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TOTALIZATION OF THE MONTGOMERY IDENTITY

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Abstract. The aim of this note is to define the total value of the Riemann integral that can be used to generalize the well-known Montgomery identity.

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1. Introduction

Let $[a, b]$ be some compact interval in $\mathbb{R}$. It is an old result that for any function $F : [a, b] \mapsto \mathbb{R}$, which is differentiable on $[a, b]$, and its derivative f is Riemann integrable on $[a, b]$, the Montgomery identity holds (see [1])

$$F(t) = \frac{1}{b-a}\int_a^b F(x)\,dx + \int_a^b P(t,x)\,f(x)\,dx, \tag{1.1}$$

where the Peano kernel $P(t, x)$ is as follows

$$P(t,x) = \begin{cases} \frac{x-a}{b-a}, a \le x < t \\ \frac{x-b}{b-a}, t < x \le b \end{cases}. \tag{1.2}$$

The aim of this note is to define the total value of the Riemann integral that can be used to extend the above mentioned result to any real-valued function F defined and differentiable on $[a, b] \setminus E$, where E is a certain subset of $[a, b]$ at whose points F can take values $\pm\infty$ or not be defined at all. Unless otherwise stated in what follows, we assume that the endpoints of $[a, b]$ do not belong to E. Now, define point functions $F_{ex} : [a, b] \mapsto \mathbb{R}$ and $D_{ex}F : [a, b] \mapsto \mathbb{R}$ by extending F and its derivative f from $[a, b] \setminus E$ to E by $F_{ex}(x) = 0$ and $D_{ex}F(x) = 0$ for $x \in E$ (see [6]), so that

$$F_{ex}(x) = \begin{cases} F(x), \text{ if } x \in [a,b] \setminus E \\ 0, \text{ if } x \in E \end{cases} \quad \text{and} \tag{1.3}$$

$$D_{ex}F(x) = \begin{cases} f(x), \text{ if } x \in [a,b] \setminus E \\ 0, \text{ if } x \in E \end{cases}.$$

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2. Preliminaries

A partition $P[a,b]$ of $[a,b] \in \mathbb{R}$ is a finite set (collection) of interval-point pairs $\{([a_i,b_i],x_i) \mid i=1,...,\nu\}$, such that the subintervals $[a_i,b_i]$ are non-overlapping, $\cup_{i\le\nu}[a_i,b_i]=[a,b]$ and $x_i\in[a_i,b_i]$. The points $\{x_i\}_{i\le\nu}$ are the tags of $P[a,b]$, [4]. It is evident that a given partition of $[a,b]$ can be tagged in infinitely many ways by choosing different points as tags. If E is a subset of $[a,b]$, then the restriction of $P[a,b]$ to E is a finite collection of $([a_i,b_i],x_i)\in P[a,b]$ such that each pair of sets $[a_i,b_i]$ and E intersects in at least one point and all x_i are tagged in E. In symbols, $P[a,b]|_E = \{([a_i,b_i],x_i)\in P[a,b] \mid [a_i,b_i]\cap E\neq\emptyset \text{ and } x_i\in E\}$. Let $\mathcal{P}[a,b]$ be the family of all partitions $P[a,b]$ of $[a,b]$. Given $\delta:[a,b]\mapsto\mathbb{R}_+$, named a gauge, a point-interval pair $([a_i,b_i],x_i)$ is called δ-fine if $[a_i,b_i]\subseteq(x_i-\delta(x_i),x_i+\delta(x_i))$.

The collection $\mathcal{I}([a,b])$ is the family of compact subintervals I of $[a,b]$. The Lebesgue measure of the interval I is denoted by $|I|$. Any real-valued function defined on $\mathcal{I}([a,b])$ is an interval function. For a function $f:[a,b]\mapsto\mathbb{R}$, the associated interval function of f is an interval function $f:\mathcal{I}([a,b])\mapsto\mathbb{R}$, again denoted by f. If $f\equiv 0$ on $[a,b]$ then its associated interval function is trivial. The function f is said to be a null function on $[a,b]$ if the set $\{x\in[a,b]\mid f(x)\neq 0\}$ is a set of Lebesgue measure zero, see Definition 2.4 in [2].

