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STABILITY OF PEXIDERIZED QUADRATIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN FUZZY NORMED SPASES

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Abstract. We determine some stability results concerning the pexiderized quadratic functional equation in non-Archimedean fuzzy normed spaces. Our result can be regarded as a generalization of the stability phenomenon in the framework of $\mathcal{L}$-fuzzy normed spaces.

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1. Introduction

The stability of functional equations is an interesting area of research for mathematicians, but it can be also of importance to persons who work outside of the realm of pure mathematics.

It seems that the stability problem of functional equations had been first raised by Ulam [14]. Moreover the approximated types of mappings have been studied extensively in several papers. (See for instance [10], [6], [3], and [4]).

Fuzzy notion introduced firstly by Zadeh [15] that has been widely involved in different subjects of mathematics. Zadeh's definition of a fuzzy set characterized by a function from a nonempty set X to $[0, 1]$. Goguen in [5] generalized the notion of a fuzzy subset of X to that of an $\mathcal{L}$-fuzzy subset, namely a function from X to a lattice L.

Later, in 1984 Katsaras [7] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space.

With [9] and by modifying the definition of a fuzzy normed space in [2], Mirmostafaee and Moslehian in [8] introduced a notion of a non-Archimedean fuzzy normed space. Shekari et al. ([12]) considered the quadratic functional equation in $\mathcal{L}$-fuzzy normed space. Also Saadati and Park considered the equation $f(lx+y)+f(lx-y) = 2l^2f(x)+2f(y)$ and proved the Hyers-Ulam-Rasssias stability of this equation in $\mathcal{L}$-fuzzy normed spaces ([13]).

Defining the class of approximate solutions of a given functional equation one can ask whether every mapping from this class can be somehow

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approximated by an exact solution of the considered equation in the non-Archimedean $\mathcal{L}$-fuzzy normed spaces. To answer this question, we established a non-Archimedean $\mathcal{L}$-fuzzy Hyers-Ulam-Rassias stability of the pexiderized quadratic functional equation $f(x+y)+f(x-y)=2g(x)+2h(y)$.

2. Preliminaries

In this section, we provide a collection of definitions and related results which are essential and used in the subsequent discussions.

Definition 2.1. Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $t, s \in \mathbb{R}$,

(N1) $N(x,c) = 0$ for $c \le 0$;

(N2) $x = 0$ if and only if $N(x,c) = 1$ for all $c > 0$;

(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N4) $N(x+y, s+t) \ge min\{N(x,s), N(y,t)\}$;

(N5) $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{t\to\infty} N(x,t) = 1$;

(N6) for $x \neq 0$, $N(x,.)$ is (upper semi) continuous on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space.

Example 2.2. Let $(X, ||.||)$ be a normed linear space. We define function N by $N(x,t) = \frac{t^2-||x||^2}{t^2+||x||^2}$ if $t > ||x||$ and $N(x,t) = 0$ if $t \le ||x||$. Then N defines a fuzzy norm on X.

Definition 2.3. Let (X,N) be a fuzzy normed linear space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N - lim_{n\to\infty} x_n = x$.

Definition 2.4. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \ge n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is known that every convergent sequence in a fuzzy normed space is Cauchy and if each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and furthermore the fuzzy normed space is called a fuzzy Banach space.

Definition 2.5. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a t-norm if it satisfies the following conditions:

($*1$) $*$ is associative,

($*2$) $*$ is commutative,

($*3$) $a * 1 = a$ for all $a \in [0,1]$ and

($*4$) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for each $a, b, c, d \in [0,1]$.

Definition 2.6. ([5]) Let $\mathcal{L} = (L, \le_L)$ be a complete lattice and let U be a non-empty set called the universe. An $\mathcal{L}$-fuzzy set in U is defined as a mapping $\mathcal{A} : U \to L$. For each u in U, $\mathcal{A}(u)$ represents the degree (in L) to which u is an element of $\mathcal{A}$.

