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LORENTZ HYPERSURFACES SATISFYING $\triangle\vec{H}=\alpha\vec{H}$ WITH COMPLEX EIGEN VALUES

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Abstract. In this paper, we study Lorentz hypersurface M_1^n in E_1^{n+1} satisfying $\triangle\vec{H}=\alpha\vec{H}$ with minimal polynomial $[(y-\lambda)^2+\mu^2](y-\lambda_1)(y-\lambda_n)$ having shape operator (2.11). We prove that every such Lorentz hypersurface in E_1^{n+1} having at most four distinct principal curvatures has a constant mean curvature.

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1. Introduction

The study of submanifolds with harmonic mean curvature vector field was initiated by B. Y. Chen in 1985 and arose in the context of his theory of submanifolds of finite type. For a survey on submanifolds of finite type and various related topics, see [8, 9]. Let M_r^n be an n-dimensional, connected submanifold of the pseudo-Euclidean space E_s^m. Denote by $\vec{x}$, $\vec{H}$, and $\triangle$ respectively the position vector field, mean curvature vector field of M_r^n, and the Laplace operator on M_r^n, with respect to the induced metric g on M_r^n, from the indefinite metric on the ambient space E_s^m. It is well known that [7]

$$\triangle\vec{x}=-n\vec{H}. \tag{1.1}$$

A submanifold M_r^n of E_s^m satisfying the condition

$$\triangle\vec{H}=0, \tag{1.2}$$

is called a biharmonic submanifold. In view of (1.1), condition (1.2) is equivalent to $\triangle^2\vec{x}=0$. Equation (1.2) is the special case of the equation

$$\triangle\vec{H}=\alpha\vec{H}. \tag{1.3}$$

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As remarked, minimal submanifolds are immediately seen to be biharmonic. Conversely, the question arises whether the class of submanifolds with harmonic mean curvature vector field is essentially larger than the class of minimal submanifolds. Concerning this problem B. Y. Chen conjectured the following:
Conjecture: *The only biharmonic submanifolds of Euclidean spaces are the minimal ones.*

In Euclidean spaces, we have the following results, which indeed support the above mentioned conjecture. B. Y. Chen proved in 1985 that every biharmonic surface in E^3 is minimal. Thereafter, I. Dimitric generalized this result [13]. In [17], it was proved that every biharmonic hypersurface in E^4 is minimal. Recently, it was proved that every biharmonic hypersurface with three distinct principal curvatures in E^{n+1} with arbitrary dimension is minimal [16].

The study of equation (1.3) for submanifolds in pseudo-Euclidean spaces was originated by Ferrandez et al. in [4, 5]. They showed that if the minimal polynomial of the shape operator of a hypersurface M_r^{n-1} (r = 0, 1) in E_1^n is at most of degree two, then M_r^{n-1} has a constant mean curvature. Also, in [8] various classification theorems for submanifolds in a Minkowski spacetime were obtained. In [1], it was proved that every hypersurface M_r^3 (r = 0, 1, 2, 3) of E_s^4 satisfying equation (1.3) whose shape operator is diagonal, has a constant mean curvature. Also, in [3] the same conclusion was obtained for every hypersurface M_1^3 in E_1^4. Recently, it was proved that every hypersurface having at most three distinct principal curvatures in E_s^{n+1} satisfying (1.3) with diagonal shape operator has a constant mean curvature [14].

In contrast to the submanifolds of Euclidean spaces, Chen's conjecture is not true always for the submanifolds of the pseudo-Euclidean spaces. For example, B. Y. Chen et al. [11, 12] obtained some examples of proper biharmonic surfaces in 4-dimensional pseudo-Euclidean spaces E_s^4 for $s = 1, 2, 3$ (see also [10]). But for hypersurfaces in pseudo-Euclidean spaces, it is reasonable that Chen's conjecture is also right. This is supported by the following facts: B. Y. Chen et al. proved in [11, 12] that biharmonic surfaces in pseudo-Euclidean 3-spaces are minimal, and A. Arvanitoyeorgos et al. [2] proved that biharmonic Lorentzian hypersurfaces in Minkowski 4-spaces are minimal.

In this paper, we study Lorentz hypersurfaces M_1^n in E_1^{n+1} satisfying (1.3) and having shape operator (2.11).

2. Preliminaries

Let (M_1^n, g) be a n-dimensional Lorentz hypersurface isometrically immersed in a $n+1$-dimensional pseudo-Euclidean space $(E_1^{n+1}, \overline{g})$ and $g = \overline{g}_{|M_1^n}$. We denote by ξ unit normal vector to M_1^n with $\overline{g}(\xi, \xi) = 1$.