In what follows we will use the following notations: $F(I)=F(v)-F(u)$, where u and v are the endpoints of I,

$$\Xi_f(P[a,b]) = \sum_{i\le\nu}[f(x_i)\,|[a_i,b_i]|] \text{ and } \Sigma_{\varphi F}(P[a,b]) = \sum_{i\le\nu}\varphi([a_i,b_i])\,F([a_i,b_i]).$$

Definition 2.1. For $E\subseteq[a,b]$ let $D_{ex}F(x):[a,b]\mapsto\mathbb{R}$ be defined by (1.3). Then, the point function f is said to be Riemann integrable to a real number A on $[a,b]$ if for every $\varepsilon>0$ there exists a gauge $\delta_\varepsilon\equiv\underline{\delta}_\varepsilon=\inf\{\delta_\varepsilon(x)\mid x\in[a,b]\}>0$, such that $|\Xi_{D_{ex}F}(P[a,b])-A|<\varepsilon$, whenever $P[a,b]|_E\subset P[a,b]$ and $P[a,b]\in\mathcal{P}[a,b]$ is a δ_ε-*fine* partition. In symbols, $A = vp\int_a^b f(x)\,dx$.

Definition 2.2. Let $\varphi:\mathcal{I}([a,b])\mapsto\mathbb{R}$ and $E\subseteq[a,b]$. A function $f:[a,b]\mapsto\mathbb{R}$ is the limit of φ on $[a,b]\backslash E$ if for every $\varepsilon>0$ there exists a gauge $\delta_\varepsilon\equiv\underline{\delta}_\varepsilon$, such that

$$|\varphi([a_i,b_i])-f(x_i)|<\varepsilon, \tag{2.1}$$

whenever $([a_i,b_i],x_i)\in P[a,b]\backslash P[a,b]|_E$ and $P[a,b]\in\mathcal{P}[a,b]$ is a δ_ε-*fine* partition.

3. Main results

For a given pair of real-valued point functions f and g with the primitives F and G, respectively, let $E\subset[a,b]$ be a set of points, of Lebesgue measure zero, at which they can take values $\pm\infty$ or not be defined at all and $\Delta\phi : \mathcal{I}([a,b]) \mapsto \mathbb{R}$ be an interval function defined by

$$\Delta\phi(I) = D_{ex}(FG)(I) - (\varphi(I)\, D_{ex}G(I) + \gamma(I)\, D_{ex}F(I)), \tag{3.1}$$

where $D_{ex}(FG)(I)$ denotes an interval function associated with the product of the point functions $D_{ex}F$ and $D_{ex}G$, $\varphi(I) = F_{ex}(I)/|I|$ and $\gamma(I) = G_{ex}(I)/|I|$.

Given $\varepsilon > 0$, we can define a set Γ_ε as follows

$$\Gamma_\varepsilon = \{(x, I) \mid x \in [a,b] \text{ is a point of } I \text{ and } |\Delta\phi(I)| < \varepsilon\}. \tag{3.2}$$

From the collection of all δ_ε-*fine* point-interval pairs $(x, I) \in \Gamma_\varepsilon$, a subset of $[a,b]$ may be obtained, as follows.

Definition 3.1. The set $\{x \in [a,b] \mid$ for every $\varepsilon > 0$ there exists a δ_ε-fine $(x, I) \in \Gamma_\varepsilon\}$ denoted by $(vp)_{\Delta\phi}[a,b]$ is said to be the null set of $\Delta\phi$ on $[a,b]$.

Definition 3.2. The set $[a,b] \setminus (vp)_{\Delta\phi}[a,b]$ denoted by $(vs)_{\Delta\phi}[a,b]$ is said to be the residual set of $\Delta\phi$ on $[a,b]$.

Accordingly, we are now in a position to define the notion of a residue of an interval function $F : \mathcal{I}([a,b]) \mapsto \mathbb{R}$ at $x \in [a,b]$.