Definition 2.7. ([1]) A t-norm on $\mathcal{L}$ is a mapping $*_L : L^2 \to L$ satisfying the following conditions:

(i) $(\forall x \in L)(x *_L 1_{\mathcal{L}} = x)$(: boundary condition);

(ii) $(\forall (x, y) \in L^2)(x *_L y = y *_L x)$(: commutativity);

(iii) $(\forall (x, y, z) \in L^3)(x *_L (y *_L z)) = ((x *_L y) *_L z)$(: associativity);

(iv) $(\forall (x, y, z, w) \in L^4)(x \leq_L x' \quad and \quad y \leq_L y' \Rightarrow x *_L y \leq_L x' *_L y')$(: monotonicity).

A t-norm $*_L$ on $\mathcal{L}$ is said to be continuous if, for any $x, y \in \mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converges to x and y, respectively, $lim_{n\to\infty}(x_n *_L y_n) = x *_L y$.

Definition 2.8. The triple $(V, \mathcal{P}, *_L)$ is said to be an $\mathcal{L}$-fuzzy normed space if V is vector space, $*_L$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $V \times (0, \infty)$ satisfying the following conditions:

for all $x, y \in V$ and $t, s \in (0, \infty)$;

(a) $\mathcal{P}(x, t) >_L 0_{\mathcal{L}}$;

(b) $\mathcal{P}(x, t) = 1_{\mathcal{L}}$ if and only if $x = 0$;

(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;

(d) $\mathcal{P}(x, t) *_L \mathcal{P}(y, s) \leq_L \mathcal{P}(x + y, t + s)$;

(e) $\mathcal{P}(x, t) : (0, \infty) \to L$ is continuous;

(f) $lim_{t\to 0}\mathcal{P}(x, t) = 0_{\mathcal{L}}$ and $lim_{t\to\infty}\mathcal{P}(x, t) = 1_{\mathcal{L}}$.

In this case, $\mathcal{P}$ is called an $\mathcal{L}$-fuzzy norm.

Definition 2.9. A negator on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$.

Definition 2.10. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$, then $\mathcal{N}$ is called an involutive negator.

In this paper, the involutive negator $\mathcal{N}$ is fixed.

Definition 2.11. A sequence (x_n) in an $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, *_L)$ is called a Cauchy sequence if, for each $\varepsilon \in L - \{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n, m \geq n_0$, $\mathcal{P}(x_n - x_m, t) >_L \mathcal{N}(\varepsilon)$, where $\mathcal{N}$ is a negator on $\mathcal{L}$.

A sequence (x_n) is said to be convergent to $x \in V$ in the $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, *_L)$, if $\mathcal{P}(x_n - x, t) \to 1_{\mathcal{L}}$, whenever $n \to +\infty$ for all $t > 0$.

An $\mathcal{L}$-fuzzy normed space $(V, \mathcal{P}, *_L)$ is said to be complete if and only if every Cauchy sequence in V is convergent.

Definition 2.12. Let $\mathbb{K}$ be a field. A non-Archimedean absolute value on $\mathbb{K}$ is a function $|.| : \mathbb{K} \to \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have

(1) $|a| \geq 0$ and equality holds if and only if $a = 0$,

(2) $|ab| = |a||b|$,

(3) $|a + b| \leq max\{|a|, |b|\}$.

Note that $|n| \leq 1$ for each integer n. We always assume, in addition, that $|.|$ is non-tivial, i.e., there exists an $a_0 \in \mathbb{K}$ such that $|a_0| \neq 0, 1$.

Definition 2.13. A non-Archimedean $\mathcal{L}$-fuzzy normed space is a triple $(V, \mathcal{P}, *_L)$, where V is a vector space over non-Archimedean field $\mathbb{K}$, $*_L$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $V \times (0, +\infty)$ such that for all $x, y \in V$ and $t, s \in (0, \infty)$, satisfying the following conditions:

(a) $\mathcal{P}(x,t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{P}(x,t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$;
(d) $\mathcal{P}(x,t) *_L \mathcal{P}(y,s) \leq_L \mathcal{P}(x+y, max\{t,s\})$;
(e) $\mathcal{P}(x,.) : (0,\infty) \to L$ is continuous;
(f) $lim_{t\to 0}\mathcal{P}(x,t) = 0_{\mathcal{L}}$ and $lim_{t\to\infty}\mathcal{P}(x,t) = 1_{\mathcal{L}}$.