Let $\overline{\nabla}$ and ∇ denote linear connections on E_1^{n+1} and M_1^n, respectively. Then, the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \ \forall \ \ X, Y \in \Gamma(TM_1^n), \tag{2.1}$$

$$\overline{\nabla}_X \xi = -S_\xi X, \ \ \forall \ \ \xi \in \Gamma(TM_1^n)^\perp, \tag{2.2}$$

where h is the second fundamental form and S is the shape operator. It is well known that the second fundamental form h and shape operator S are related by

$$\overline{g}(h(X,Y),\xi) = g(S_\xi X, Y). \tag{2.3}$$

The mean curvature vector is given by

$$\vec{H} = \frac{1}{n} traceh. \tag{2.4}$$

The Gauss and Codazzi equations are given by

$$R(X,Y)Z = g(SY,Z)SX - g(SX,Z)SY, \tag{2.5}$$

$$(\nabla_X S)Y = (\nabla_Y S)X, \tag{2.6}$$

respectively, where R is the curvature tensor, $S = S_\xi$ for some unit normal vector field ξ and

$$(\nabla_X S)Y = \nabla_X(SY) - S(\nabla_X Y), \tag{2.7}$$

for all $X, Y, Z \in \Gamma(TM_1^n)$.

The necessary and sufficient conditions for M_1^n to have proper mean curvature in E_1^{n+1} [1] are

$$\triangle H + H trace S^2 = \alpha H, \tag{2.8}$$

$$S(gradH) + \frac{n}{2} H gradH = 0, \tag{2.9}$$

where H denotes the mean curvature. Also, the Laplace operator $\triangle$ of a scalar valued function f is given by [11]

$$\triangle f = -\sum_{i=1}^{n} \epsilon_i (e_i e_i f - \nabla_{e_i} e_i f), \tag{2.10}$$

where $\{e_1, e_2, ..., e_n\}$ is an orthonormal local tangent frame on M_1^n with $\epsilon_i = \pm 1$.

A vector X in E_s^{n+1} is called spacelike, timelike or lightlike according as $\overline{g}(X,X) > 0$, $\overline{g}(X,X) < 0$ or $\overline{g}(X,X) = 0$, respectively. A non-degenerate hypersurface M_r^n of E_s^{n+1} is called Riemannian or pseudo-Riemannian if the induced metric on M_r^{n+1} from the indefinite metric on E_s^{n+1} is definite or indefinite, respectively. A shape operator of pseudo-Riemannian hypersurfaces is not diagonalizable always unlike the Riemannian hypersurfaces.

The matrix representation of shape operator of M_1^n in E_1^{n+1} having minimal polynomial $[(y-\lambda)^2+\mu^2](y-\lambda_1)(y-\lambda_n)$ with respect to a suitable orthonormal base field of the tangent bundle takes the form [6, 15]

$$S = \begin{pmatrix} \lambda & -\mu & & & & & \\ \mu & \lambda & & & & & \\ & & \lambda_1 & & & & \\ & & & \cdots & & & \\ & & & & \cdots & & \\ & & & & & \lambda_1 & \\ & & & & & & \lambda_n \end{pmatrix}, \tag{2.11}$$

for some smooth functions $\lambda, \lambda_1, \lambda_n$ and μ.

3. Lorentz Hypersurfaces in E_1^{n+1} satisfying $\triangle\vec{H} = \alpha\vec{H}$

We assume that H is not constant and grad$H \neq 0$. Assuming non constant mean curvature implies the existence of an open connected subset U of M_1^n, with grad$_pH \neq 0$ for all $p \in U$. From (2.9), it is easy to see that grad$H$ is an eigenvector of the shape operator S with the corresponding principal curvature $-\frac{n}{2}H$. In view of (2.11), the shape operator S of hypersurfaces will take the following form

$$S(e_1) = \lambda e_1 + \mu e_2, \quad S(e_2) = -\mu e_1 + \lambda e_2, \quad S(e_b) = \lambda_1 e_b, \quad S(e_n) = \lambda_n e_n, \tag{3.1}$$

for $b = 3, 4, \ldots, n-1$, with respect to orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $T_pM_1^n$, which satisfies

$$g(e_1, e_1) = -1, \quad g(e_i, e_i) = 1, \quad i = 2, 3, ..., n, \tag{3.2}$$

and

$$g(e_i, e_j) = 0, \quad for \;\; i \neq j. \tag{3.3}$$

We write

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_{ij}^k e_k, \qquad i, j = 1, 2, ..., n. \tag{3.4}$$

Using (3.4) and taking covariant derivatives of (3.2) and (3.3) with e_k, we find

$$\omega_{ki}^i = 0, \qquad \omega_{kj}^i = -\omega_{ki}^j, \tag{3.5}$$

for $i \neq j$ and $i, j, k = 1, 2, ..., n$.