Definition 3.3. An interval function $F : \mathcal{I}([a,b]) \mapsto \mathbb{R}$ is said to have a residue at $x \in [a,b]$ with residual value $\mathcal{R}(x)$ if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon \equiv \underline{\delta}_\varepsilon$, such that

$$|F(I) - \mathcal{R}(x)| < \varepsilon, \tag{3.3}$$

whenever (x, I) is a δ_ε-fine point-interval pair and x is a point of $I \in \mathcal{I}([a,b])$.

A real-valued point function $\mathcal{R}$, which is the limit of F on $[a,b]$, is called a residual function of F on $[a,b]$.

Definition 3.4. For $F : \mathcal{I}([a,b]) \mapsto \mathbb{R}$ let $E \subset [a,b]$ be its residual set. Then, the residual function $\mathcal{R}$ of F is said to be basically summable $(BS_{\delta_\varepsilon})$ on E with the sum $\Re \in \mathbb{R}$, if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon \equiv \underline{\delta}_\varepsilon$, such that $|\Sigma_F(P[a,b]|_E) - \Re| < \varepsilon$, whenever $P[a,b]|_E \subset P[a,b]$ and $P[a,b] \in \mathcal{P}[a,b]$ is a δ_ε-*fine* partition. The residual function $\mathcal{R}$ of F is BSG_{δ_ε} on E if E can be written as a countable union of sets on each of which F is BS_{δ_ε}. In symbols, $\Re = \sum_{x \in E} \mathcal{R}(x)$.

Remark 3.5. By Definition 5.11 in [2], if $\Re = 0$ above, then F has negligible variation on E. However, if there is a set $E \subset [a,b]$ of variation zero, then F does not satisfy the variational Strong Lusin condition on $[a,b]$. Here E is of variation zero if, given $\varepsilon > 0$ there is a gauge $\delta_\varepsilon \equiv \underline{\delta}_\varepsilon$ such that $|\Sigma_I(P[a,b]|_E)| < \varepsilon$, whenever $P[a,b]|_E \subset P[a,b]$ and $P[a,b] \in \mathcal{P}[a,b]$ is δ_ε-fine partition, [3]; on which $\mathcal{R}$ of F is BS_{δ_ε} with $\Re \neq 0$. On the other hand, since for every $\varepsilon > 0$ there exists a gauge δ_ε such that $|F(I)| < \varepsilon$, whenever (x, I) is a δ_ε-fine point-interval pair tagged in $(vp)_F[a,b]$ and x is a point of $I \in \mathcal{I}([a,b])$, it follows immediately that $\mathcal{R}(x) \equiv 0$ on $(vp)_F[a,b]$. In addition, for a given pair of functions F and $\mathcal{R}$, if F is an additive function, and $\mathcal{R}$ vanishes identically on the whole interval $[a,b]$, then $F([a,b]) = \sum_{x\in[a,b]}\mathcal{R}(x)$. So, if $F_{ex} : [a,b] \mapsto \mathbb{R}$ is the primitive defined by (1.3), then using the Newton-Leibniz formula we may obtain that for any compact interval $I \subset [a,b] \setminus E$

$$\sum_{x\in I}\mathcal{R}(x) = F(I) = \int_I f dx.$$

Therefore, if $E \subset [a,b]$ is a set of points of Lebesgue measure zero at which a real-valued function F can take values $\pm\infty$ or not be defined at all and, in addition, E is the residual set of the interval function $F_{ex} : \mathcal{I}([a,b]) \mapsto \mathbb{R}$ associated to the point function $F_{ex} : [a,b] \mapsto \mathbb{R}$ ($E = (vs)_{F_{ex}}[a,b]$), then we can divide the infinite sum of all values of the null function $\mathcal{R}$ as a residual function of F_{ex} on $[a,b]$ into two sums $\sum_{x\in(vp)_{F_{ex}}[a,b]}\mathcal{R}(x) = vp\int_a^b f dx$ and $\sum_{x\in E}\mathcal{R}(x)$, so that

$$F([a,b]) = \sum_{x\in[a,b]}\mathcal{R}(x) = vp\int_a^b f dx + \sum_{x\in E}\mathcal{R}(x).$$

In what follows, we will prove the lemma that gives us this result explicitly. Clearly, if $vp\int_a^b f dx$ does not exist, then the right-hand side of the previous equation is reduced to the so-called indeterminate expression $\infty - \infty$ that actually have, in this situation, the real numerical value of $F([a,b])$.