Example 2.14. Let $(X, ||.||)$ be a non-Archimedean normed linear space. Then the triple $(X, \mathcal{P}, min)$, where

$$\mathcal{P}(x,t) = \begin{cases} 0, & t \leq ||x||; \\ 1, & t > ||x||. \end{cases}$$

in a non-Archimedean $\mathcal{L}$-fuzzy normed space in which $L = [0,1]$.

3. Stability of pexiderized quadratic equation in $\mathcal{L}$-fuzzy normed spaces

Let $\mathbb{K}$ be a non-Archimedean field, X a vector space over $\mathbb{K}$ and $(Y, \mathcal{P}, *_L)$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathbb{K}$.

In this section we investigate the pexiderized quadratic functional equation. We define an $\mathcal{L}$-fuzzy approximately pexiderized quadratic mapping. Let Ψ be an $\mathcal{L}$-fuzzy set on $X \times X \times [0,\infty)$ such that $\Psi(x,y,.)$ is nondecreasing,

$$\Psi(cx, cx, t) \geq_L \Psi(x, x, \tfrac{t}{|c|}), \qquad \forall x \in X, \qquad c \neq 0$$

and

$$lim_{t\to\infty}\Psi(x,y,t) = 1_{\mathcal{L}}, \qquad \forall x, y \in X, \qquad t > 0.$$

Throughout this paper, we use the notation $\prod_{j=1}^{n} a_j$ for the expression $a_1 *_L a_2 *_L ... *_L a_n$.

Definition 3.1. A mapping $f : X \to Y$ is said to be Ψ-approximately pexiderized quadratic if

(3.1)
$$\mathcal{P}(f(x+y) + f(x-y) - 2g(x) - 2h(y), t) \geq_L \Psi(x,y,t), \qquad \forall x, y \in X, t > 0.$$

Proposition 3.2. *Let $\mathbb{K}$ be a non-Archimedean field, X a vector space over $\mathbb{K}$ and $(Y, \mathcal{P}, *_L)$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathbb{K}$. Let $f : X \to Y$ be a Ψ-approximately pexiderized quadratic mapping. Suppose that f, g and* *h are odd. If there exist an* $\alpha \in \mathbb{R}$ $(\alpha > 0)$*, an integer* k*,* $k \geq 2$ *with* $|2^k| < \alpha$ *and* $|2| \neq 0$ *such that*

$$\Psi(2^{-k}x, 2^{-k}y, t) \geq_L \Psi(x, y, \alpha t), \qquad \forall x \in X, t > 0, \tag{3.2}$$

and

$$lim_{n\to\infty} \prod_{j=n}^{\infty} \mathcal{M}(x, \tfrac{\alpha^j t}{|2|^{kj}}) = 1_{\mathcal{L}}, \qquad \forall x \in X, t > 0,$$

then there exists an additive mapping $T : X \to Y$ *such that*

$$\mathcal{P}(f(x) - T(x), t) \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, \frac{\alpha^{i+1} t}{|2|^{ki}}), \qquad \forall x \in X, t > 0, \tag{3.3}$$

and

$$\mathcal{P}(g(x) + h(x) - T(x), t) \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, \tfrac{\alpha^{i+1} t}{|2|^{ki}}),$$

where

$$\mathcal{M}(x,t) = \Psi(x, x, t) *_L \Psi(x, 0, t) *_L \Psi(x, 0, t) *_L \Psi(2x, 2x, t) *_L \Psi(2x, 0, t) *_L \Psi(0, 2x, t) *_L \ldots *_L \Psi(2^{k-1}x, 2^{k-1}x, t) *_L \Psi(2^{k-1}x, 0, t) *_L \Psi(0, 2^{k-1}x, t), \qquad \forall x \in X, t > 0.$$

Proof. By changing the roles of x and y (3.1) we get

$$\mathcal{P}(f(x+y) - f(x-y) - 2g(y) - 2h(x), t) \geq_L \Psi(y, x, t). \tag{3.4}$$

