

CONFORMAL TRANSFORMATION IN AN ALMOST HERMITIAN MANIFOLD

B. B. Chaturvedi¹ and Pankaj Pandey²

Abstract. In this paper, we have studied a conformal transformation between two almost Hermitian manifolds and shown that the associated Nijenhuis tensor is conformally invariant under this transformation. We have also discussed the properties of contravariant almost analytic vector field, covariant almost analytic vector field and some other properties in almost Hermitian manifold under this transformation.

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1. Introduction

The conformal transformation in almost Hermitian manifolds is studied by different differential geometers time to time. Franco Tricerri and Lieven Vanhecke [7] found an expression for conformally invariant Bochner curvature tensor in an almost Hermitian manifold. In 2002, a conformally flat almost Hermitian manifold with anti-invariant Ricci tensor has been discussed by Toulas and Philippos J. Xenon [6]. K. Matsuo [3] obtained certain results related to local conformal flat Hermitian manifolds and derived a necessary and sufficient condition for a Hermitian manifold to be locally conformally Hermitian flat-manifold for dimensions not less than 4. Also In 2008, a semi-symmetric non-metric connection has been considered by B. B. Chaturvedi and P. N. Pandey [2] and studied contravariant almost analytic vector fields on a Kähler manifold.

Recently, Falleh R. Al-Solamy and Reem Al-Ghefari [1] and Marta Teofilova [5] extended the ideas of conformal transformation by studying the conformal vector fields in Kähler manifolds and complex connections on conformal Kähler manifolds with Norden metric respectively.

A transformation ϕ from manifold (M, F, g) to (M, F, \bar{g}) such that

$$(1.1) \quad \phi * g = \bar{g} = e^{-2f} g,$$

is called a conformal transformation, where $f: M \rightarrow \mathbb{R}$ is a smooth function and (M, F, g) is an almost complex manifold with almost complex structure F and Riemannian metric g satisfying $g(FX, FY) = g(X, Y)$.

¹Department of Pure and Applied Mathematics, Guru Ghasidas Vishwavidyalaya Bilaspur (C.G.), India, e-mail: brajbhushan25@gmail.com

²Department of Pure and Applied Mathematics, Guru Ghasidas Vishwavidyalaya Bilaspur (C.G.), India, e-mail: pankaj.anvarat@gmail.com

Two almost Hermitian manifolds (M, F, g) and (M, F, \bar{g}) are said to be conformally related if there exist a conformal transformation between them. We know that, if two almost Hermitian manifolds are conformally related then the following relation holds:

$$(1.2) \quad \bar{\nabla}_X Y = \nabla_X Y - X(f)Y - Y(f)X + g(X, Y)\sigma(f),$$

where X, Y are vector fields, ∇ and $\bar{\nabla}$ are connections with respect to the metric g and \bar{g} respectively. $\sigma(f)$ denotes the gradient of f and defined by $X(f) = g(X, \sigma(f))$.

Definition 1.1. The conformal transformation ϕ from manifold (M, F, g) to (M, F, \bar{g}) is called homothetic if the function $f: M \rightarrow \mathbb{R}$, given in equation (1.1), is a constant function.

Definition 1.2. An almost Hermitian manifold (M, F, g) is said to be Kähler-like manifold if it satisfies

$$(1.3) \quad (\nabla_X F)(FY) + (\nabla_{FX} F)Y = 0.$$

2. Associated Nijenhuis tensor

We know that the Nijenhuis tensor in an almost Hermitian manifold is defined by

$$(2.1) \quad N(X, Y) = [(\nabla_X F)(FY) + (\nabla_{FX} F)Y] - [(\nabla_Y F)(FX) + (\nabla_{FY} F)X].$$

Therefore, the Nijenhuis tensor with respect to connection $\bar{\nabla}$ is given by

$$(2.2) \quad \bar{N}(X, Y) = [(\bar{\nabla}_X F)(FY) + (\bar{\nabla}_{FX} F)Y] - [(\bar{\nabla}_Y F)(FX) + (\bar{\nabla}_{FY} F)X].$$

