ab)_{\mathcal{N}}$  or  $(b)_{\mathcal{N}} = (ab)_{\mathcal{N}}$ . We have  $(a)_{\mathcal{N}} \preceq (b)_{\mathcal{N}}$  or  $(b)_{\mathcal{N}} \preceq (a)_{\mathcal{N}}$ .  $\square$

**Corollary 3.9.** *Let  $S$  be an ordered semigroup. Then the following conditions are equivalent:*

- (i)  *$S$  is a chain of nil-extensions of simple completely  $\pi$ -regular ordered semigroups.*
- (ii)  *$(\forall a, b \in S)(\exists n \in \mathbb{Z}^+)a^n \in (a^{2n}Sa^mb^rSa^{2n})$  or  $b^n \in (b^{2n}Sa^mb^rSb^{2n})$  for every  $m, r \in \mathbb{Z}^+$ .*
- (iii)  *$(\forall a, b \in S)(\exists n \in \mathbb{Z}^+)a^n \in (a^{2n}SabSa^{2n})$  or  $b^n \in (b^{2n}SabSb^{2n})$ .*

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