It follows from (3.1), (3.4) and $|2| \leq 1$ that

$$\mathcal{P}(f(x+y) - g(x) - h(y) - g(y) - h(x), t) \geq_L \tag{3.5}$$

$$\mathcal{P}(f(x+y) - g(x) - h(y) - g(y) - h(x), \tfrac{t}{|2|}) \geq_L \mathcal{P}(f(x+y) + f(x-y) - 2g(x) - 2h(y), t) *_L \mathcal{P}(f(x+y) - f(x-y) - 2g(y) - 2h(x), t) \geq_L \Psi(x, y, t) *_L \Psi(y, x, t).$$

If we put $y = 0$ in (3.5), we get

$$\mathcal{P}(f(x) - g(x) - h(x), t) \geq_L \Psi(x, 0, t) *_L \Psi(0, x, t). \tag{3.6}$$

Similarly by putting $x = 0$ in (3.5), we have

$$\mathcal{P}(f(y) - g(y) - h(y), t) \geq_L \Psi(0, y, t) *_L \Psi(y, 0, t). \tag{3.7}$$

From (3.5), (3.6) and (3.7) we conclude that

$$\mathcal{P}(f(x+y) - f(x) - f(y), t) \geq_L \mathcal{P}(f(x+y) - g(x) - h(y) - g(y) - h(x), t) *_L \tag{3.8}$$

$$\mathcal{P}(f(x)-g(x)-h(x),t) *_L \mathcal{P}(f(y)-h(y)-g(y),t) \geq_L$$
$$\Psi(x,y,t) *_L \Psi(y,x,t) *_L \Psi(x,0,t) *_L \Psi(0,x,t) *_L \Psi(0,y,t) *_L \Psi(y,0,t).$$

We show, by induction on j, that, for all $x \in X$, $t > 0$ and $j \geq 1$,

$$\mathcal{P}(f(2^j x) - 2^j f(x), t) \geq_L \mathcal{M}_j(x,t). \tag{3.9}$$

If we put $x = y$ in (3.8), we get

$$\mathcal{P}(f(2x) - 2f(x), t) \geq_L \Psi(x,x,t) *_L \Psi(x,0,t) *_L \Psi(0,x,t). \tag{3.10}$$

This proves (3.9) for $j = 1$. Let (3.9) holds for some $j > 1$. Replacing x by $2^j x$ in (3.10), we get

$$\mathcal{P}(f(2^{j+1}x) - 2f(2^j x), t) \geq_L \Psi(2^j x, 2^j x, t) *_L \Psi(2^j x, 0, t) *_L \Psi(0, 2^j x, t).$$

Since $|2| \leq 1$, it follows that

$$\mathcal{P}(f(2^{j+1}x) - 2^{j+1}f(x), t) \geq_L \mathcal{P}(f(2^{j+1}x) - 2f(2^j x), t) *_L \mathcal{P}(2f(2^j x) -$$
$$2^{j+1}f(x), t) = \mathcal{P}(f(2^{j+1}x) - 2f(2^j x), t) *_L \mathcal{P}(f(2^j x) - 2^j f(x), t/|2|) \geq_L$$
$$\mathcal{P}(f(2^{j+1}x) - 2f(2^j x), t) *_L \mathcal{P}(f(2^j x) - 2^j f(x), t) \geq_L$$
$$\Psi(2^j x, 2^j x, t) *_L \Psi(2^j x, 0, t) *_L \Psi(0, 2^j x, t) *_L \mathcal{M}_j(x,t) = \mathcal{M}_{j+1}(x,t).$$

Thus (3.9) holds for all $j \geq 1$. In particular, we have

$$\mathcal{P}(f(2^k x) - 2^k f(x), t) \geq_L \mathcal{M}(x,t). \tag{3.11}$$

Replacing x by $2^{-(kn+k)}x$ in (3.11) and using the inequality (3.2), we obtain

$$\mathcal{P}(f(\tfrac{x}{2^{kn}}) - 2^k f(\tfrac{x}{2^{kn+k}}), t) \geq_L \mathcal{M}(\tfrac{x}{2^{kn+k}}, t) \geq_L \mathcal{M}(x, \alpha^{n+1}t)$$

and so

$$\mathcal{P}(2^{kn} f(\tfrac{x}{2^{kn}}) - 2^{k(n+1)} f(\tfrac{x}{2^{k(n+1)}}), t) \geq_L \mathcal{M}(x, \tfrac{\alpha^{n+1}t}{|2^{kn}|}).$$