Now, taking covariant derivative of FY with respect to connection $\bar{\nabla}$, we can write

$$(2.3) \quad (\bar{\nabla}_X F)Y = \bar{\nabla}_X(FY) - F(\bar{\nabla}_X Y).$$

Using (1.2) in equation (2.3), we have

$$(2.4) \quad (\bar{\nabla}_X F)Y = \nabla_X(FY) - (FY)(f)X + g(X, FY)\sigma(f) - F(\nabla_X Y) \\ + Y(f)FX - g(X, Y)F\sigma(f).$$

Replacing Y by FY in equation (2.4), we have

$$(2.5) \quad (\bar{\nabla}_X F)(FY) = -\nabla_X Y + Y(f)X - g(X, Y)\sigma(f) - F(\nabla_X(FY)) \\ + (FY)(f)FX - g(X, FY)F\sigma(f).$$

Replacing X by FX in equation (2.4), we get

$$(2.6) \quad (\bar{\nabla}_{FX} F)Y = \nabla_{FX}(FY) - (FY)(f)FX + g(X, Y)\sigma(f) \\ - F(\nabla_{FX} Y) - Y(f)X - g(FX, Y)F\sigma(f).$$

Adding equations (2.5) and (2.6), we obtain

$$(2.7) \quad (\bar{\nabla}_X F)(FY) + (\bar{\nabla}_{FX} F)Y = (\nabla_X F)(FY) + (\nabla_{FX} F)Y.$$

Using equation (2.7) in (2.2), we get

$$(2.8) \quad \bar{N}(X, Y) = [(\nabla_X F)(FY) + (\nabla_{FX} F)Y] - [(\nabla_Y F)(FX) + (\nabla_{FY} F)X].$$

From equations (2.1) and (2.8), we have

$$(2.9) \quad \bar{N}(X, Y) = N(X, Y).$$

Now, if we denote the associated Nijenhuis tensors of type (0,3) with respect to connections ∇ and $\bar{\nabla}$ respectively by

$$(2.10) \quad N(X, Y, Z) = g(N(X, Y), Z) \text{ and } \bar{N}(X, Y, Z) = \bar{g}(\bar{N}(X, Y), Z),$$

then from equation (2.9), we can write

$$(2.11) \quad \bar{g}(\bar{N}(X, Y), Z) = \bar{g}(N(X, Y), Z).$$

By the use of (1.1), above equation implies

$$(2.12) \quad \bar{g}(\bar{N}(X, Y), Z) = e^{-2f} g(N(X, Y), Z).$$

From equations (2.10) and (2.12), we get

$$(2.13) \quad \bar{N}(X, Y, Z) = e^{-2f} N(X, Y, Z).$$

Hence, from (2.7) and (2.13), we can state:

Theorem 2.1. *Let (M, F, g) and (M, F, \bar{g}) be almost Hermitian manifolds, conformally related by equation (1.1), then*

(i) *the manifold (M, F, \bar{g}) is a Kähler-like manifold if and only if (M, F, g) is a Kähler-like manifold.*

(ii) *the associated Nijenhuis tensor $N(X, Y, Z)$, defined in (2.10), is conformally invariant.*

Again, if we denote

$$(2.14) \quad N(X, Y, Z) + N(Y, Z, X) + N(Z, X, Y) = \tilde{N}(X, Y, Z),$$

and

$$(2.15) \quad \bar{N}(X, Y, Z) + \bar{N}(Y, Z, X) + \bar{N}(Z, X, Y) = \tilde{\bar{N}}(X, Y, Z),$$

then by interchanging X, Y, Z in cyclic order in equation (2.13) and adding, we obtain

$$(2.16) \quad \begin{aligned} \bar{N}(X, Y, Z) + \bar{N}(Y, Z, X) + \bar{N}(Z, X, Y) &= e^{-2f} [N(X, Y, Z) \\ &+ N(Y, Z, X) + N(Z, X, Y)]. \end{aligned}$$

from equations (2.14), (2.15) and (2.16), we can write

$$(2.17) \quad \tilde{\bar{N}}(X, Y, Z) = e^{-2f} \tilde{N}(X, Y, Z).$$

Thus we compose:

Theorem 2.2. *Let (M, F, g) and (M, F, \bar{g}) be almost Hermitian manifolds, conformally related by equation (1.1), then the tensor $\tilde{N}(X, Y, Z)$ defined in equation (2.14) is conformally invariant.*

3. Contravariant almost analytic vector field

A vector field V is said to be a contravariant almost analytic vector field in an almost Hermitian manifold if

$$(3.1) \quad (L_V F)X = 0,$$

and the vector field V is called a strictly contravariant almost analytic vector field if V and FV are both contravariant almost analytic vector fields, where X is a vector field and L denotes the Lie derivative defined by

$$(3.2) \quad L_V X = [V, X] = \nabla_V X - \nabla_X V.$$

Now, taking Lie differentiation with respect to V of the vector field FX , we can write

$$(3.3) \quad (L_V F)X = L_V(FX) - F(L_V X).$$

Using equation (3.2) in (3.3), we have

$$(3.4) \quad (L_V F)X = \nabla_V(FX) - F(\nabla_V X) - \nabla_{FX} V + F(\nabla_X V).$$

Similarly, for the conformal connection $\bar{\nabla}$, equation (3.4) can be written as

$$(3.5) \quad (\bar{L}_V F)X = \bar{\nabla}_V(FX) - F(\bar{\nabla}_V X) - \bar{\nabla}_{FX} V + F(\bar{\nabla}_X V),$$

where, \bar{L} denotes the Lie derivative with respect to connection $\bar{\nabla}$ defined as $\bar{L}_V X = \bar{\nabla}_V X - \bar{\nabla}_X V$. Using (1.2) in (3.5), we get

$$(3.6) \quad (\bar{L}_V F)X = \nabla_V(FX) - F(\nabla_V X) - \nabla_{FX} V + F(\nabla_X V).$$

From (3.4) and (3.5), we have

$$(3.7) \quad (\bar{L}_V F)X = (L_V F)X.$$

Thus, we conclude:

Theorem 3.1. *Let (M, F, g) and (M, F, \bar{g}) be almost Hermitian manifolds, conformally related by equation (1.1), then the vector field V is contravariant almost analytic in (M, F, \bar{g}) if and only if it is contravariant almost analytic in (M, F, g) .*

Replacing Y by V in equation (2.9), we get

$$(3.8) \quad \bar{N}(X, V) = N(X, V).$$

Equation (3.8) shows that $\bar{N}(X, V)$ vanishes if and only if $N(X, V)$ vanishes. Also, we know [4] that the necessary and sufficient condition for a contravariant almost analytic vector field V to be strictly contravariant almost analytic vector field, is that the Nijenhuis tensor vanishes. But the condition of Kähler-like manifold is the necessary and sufficient condition [4] for vanishing of Nijenhuis tensor.

Hence, we can state:

Theorem 3.2. *Let (M, F, g) and (M, F, \bar{g}) be almost Hermitian manifolds, conformally related by equation (1.1), then the necessary and sufficient condition for a contravariant almost analytic vector field V to be strictly contravariant almost analytic is that the manifold (M, F, g) is a Kähler-like manifold.*

4. Covariant almost analytic vector field

A 1-form ω is said to be a covariant almost analytic vector field [4] in an almost Hermitian manifold if the following condition holds:

$$(4.1) \quad \omega((\nabla_X F)Y - (\nabla_Y F)X) - (\nabla_{FX}\omega)Y + (\nabla_X\omega)(FY) = 0,$$

where X, Y are vector fields.

Let us define 1-form $\bar{\omega}$ in almost Hermitian manifold (M, F, \bar{g}) such that

$$(4.2) \quad \bar{\omega}(X) = \bar{g}(X, \rho).$$

By using (1.1), equation (4.2) can be written as

$$(4.3) \quad \bar{\omega}(X) = e^{-2f}g(X, \rho) = e^{-2f}\omega(X),$$

where ρ is the associated vector field defined by $\omega(X) = g(X, \rho)$.