Now, we consider following two cases:

Case A: If $\lambda_1 \neq \lambda_n$. We can choose grad$H$ in the direction of e_n or e_k for $k = 3, 4, \dots, n-1$. In both the cases, grad$H$ is spacelike. We can express grad$H = \sum_{i=1}^{n} e_i(H)e_i$. Assuming grad$H$ in the direction of e_n, we get grad$H$= $e_n(H)e_n$. Also, from (2.9), we obtain the corresponding eigenvalue $\lambda_n = -\frac{nH}{2}$. Using (2.4) and (2.11), we find $\lambda_1 = \frac{3nH}{2(n-3)} - \frac{2\lambda}{n-3}$. Also, we have

$$e_n(H) \neq 0, \quad e_1(H) = e_2(H) = ... = e_{n-1}(H) = 0. \tag{3.6}$$

Using (3.4), (3.6) and the fact that $[e_i\, e_j](H) = 0 = \nabla_{e_i} e_j(H) - \nabla_{e_j} e_i(H)$, for $i \neq j$ and $i, j \neq n$, we find

$$\omega_{ij}^n = \omega_{ji}^n. \tag{3.7}$$

Taking inner product of (2.6) with $Z \in TM_1^n$, we get

$$g((\nabla_X S)Y, Z) = g((\nabla_Y S)X, Z). \tag{3.8}$$

Using (3.1), (3.4), (3.6), (3.8) and the value of λ_1 and λ_n, we get equations (3.9)∼(3.26), where $j \neq k$ and $j, k = 3, 4, ..., n-1$:

For X = e_1, Y = e_2, Z = e_n

$$[\lambda + \frac{nH}{2}](\omega_{12}^n - \omega_{21}^n) = \mu(\omega_{22}^n + \omega_{11}^n). \tag{3.9}$$

For X = e_1, Y = e_k, Z = e_k

$$e_1(-\frac{2\lambda}{n-3}) = [\frac{(n-1)\lambda}{n-3} - \frac{3nH}{2(n-3)}]\omega_{k1}^k + \mu\omega_{k2}^k. \tag{3.10}$$

For X = e_1, Y = e_k, Z = e_j

$$[\frac{(n-1)\lambda}{n-3} - \frac{3nH}{2(n-3)}]\omega_{k1}^j + \mu\omega_{k2}^j = 0. \tag{3.11}$$

For X = e_1, Y = e_k, Z = e_n

$$[\frac{n^2 H}{2(n-3)} - \frac{2\lambda}{n-3}]\omega_{1k}^n = [\lambda + \frac{nH}{2}]\omega_{k1}^n + \mu\omega_{k2}^n. \tag{3.12}$$

For X = e_2, Y = e_k, Z = e_k

$$e_2(-\frac{2\lambda}{n-3}) = [\frac{(n-1)\lambda}{n-3} - \frac{3nH}{2(n-3)}]\omega_{k2}^k - \mu\omega_{k1}^k. \tag{3.13}$$

For X = e_2, Y = e_k, Z = e_j

$$[\frac{(n-1)\lambda}{n-3} - \frac{3nH}{2(n-3)}]\omega_{k2}^j - \mu\omega_{k1}^j = 0. \tag{3.14}$$

For X $= e_2$, Y $= e_k$, Z $= e_n$

$$[\frac{n^2H}{2(n-3)} - \frac{2\lambda}{n-3}]\omega^n_{2k} = [\lambda + \frac{nH}{2}]\omega^n_{k2} - \mu\omega^n_{k1}. \tag{3.15}$$

For X $= e_1$, Y $= e_n$, Z $= e_1$

$$-(\lambda + \frac{nH}{2})\omega^1_{1n} + \mu\omega^2_{1n} = e_n(\lambda). \tag{3.16}$$

For X $= e_1$, Y $= e_n$, Z $= e_2$

$$-(\lambda + \frac{nH}{2})\omega^2_{1n} - \mu\omega^1_{1n} = e_n(\mu). \tag{3.17}$$

For X $= e_1$, Y $= e_n$, Z $= e_n$

$$(\lambda + \frac{nH}{2})\omega^n_{n1} + \mu\omega^n_{n2} = 0. \tag{3.18}$$

For X $= e_2$, Y $= e_n$, Z $= e_1$

$$-(\lambda + \frac{nH}{2})\omega^1_{2n} + \mu\omega^2_{2n} = -e_n(\mu). \tag{3.19}$$