Hence it follows that

$$\mathcal{P}(2^{kn} f(\tfrac{x}{2^{kn}}) - 2^{k(n+p)} f(\tfrac{x}{2^{k(n+p)}}), t) \geq_L$$
$$\prod_{j=n}^{n+p} (\mathcal{P}(2^{kj} f(\tfrac{x}{2^{kj}}) - 2^{k(j+1)} f(\tfrac{x}{2^{k(j+1)}}), t) \geq_L \prod_{j=n}^{n+p} \mathcal{M}(x, \tfrac{\alpha^{j+1}t}{|2^{kj}|}).$$

Since $lim_{n\to\infty} \prod_{j=n}^{\infty} \mathcal{M}(x, \frac{\alpha^{j+1}t}{2^{kj}}) = 1_{\mathcal{L}}$ for all $x \in X$ and $t > 0$, $\{2^{kn} f(\frac{x}{2^{kn}})\}$ is a Cauchy sequence in the non-Archimedean $\mathcal{L}$-fuzzy Banach space $(Y, \mathcal{P}, *_L)$. Hence we can define a mapping $T : X \to Y$ such that

$$lim_{n\to\infty} \mathcal{P}(2^{kn} f(\frac{x}{2^{kn}}) - T(x), t) = 1_{\mathcal{L}}. \tag{3.12}$$

Next, for all $n \geq 1$, $x \in X$ and $t > 0$, we have

$$\mathcal{P}(f(x)-2^{kn}f(\tfrac{x}{2^{kn}}),t)=\mathcal{P}(\textstyle\sum_{i=0}^{n-1}2^{ki}f(\tfrac{x}{2^{ki}})-2^{k(i+1)}f(\tfrac{x}{2^{k(i+1)}}),t)\geq_L \\ \textstyle\prod_{i=0}^{n-1}\mathcal{P}(2^{ki}f(\tfrac{x}{2^{ki}})-2^{k(i+1)}f(\tfrac{x}{2^{k(i+1)}}),t)\geq_L \prod_{i=0}^{n-1}\mathcal{M}(x,\tfrac{\alpha^{i+1}t}{|2^{ki}|})$$

and so

$$\mathcal{P}(f(x)-T(x),t)\geq_L \mathcal{P}(f(x)-2^{kn}f(\frac{x}{2^{kn}}),t)*_L \tag{3.13}$$

$$\mathcal{P}(2^{kn}f(\tfrac{x}{2^{kn}})-T(x),t)\geq_L \textstyle\prod_{i=0}^{n-1}\mathcal{M}(x,\tfrac{\alpha^{i+1}t}{|2^{ki}|})*_L \mathcal{P}(2^{kn}f(\tfrac{x}{2^{kn}})-T(x),t).$$

Taking the limit as $n\to\infty$ in (3.13), we obtain

$$\mathcal{P}(f(x)-T(x),t)\geq_L \prod_{i=1}^{\infty}\mathcal{M}(x,\frac{\alpha^{i+1}t}{|2^{ki}|}), \tag{3.14}$$

which proves (3.3). As $*_L$ is continuous, from a well known result in $\mathcal{L}$-fuzzy normed space(see [11], Chapter 12), it follows that

$$lim_{n\to\infty}\mathcal{P}(2^{kn}f(2^{-kn}(x+y))-2^{kn}f(2^{-kn}x)-2^{kn}f(2^{-kn}y),t)= \\ \mathcal{P}(T(x+y)-T(x)-T(y),t)$$

for almost all $t>0$.