The condition (4.1), for $\bar{\omega}$ to be a covariant almost analytic vector field in (M, F, \bar{g}) , becomes

$$(4.4) \quad \bar{\omega}((\bar{\nabla}_X F)Y - (\bar{\nabla}_Y F)X) - (\bar{\nabla}_{FX}\bar{\omega})Y + (\bar{\nabla}_X\bar{\omega})(FY) = 0.$$

Now we propose:

Theorem 4.1. *Let (M, F, g) and (M, F, \bar{g}) be almost Hermitian manifolds, conformally related by equation (1.1), and $\omega, \bar{\omega}$ are 1-forms in (M, F, g) and (M, F, \bar{g}) respectively. If ω is covariant almost analytic vector field in (M, F, g) then $\bar{\omega}$ is covariant almost analytic vector field in manifold (M, F, \bar{g}) if and only if one of the following conditions hold:*

$$(i) \quad X(f)\omega(FY) = (FX)(f)\omega(Y).$$

or

$$(ii) \quad \text{The manifold } (M, F, g) \text{ is homothetic.}$$

Proof. Using equation (2.4) in (4.4), we can write

$$(4.5) \quad \begin{aligned} \bar{\omega}((\bar{\nabla}_X F)Y - (\bar{\nabla}_Y F)X) &= \bar{\omega}((\nabla_X F)Y - (\nabla_Y F)X) + Y(f)\bar{\omega}(FX) \\ &- X(f)\bar{\omega}(FY) + (FX)(f)\bar{\omega}(Y) - (FY)(f)\bar{\omega}(X) \\ &+ 2g(X, FY)\bar{\omega}(\sigma(f)). \end{aligned}$$

Using equation (4.3) in (4.5), we can write

$$(4.6) \quad \begin{aligned} \bar{\omega}((\bar{\nabla}_X F)Y - (\bar{\nabla}_Y F)X) &= e^{-2f}\omega((\nabla_X F)Y - (\nabla_Y F)X) + Y(f)\bar{\omega}(FX) \\ &- X(f)\bar{\omega}(FY) + (FX)(f)\bar{\omega}(Y) - (FY)(f)\bar{\omega}(X) \\ &+ 2g(X, FY)\bar{\omega}(\sigma(f)). \end{aligned}$$

Using equation (4.1) in (4.6), we have

$$(4.7) \quad \begin{aligned} \bar{\omega}((\bar{\nabla}_X F)Y - (\bar{\nabla}_Y F)X) &= e^{-2f}[(\nabla_{FX}\omega)Y - (\nabla_X\omega)(FY)] \\ &+ Y(f)\bar{\omega}(FX) - X(f)\bar{\omega}(FY) + (FX)(f)\bar{\omega}(Y) \\ &- (FY)(f)\bar{\omega}(X) + 2g(X, FY)\bar{\omega}(\sigma(f)). \end{aligned}$$

Again by straightforward calculation, we can write

$$(4.8) \quad \begin{aligned} (\bar{\nabla}_{FX}\bar{\omega})Y - (\bar{\nabla}_X\bar{\omega})(FY) &= \bar{\nabla}_{FX}\bar{\omega}(Y) - \bar{\omega}(\bar{\nabla}_{FX}Y) \\ &- \bar{\nabla}_X\bar{\omega}(FY) + \bar{\omega}(\bar{\nabla}_X(FY)). \end{aligned}$$

Using (1.2) in (4.8), we have

$$(4.9) \quad \begin{aligned} (\bar{\nabla}_{FX}\bar{\omega})Y - (\bar{\nabla}_X\bar{\omega})(FY) &= \nabla_{FX}\bar{\omega}(Y) - \bar{\omega}(\nabla_{FX}Y) + Y(f)\bar{\omega}(FX) \\ &- \nabla_X\bar{\omega}(FY) + \bar{\omega}(\nabla_X(FY)) - (FY)(f)\bar{\omega}(X) \\ &+ 2g(X, FY)\bar{\omega}(\sigma(f)). \end{aligned}$$