For X $= e_2$, Y $= e_n$, Z $= e_2$

$$-(\lambda + \frac{nH}{2})\omega^2_{2n} - \mu\omega^1_{2n} = e_n(\lambda). \tag{3.20}$$

For X $= e_2$, Y $= e_n$, Z $= e_n$

$$(\lambda + \frac{nH}{2})\omega^n_{n2} - \mu\omega^n_{n1} = 0. \tag{3.21}$$

For X $= e_k$, Y $= e_n$, Z $= e_1$

$$-(\lambda + \frac{nH}{2})\omega^1_{kn} + \mu\omega^2_{kn} = -[\frac{(n-1)\lambda}{n-3} - \frac{3nH}{2(n-3)}]\omega^1_{nk} + \mu\omega^2_{nk}. \tag{3.22}$$

For X $= e_k$, Y $= e_n$, Z $= e_2$

$$-(\lambda + \frac{nH}{2})\omega^2_{kn} - \mu\omega^1_{kn} = -[\frac{(n-1)\lambda}{n-3} - \frac{3nH}{2(n-3)}]\omega^2_{nk} - \mu\omega^1_{nk}. \tag{3.23}$$

For X $= e_k$, Y $= e_n$, Z $= e_k$

$$e_n(\frac{3nH}{2(n-3)} - \frac{2\lambda}{n-3}) = -[\frac{n^2H}{2(n-3)} - \frac{2\lambda}{n-3}]\omega^k_{kn}. \tag{3.24}$$

For X $= e_k$, Y $= e_n$, Z $= e_j$

$$\omega^j_{kn} = 0. \tag{3.25}$$

For X $=e_k$, Y $=e_n$, Z $=e_n$

$$\omega_{nk}^{n}=0. \tag{3.26}$$

From (3.25), (3.26) and (3.5), we have

$$\omega_{kn}^{j}=\omega_{nk}^{n}=\omega_{kj}^{n}=\omega_{nn}^{k}=0, \tag{3.27}$$

for $j\neq k$ and $j,k=3,4,...,n-1$.

Using (3.12), (3.15), (3.7) and (3.5), we find

$$\omega_{2k}^{n}=\omega_{2n}^{k}=\omega_{k2}^{n}=\omega_{kn}^{2}=\omega_{k1}^{n}=\omega_{1k}^{n}=\omega_{kn}^{1}=\omega_{1n}^{k}=0, \tag{3.28}$$

for $k=3,4,...,n-1$.

Also, from (3.16), (3.17), (3.19), (3.20), (3.9), (3.7) and (3.5), we obtain

$$\omega_{2n}^{1}=\omega_{21}^{n}=\omega_{12}^{n}=\omega_{1n}^{2}=\omega_{22}^{n}=\omega_{11}^{n}=\omega_{2n}^{2}=\omega_{1n}^{1}=0. \tag{3.29}$$

Using (3.11), (3.14) and (3.5), we get

$$\omega_{k1}^{j}=\omega_{k2}^{j}=\omega_{kj}^{1}=\omega_{kj}^{2}=0, \tag{3.30}$$

for $j\neq k$ and $j,k=3,4,...,n-1$.

Similarly, from (3.18), (3.21) and (3.5), we find

$$\omega_{n2}^{n}=\omega_{nn}^{2}=\omega_{n1}^{n}=\omega_{nn}^{1}=0. \tag{3.31}$$

Also, from (3.22), (3.23), (3.28) and (3.5), we obtain

$$\omega_{nk}^{1}=\omega_{nk}^{2}=\omega_{n1}^{k}=\omega_{n2}^{k}=0, \tag{3.32}$$

for $k=3,4,...,n-1$.

Now, we have the following:

Lemma 3.1. *Let M_1^n be a Lorentz hypersurface in E_1^{n+1}, having the shape operator (2.11) with respect to suitable orthonormal basis $\{e_1,e_2,...,e_n\}$. If gradH is spacelike and in the direction of e_n, then*

$$\nabla_{e_1}e_1=\sum_{p\neq 1,n}\omega_{11}^{p}e_p,\ \ \nabla_{e_1}e_2=\sum_{p\neq 2,n}\omega_{12}^{p}e_p,\ \ \nabla_{e_2}e_1=\sum_{p\neq 1,n}\omega_{21}^{p}e_p,$$