On the other hand, replacing x,y by $2^{-kn}x, 2^{-kn}y$ in (3.8), we get

$$\mathcal{P}(2^{kn}f(2^{-kn}(x+y))-2^{kn}f(2^{-kn}x)-2^{kn}f(2^{-kn}y),t)\geq_L \\ \Psi(2^{-kn}x,2^{-kn}y,\tfrac{t}{|2^{kn}|})*_L\Psi(2^{-kn}y,2^{-kn}x,\tfrac{t}{|2^{kn}|})*_L\Psi(2^{-kn}x,0,\tfrac{t}{|2^{kn}|})*_L \\ \Psi(0,2^{-kn}x,\tfrac{t}{|2^{kn}|})*_L\Psi(0,2^{-kn}y,\tfrac{t}{|2^{kn}|})*_L\Psi(2^{-kn}y,0,\tfrac{t}{|2^{kn}|})\geq_L\Psi(x,y,\tfrac{\alpha^n t}{|2^{kn}|})*_L \\ \Psi(y,x,\tfrac{\alpha^n t}{|2^{kn}|})*_L\Psi(x,0,\tfrac{\alpha^n t}{|2^{kn}|})*_L\Psi(0,x,\tfrac{\alpha^n t}{|2^{kn}|})*_L\Psi(0,y,\tfrac{\alpha^n t}{|2^{kn}|})*_L\Psi(y,0,\tfrac{\alpha^n t}{|2^{kn}|}).$$

Since All terms of the right hand side of above inequality tend to 1 as $n\to\infty$, it follows from (3.6) and (3.14) that

$$\mathcal{P}(g(x)+h(x)-T(x),t)\geq_L\mathcal{P}(f(x)-T(x),t)*_L\mathcal{P}(g(x)+h(x)-f(x),t)\geq_L \\ \textstyle\prod_{i=1}^{\infty}\mathcal{M}(x,\tfrac{\alpha^{i+1}t}{|2^{ki}|})*_L\Psi(x,0,t)*_L\Psi(0,x,t)\geq_L\prod_{i=1}^{\infty}\mathcal{M}(x,\tfrac{\alpha^{i+1}t}{|2^{ki}|}).$$

For the uniqueness of T, let $T':X\to Y$ be another additive mapping such that

$$\mathcal{P}(T'(x)-f(x),t)\geq_L\mathcal{M}(x,t).$$

Then for all $x,y\in X$ and $t>0$, we have

$$\mathcal{P}(T(x)-T'(x),t)\geq_L\mathcal{P}(T(x)-2^{kn}f(\tfrac{x}{2^{kn}}),t)*_L\mathcal{P}(2^{kn}f(\tfrac{x}{2^{kn}})-T'(x),t).$$

Therefore from (3.12), we have $T=T'$. □

Proposition 3.3. *Let $\mathbb{K}$ be a non-Archimedean field, X a vector space over $\mathbb{K}$ and $(Y, \mathcal{P}, *_L)$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathbb{K}$. Let $f : X \to Y$ be a Ψ-approximately pexiderized quadratic mapping. Suppose that f, g and h are even and $f(0) = g(0) = h(0) = 0$. If there exist an $\alpha \in \mathbb{R}$ $(\alpha > 0)$, an integer k, $k \geq 2$ with $|2^k| < \alpha$ and $|2| \neq 0$ such that*

$$\Psi(2^{-k}x, 2^{-k}y, t) \geq_L \Psi(x, y, \alpha t), \qquad \forall x \in X, t > 0,$$

and

$$lim_{n\to\infty} \prod_{j=n}^{\infty} \mathcal{M}(x, \tfrac{\alpha^j t}{|2|^{kj}}) = 1_{\mathcal{L}}, \qquad \forall x \in X, t > 0,$$

then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mathcal{P}(f(x) - Q(x), t) \geq_L, \prod_{i=1}^{\infty} \mathcal{M}(x, \frac{\alpha^{i+1}t}{|2|^{ki}}) \qquad \forall x \in X, t > 0, \tag{3.15}$$

$$\mathcal{P}(Q(x) - g(x), t) \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, \tfrac{\alpha^{i+1}t}{|2|^{ki}}),$$

and

$$\mathcal{P}(Q(x) - h(x), t) \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, \tfrac{\alpha^{i+1}t}{|2|^{ki}}),$$

where

$$\mathcal{M}(x, t) = \Psi(x, x, t) *_L \Psi(x, 0, t) *_L \Psi(x, 0, t) *_L \Psi(2x, 2x, t) *_L \Psi(2x, 0, t) *_L \Psi(0, 2x, t) *_L \ldots *_L \Psi(2^{k-1}x, 2^{k-1}x, t) *_L \Psi(2^{k-1}x, 0, t) *_L \Psi(0, 2^{k-1}x, t), \qquad \forall x \in X, t > 0.$$