Using (4.3) in (4.9), we get

$$(4.10) \quad \begin{aligned} (\bar{\nabla}_{FX}\bar{\omega})Y - (\bar{\nabla}_X\bar{\omega})(FY) &= \nabla_{FX} e^{-2f}\omega(Y) - \bar{\omega}(\nabla_{FX}Y) \\ &+ Y(f)\bar{\omega}(FX) - \nabla_X e^{-2f}\omega(FY) + \bar{\omega}(\nabla_X(FY)) \\ &- (FY)(f)\bar{\omega}(X) + 2g(X, FY)\bar{\omega}(\sigma(f)). \end{aligned}$$

Above equation implies

$$(4.11) \quad \begin{aligned} (\bar{\nabla}_{FX}\bar{\omega})Y - (\bar{\nabla}_X\bar{\omega})(FY) &= 2X(f)\bar{\omega}(FY) - 2(FX)(f)\bar{\omega}(Y) \\ &+ Y(f)\bar{\omega}(FX) - (FY)(f)\bar{\omega}(X) \\ &+ e^{-2f}[(\nabla_{FX}\omega)Y - (\nabla_X\omega)(FY)] + 2g(X, FY)\bar{\omega}(\sigma(f)). \end{aligned}$$

Subtracting (4.11) from (4.7), we have

$$(4.12) \quad \begin{aligned} (\bar{\nabla}_{FX}\bar{\omega})Y - (\bar{\nabla}_X\bar{\omega})(FY) - (\bar{\nabla}_{FX}\bar{\omega})Y + (\bar{\nabla}_X\bar{\omega})(FY) &= \\ &- 3[X(f)\bar{\omega}(FY) - (FX)(f)\bar{\omega}(Y)]. \end{aligned}$$

Equation (4.12) can also be written as

$$(4.13) \quad \begin{aligned} (\bar{\nabla}_{FX}\bar{\omega})Y - (\bar{\nabla}_X\bar{\omega})(FY) - (\bar{\nabla}_{FX}\bar{\omega})Y + (\bar{\nabla}_X\bar{\omega})(FY) &= \\ &- 3e^{-2f}[X(f)\omega(FY) - (FX)(f)\omega(Y)]. \end{aligned}$$

From above equation, we can say that if

$$(4.14) \quad X(f)\omega(FY) = (FX)(f)\omega(Y),$$

then $\bar{\omega}$ is a covariant almost analytic vector field in (M, F, g) .

Hence, the first part of the theorem is completed.

Again, if the manifold (M, F, g) be homothetic then by definition (1.1) the function $f:M \rightarrow \mathbb{R}$ will be a constant function and then we have

$$(4.15) \quad X(f) = 0.$$

Hence, from (4.15) we can say that if the manifold (M, F, g) be homothetic then $\bar{\omega}$ is a covariant almost analytic vector field in (M, F, \bar{g}) .

Hence, the second part of the theorem is completed. \square

5. Some properties of conformal transformation

Let us denote

$$(5.1) \quad 'F(X, Y) = g(FX, Y), \quad \text{and} \quad 'F(X, Y) = \bar{g}(FX, Y).$$

Now, taking covariant derivative of $'F(Y, Z)$ with respect to X , easily we can conclude

$$(5.2) \quad \begin{aligned} (\bar{\nabla}_X 'F)(Y, Z) &= \bar{\nabla}_X 'F(Y, Z) - 'F(\bar{\nabla}_X Y, Z) \\ &- 'F(Y, \bar{\nabla}_X Z). \end{aligned}$$

Using equation (5.1) in (5.2), we get

$$(5.3) \quad \begin{aligned} (\bar{\nabla}_X 'F)(Y, Z) &= \bar{\nabla}_X \bar{g}(FY, Z) - \bar{g}(F(\bar{\nabla}_X Y), Z) \\ &- \bar{g}(FY, \bar{\nabla}_X Z). \end{aligned}$$