$$\nabla_{e_2}e_2=\sum_{p\neq 2,n}\omega_{22}^{p}e_p,\ \ \nabla_{e_2}e_n=0,\ \ \nabla_{e_k}e_k=\sum_{p\neq k}\omega_{kk}^{p}e_p,\ \ \nabla_{e_k}e_j=\sum_{p\neq 1,2,j,n}\omega_{kj}^{p}e_p,$$

$$\nabla_{e_k}e_n=\sum_{p\neq 1,2,j,n}\omega_{kn}^{p}e_p,\ \ \nabla_{e_n}e_1=\omega_{n1}^{2}e_2,\ \ \nabla_{e_n}e_2=\omega_{n2}^{1}e_1,$$

$$\nabla_{e_n}e_k=\sum_{p\neq 1,2,k,n}\omega_{nk}^{p}e_p,\ \ \nabla_{e_n}e_n=0,$$

for $j \neq k$ *and* $j, k = 3, 4, ..., n-1$.

Using Lemma 3.1 and (2.5) to evaluate $g(R(e_2, e_n)e_2, e_n)$, we obtain

$$\frac{nH}{2}\lambda = 0 \quad \text{or} \quad \lambda = 0, \tag{3.33}$$

as $H \neq 0$.

Using (3.33) and (3.5) in (3.10) and (3.13), we find

$$\omega_{k2}^{k} = \omega_{kk}^{2} = \omega_{k1}^{k} = \omega_{kk}^{1} = 0, \tag{3.34}$$

for $k = 3, 4, ..., n-1$.

Also, using $traceS^2 = \frac{n^2(n+6)H^2}{4(n-3)} - 2\mu^2$, (2.10) and Lemma 3.1, the equation (2.8) with respect to the basis $\{e_1, e_2, ..., e_n\}$ reduces

$$-e_ne_n(H) + \sum_{k=3}^{n-1}\omega_{kk}^{n}e_n(H) + H\left(\frac{n^2(n+6)H^2}{4(n-3)} - 2\mu^2\right) = \alpha H. \tag{3.35}$$

Using (2.5) and Lemma 3.1 to evaluate $g(R(e_k, e_n)e_n, e_k)$, we get

$$e_n(\omega_{kk}^{n}) - (\omega_{kk}^{n})^2 = -\frac{3n^2H^2}{4(n-3)}, \tag{3.36}$$

for $k = 3, 4, ..., n-1$.

Using (3.33) in (3.24), we have

$$\omega_{kk}^{n} = \frac{3e_n(H)}{nH}, \tag{3.37}$$

for $k = 3, 4, ..., n-1$.

From (3.37), we find

$$\omega_{33}^{n} = \omega_{44}^{n} = \cdots = \omega_{(n-1)(n-1)}^{n}. \tag{3.38}$$

Differentiating (3.37) along e_n and using (3.36), we get

$$e_ne_n(H) = \frac{n(n+3)H}{9}(\omega_{kk}^{n})^2 - \frac{n^3H^3}{4(n-3)}, \tag{3.39}$$

for $k = 3, 4, ..., n-1$.

Using (3.38) and (3.39) in (3.35), we obtain

$$\frac{2}{9}n(n-6)(\omega_{kk}^{n})^2 + \frac{n^2(n+3)}{2(n-3)}H^2 - 2\mu^2 = \alpha, \tag{3.40}$$

for $k = 3, 4, ..., n-1$.

Differentiating (3.40) with respect to e_n and using (3.17), (3.29), (3.36) and (3.37), we find

$$\frac{4}{9}(n-6)(\omega_{kk}^{n})^2 + \frac{3n^2}{n-3}H^2 = 0, \tag{3.41}$$

for $k = 3, 4, ..., n-1$.

Again, acting along e_n on (3.41) and using (3.36) and (3.37), we get

$$\frac{8(n-6)}{9}(\omega_{kk}^n)^2 + \frac{4n^2(n+3)}{3(n-3)}H^2 = 0, \tag{3.42}$$

for $k = 3, 4, ..., n-1$.

Hence, from (3.41) and (3.42), we obtain that H must be zero.

Case B: If $\lambda_1 = \lambda_n$, then we get $\lambda_1 = \lambda_n = \frac{-nH}{2}$. Using (2.11) and (2.4), we find $\lambda = \frac{n^2H}{4}$. Then, from (3.24), we get $e_n(H) = 0$, which is a contradiction of (3.6), and therefore H must be constant.

Combining case A and case B, we have

Theorem 3.2. *Every Lorentz hypersurface M_1^n in E_1^{n+1} satisfying $\triangle\vec{H} = \alpha\vec{H}$, having shape operator given by* (2.11) *with at most four distinct principal curvatures has a constant mean curvature.*

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