Proof. Put $y = x$ in (3.1). Then for all $x \in X$ and $t > 0$,

$$\mathcal{P}(f(2x) - 2g(x) - 2h(x), t) \geq_L \Psi(x, x, t).$$

Put $x = 0$ in (3.1), we get

$$\mathcal{P}(2f(y) - 2h(y), t) \geq_L \Psi(0, y, t), \tag{3.16}$$

for all $x \in X$ and $t > 0$. For $y = 0$, (3.1) becomes

$$\mathcal{P}(2f(x) - 2g(x), t) \geq_L \Psi(x, 0, t). \tag{3.17}$$

Combining (3.1), (3.16) and (3.17) we get

$$\mathcal{P}(f(x + y) + f(x - y) - 2f(x) - 2f(y), t) \geq_L \tag{3.18}$$

$$\mathcal{P}(f(x + y) + f(x - y) - 2g(x) - 2h(y), t) *_L \mathcal{P}(2f(y) - 2h(y), t) *_L \mathcal{P}(2f(x) - 2g(x), t) \geq_L \Psi(x, y, t) *_L \Psi(0, y, t) *_L \Psi(x, 0, t).$$

We show, by induction on j, that, for all $x \in X$, $t > 0$ and $j \geq 1$,

$$\mathcal{P}(f(2^j x) - 4^j f(x), t) \geq_L \mathcal{M}_j(x, t). \tag{3.19}$$

Similar the proof of Proposition (3.2) we can obtain the results. Here, by (3.15) and (3.17) we get

$$\mathcal{P}(Q(x) - g(x), t) \geq_L \mathcal{P}(Q(x) - f(x), t) *_L \mathcal{P}(f(x) - g(x), t) \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, \tfrac{\alpha^{i+1} t}{|2|^{ki}}) *_L \Psi(x, 0, t/|2|) \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, \tfrac{\alpha^{i+1} t}{|2|^{ki}}) *_L \Psi(x, 0, t) = \prod_{i=1}^{\infty} \mathcal{M}(x, \tfrac{\alpha^{i+1} t}{|2|^{ki}}).$$

A similar inequality holds for h. □

Theorem 3.4. *Let $\mathbb{K}$ be a non-Archimedean field, X a vector space over $\mathbb{K}$ and $(Y, \mathcal{P}, *_L)$ a non-Archimedean $\mathcal{L}$-fuzzy Banach space over $\mathbb{K}$. Let $f : X \to Y$ be a Ψ-approximately quadratic mapping (i.e. $\mathcal{P}(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \geq_L \Psi(x, y, t)$). Suppose that $f(0) = 0$. If there exist an $\alpha \in \mathbb{R}$ ($\alpha > 0$), an integer k, $k \geq 2$ with $|2^k| < \alpha$ and $|2| \neq 0$ such that*

$$\Psi(2^{-k} x, 2^{-k} y, t) \geq_L \Psi(x, y, \alpha t), \qquad \forall x \in X, t > 0, \tag{3.20}$$

and

$$lim_{n\to\infty} \prod_{j=n}^{\infty} \mathcal{M}(x, \tfrac{\alpha^j t}{|2|^{kj}}) = 1_{\mathcal{L}}, \qquad \forall x \in X, t > 0.$$