Using equation (1.2) in equation (5.3), we have

$$(5.4) \quad \begin{aligned} (\overline{\nabla}_X ' \overline{F})(Y, Z) &= \nabla_X \overline{g}(FY, Z) - \overline{g}(F(\nabla_X Y), Z) \\ &+ Y(f) \overline{g}(FX, Z) - g(X, Y) \overline{g}(F\sigma(f), Z) - \overline{g}(FY, \nabla_X Z) \\ &+ X(f) \overline{g}(FY, Z) + Z(f) \overline{g}(FY, X) - g(X, Z) \overline{g}(FY, \sigma(f)). \end{aligned}$$

By interchanging Y and Z, above equation gives

$$(5.5) \quad \begin{aligned} (\overline{\nabla}_X ' \overline{F})(Z, Y) &= \nabla_X \overline{g}(FZ, Y) - \overline{g}(F(\nabla_X Z), Y) + Z(f) \overline{g}(FX, Y) \\ &- g(X, Z) \overline{g}(F\sigma(f), Y) - \overline{g}(FZ, \nabla_X Y) + X(f) \overline{g}(FZ, Y) \\ &+ Y(f) \overline{g}(FZ, X) - g(X, Y) \overline{g}(FZ, \sigma(f)). \end{aligned}$$

Adding equations (5.4) and (5.5), we get

$$(5.6) \quad (\overline{\nabla}_X ' \overline{F})(Y, Z) + (\overline{\nabla}_X ' \overline{F})(Z, Y) = 0.$$

Equation (5.6) can be written as

$$(5.7) \quad (\overline{\nabla}_X ' \overline{F})(Y, Z) = -(\overline{\nabla}_X ' \overline{F})(Z, Y).$$

Thus, we conclude:

Theorem 5.1. *Let (M, F, g) and (M, F, \overline{g}) be almost Hermitian manifolds, conformally related by equation (1.1), then $(\overline{\nabla}_X ' \overline{F})(Y, Z)$ is skew-symmetric in Y and Z.*

Now, we propose:

Theorem 5.2. *Let (M, F, g) and (M, F, \overline{g}) be almost Hermitian manifolds conformally related by equation (1.1). If the manifold (M, F, g) is homothetic then $\sum (\nabla_X ' F)(Y, Z)$ is conformally invariant.*

Proof. Interchanging X, Y, Z in cyclic order in equation (5.4), we get

$$(5.8) \quad \begin{aligned} (\overline{\nabla}_Y ' \overline{F})(Z, X) &= \nabla_Y \overline{g}(FZ, X) - \overline{g}(F(\nabla_Y Z), X) + Z(f) \overline{g}(FY, X) \\ &- g(Y, Z) \overline{g}(F\sigma(f), X) - \overline{g}(FZ, \nabla_Y X) + Y(f) \overline{g}(FZ, X) \\ &+ X(f) \overline{g}(FZ, Y) - g(Y, X) \overline{g}(FZ, \sigma(f)), \end{aligned}$$

and

$$(5.9) \quad \begin{aligned} (\overline{\nabla}_Z ' \overline{F})(X, Y) &= \nabla_Z \overline{g}(FX, Y) - \overline{g}(F(\nabla_Z X), Y) + X(f) \overline{g}(FZ, Y) \\ &- g(Z, X) \overline{g}(F\sigma(f), Y) - \overline{g}(FX, \nabla_Z Y) + Z(f) \overline{g}(FX, Y) \\ &+ Y(f) \overline{g}(FX, Z) - g(Z, Y) \overline{g}(FX, \sigma(f)). \end{aligned}$$