Then there are unique mappings T and Q from X to Y such that T is additive, Q is quadratic and

$$\mathcal{P}(f(x) - T(x) - Q(x), t) \geq_L \prod_{i=1}^{\infty} \mathcal{M}(x, \frac{\alpha^{i+1} t}{|2|^{ki}}) \qquad \forall x \in X, t > 0, \tag{3.21}$$

where

$$\mathcal{M}(x, t) = \Psi(x, x, t) *_L \Psi(x, 0, t) *_L \Psi(x, 0, t) *_L \Psi(2x, 2x, t) *_L \Psi(2x, 0, t) *_L \Psi(0, 2x, t) *_L \ldots *_L \Psi(2^{k-1}x, 2^{k-1}x, t) *_L \Psi(2^{k-1}x, 0, t) *_L \Psi(0, 2^{k-1}x, t), \qquad \forall x \in X, t > 0.$$

Proof. Passing to the odd part f^o and even part f^e of f we deduce from (3.1) that

$$\mathcal{P}(f^o(x+y) + f^o(x-y) - 2f^o(x) - 2f^o(y), t) \geq_L \Psi(x, y, t)$$

and

$$\mathcal{P}(f^e(x+y) + f^e(x-y) - 2f^e(x) - 2f^e(y), t) \geq_L \Psi(x, y, t).$$

Using the proofs of Propositions (3.2) and (3.3) we get unique additive mapping T and unique quadratic mapping Q satisfying

$$\mathcal{P}(f^o(x)-T(x),t)\geq_L \prod_{i=1}^{\infty}\mathcal{M}(x,\tfrac{\alpha^{i+1}t}{|2|^{ki}}),\qquad \forall x\in X, t>0,$$

and

$$\mathcal{P}(f^e(x)-Q(x),t)\geq_L \prod_{i=1}^{\infty}\mathcal{M}(x,\tfrac{\alpha^{i+1}t}{|2|^{ki}})\qquad \forall x\in X, t>0.$$

Therefore

$$\mathcal{P}(f(x)-T(x)-Q(x),t)\geq_L \mathcal{P}(f^o(x)-T(x),t)*_L\mathcal{P}(f^e(x)-Q(x),t)\geq_L \prod_{i=1}^{\infty}\mathcal{M}(x,\tfrac{\alpha^{i+1}t}{|2|^{ki}}).$$

□

Example 3.5. Let $(X,||.||)$ be a non-Archimedean Banach space. Denote $\mathcal{T}_M(a,b)=(min\{a_1,b_1\},max\{a_2,b_2\})$ for all $a=(a_1,a_2), b=(b_1,b_2)\in L$ and let $\mathcal{P}_{\mu,\nu}$ be the fuzzy set on $X\times(0,\infty)$ defined as follows:

$$\mathcal{P}_{\mu,\nu}(x,t)=(\tfrac{t}{t+||x||},\tfrac{||x||}{t+||x||})$$

for all $t\in\mathbb{R}^+$. Then $(X,\mathcal{P}_{\mu,\nu},\mathcal{T}_M)$ is a complete non-Archimedean fuzzy normed space. Define

$$\Psi(x,y,t)=(\tfrac{t}{t+1},\tfrac{1}{t+1}).$$

It is easy to see that (3.20) holds for $\alpha=1$. Also, since

$$\mathcal{M}(x,t)=(\tfrac{t}{1+t},\tfrac{1}{1+t}),$$

we have

$$lim_{n\to\infty}\prod_{j=n}^{\infty}\mathcal{M}(x,\tfrac{\alpha^j t}{|2|^{kj}})=lim_{n\to\infty}(lim_{m\to\infty}\prod_{j=n}^{\infty}\mathcal{M}(x,\tfrac{t}{|2|^{kj}}))= lim_{n\to\infty}lim_{m\to\infty}(\tfrac{t}{t+|2^k|^n},\tfrac{|2^k|^n}{t+|2^k|})=(1,0)=1_L$$

for all $x\in X, t>0$. Let $f:X\to Y$ be a Ψ-approximately quadratic mapping. Thus all the conditions of Theorem (3.4) hold and so there exists a unique quadratic mapping $Q:X\to Y$ such that

$$\mathcal{P}_{\mu,\nu}(f(x)-Q(x),t)\geq_L(\tfrac{t}{t+|2^k|},\tfrac{|2^k|}{t+|2^k|}).$$

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