Adding equations (5.4), (5.8) and (5.9), we have

$$(5.10) \quad \begin{aligned} (\overline{\nabla}_X ' \overline{F})(Y, Z) + (\overline{\nabla}_Y ' \overline{F})(Z, X) + (\overline{\nabla}_Z ' \overline{F})(X, Y) &= \nabla_X \overline{g}(FY, Z) \\ &+ \overline{g}(\nabla_X Y, FZ) - \overline{g}(\nabla_X Z, FY) + \nabla_Y \overline{g}(FZ, X) + Z(f) \overline{g}(FY, X) \\ &+ \overline{g}(\nabla_Y Z, FX) - \overline{g}(\nabla_Y X, FZ) + \nabla_Z \overline{g}(FX, Y) + X(f) \overline{g}(FZ, Y) \\ &+ Y(f) \overline{g}(FX, Z) + \overline{g}(\nabla_Z X, FY) - \overline{g}(\nabla_Z Y, FX). \end{aligned}$$

Using equation (1.1) in (5.10), we can write

$$\begin{aligned}
 (\bar{\nabla}_X 'F)(Y, Z) + (\bar{\nabla}_Y 'F)(Z, X) + (\bar{\nabla}_Z 'F)(X, Y) &= \nabla_X e^{-2f} g(FY, Z) \\
 &+ \bar{g}(\nabla_X Y, FZ) - \bar{g}(\nabla_X Z, FY) + \nabla_Y e^{-2f} g(FZ, X) \\
 (5.11) \quad &+ Z(f)\bar{g}(FY, X) + \bar{g}(\nabla_Y Z, FX) - \bar{g}(\nabla_Y X, FZ) \\
 &+ \nabla_Z e^{-2f} g(FX, Y) + X(f)\bar{g}(FZ, Y) + Y(f)\bar{g}(FX, Z) \\
 &+ \bar{g}(\nabla_Z X, FY) - \bar{g}(\nabla_Z Y, FX).
 \end{aligned}$$

The above equation implies

$$\begin{aligned}
 (\bar{\nabla}_X 'F)(Y, Z) + (\bar{\nabla}_Y 'F)(Z, X) + (\bar{\nabla}_Z 'F)(X, Y) &= \\
 (5.12) \quad &e^{-2f} [g((\nabla_X F)Y, Z) + g((\nabla_Y F)Z, X) + g((\nabla_Z F)X, Y)] \\
 &- 3e^{-2f} [X(f)g(FY, Z) + Y(f)g(FZ, X) + Z(f)g(FX, Y)].
 \end{aligned}$$

Now, we know that in an almost Hermitian manifold

$$(5.13) \quad (\nabla_X 'F)(Y, Z) = g((\nabla_X F)Y, Z).$$

Using (5.13) in (5.12), we get

$$\begin{aligned}
 (\bar{\nabla}_X 'F)(Y, Z) + (\bar{\nabla}_Y 'F)(Z, X) + (\bar{\nabla}_Z 'F)(X, Y) &= \\
 (5.14) \quad &e^{-2f} [(\nabla_X 'F)(Y, Z) + (\nabla_Y 'F)(Z, X) + (\nabla_Z F)(X, Y)] \\
 &- 3e^{-2f} [X(f)g(FY, Z) + Y(f)g(FZ, X) + Z(f)g(FX, Y)].
 \end{aligned}$$

Now, if the manifold (M, F, g) is homothetic then $X(f) = 0$ and above equation gives

$$\begin{aligned}
 (\bar{\nabla}_X 'F)(Y, Z) + (\bar{\nabla}_Y 'F)(Z, X) + (\bar{\nabla}_Z 'F)(X, Y) &= \\
 (5.15) \quad &e^{-2f} [(\nabla_X 'F)(Y, Z) + (\nabla_Y 'F)(Z, X) + (\nabla_Z 'F)(X, Y)].
 \end{aligned}$$

i.e.

$$(5.16) \quad \sum (\bar{\nabla}_X 'F)(Y, Z) = e^{-2f} \sum (\nabla_X 'F)(Y, Z).$$

From (5.16), we can say that $\sum (\nabla_X 'F)(Y, Z)$ is invariant under the conformal transformation defined in (1.1).

Hence, the theorem is proved. □

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