Now, we are in a position to define the total value (vt) of the Riemann integral of a real-valued function f with the primitive F (f is the limit of the interval function $\varphi : \mathcal{I}([a,b]) \mapsto \mathbb{R}$ on $[a,b] \setminus E$, defined by $\varphi(I) = F_{ex}(I)/|I|$, where E be a non-empty subset of $[a,b]$ of Lebesgue measure zero), [5].

Definition 3.6. For a compact interval $[a,b] \in \mathbb{R}$ let $E_f \subset [a,b]$ and $E_g \subset [a,b]$ be non-empty sets of Lebesgue measure zero, such that $E_f \cap E_g = \emptyset$. In addition, let $D_{ex}F : [a,b] \mapsto \mathbb{R}$ and $D_{ex}G : [a,b] \mapsto \mathbb{R}$ be defined according to (1.3) via any pair of real-valued functions f and g, with their primitives F and G, respectively, each of which is the limit of the corresponding interval function $\varphi(I) = F_{ex}(I)/|I|$ or $\gamma(I) = G_{ex}(I)/|I|$ on the corresponding set of points $[a,b] \setminus E_f$ or $[a,b] \setminus E_g$, respectively. The function f is said to be totally Riemann integrable, with respect to $dG = gdx$, to a real number $\Im$ on $[a,b]$, if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon \equiv \underline{\delta}_\varepsilon$, such that $|\Sigma_{\varphi G_{ex}}(P[a,b]) - \Im| < \varepsilon$, whenever $P[a,b]\,|_{E_f\cup E_g} \subset P[a,b]$ and $P[a,b] \in \mathcal{P}[a,b]$ is a δ_ε-fine partition. In symbols, $\Im = vt\int_a^b f dG$.

Clearly, if $G = x$, then $vt\int_a^b f dx = F([a,b])$, that is,

$$vt\int_a^b f dx = vp\int_a^b f dx + \sum_{x\in E_f}\mathcal{R}(x). \tag{3.4}$$

Our main result reads as follows.

Theorem 3.7. *For any compact interval $[a,b] \in \mathbb{R}$ and any pair of real-valued functions f and g with their primitives F and G, let E be a non-empty subset of $[a,b]$ at whose points the null function $d\phi$, as the limit of the interval function $\Delta\phi(I)$ on $[a,b]$, defined by (3.1), does not vanish. If E is a set of Lebesgue measure zero and the residual function $\mathcal{R}$ of $\Delta\phi$ is basically summable $(BS_{\delta_\varepsilon})$ on E with the sum $\Re$, then the function $d\phi$ is totally Riemann integrable on $[a,b]$ and*

$$vt\int_a^b d\phi = \Re. \tag{3.5}$$

In addition,

$$vp\int_a^b d\phi = 0. \tag{3.6}$$

Before starting with the proof we give the following lemma.

Lemma 3.8. *Let E be a non-empty subset of $[a,b]$. If a function f with primitive F (both are extended from $[a,b]\backslash E$ to $[a,b]$ by $D_{ex}F : [a,b] \mapsto \mathbb{R}$ and $F_{ex} : [a,b] \mapsto \mathbb{R}$, respectively) is totally Riemann integrable to the real number $\Im$ on $[a,b]$ and the null function $\mathcal{R}$, as a residual function of $F_{ex} : \mathcal{I}([a,b]) \mapsto \mathbb{R}$ on $[a,b]$, is basically summable $(BS_{\delta_\varepsilon})$ to the sum $\Re$ on E, then f is Riemann integrable on $[a,b]$ and*

$$vp\int_a^b f dx = vt\int_a^b f dx - \Re. \tag{3.7}$$

Proof. Given $\varepsilon > 0$ we will construct a gauge for f as follows. Since f is the limit of φ on $[a,b]\backslash E$ it follows from Definition 2.2 that for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon^\bigstar \equiv \underline{\delta}_\varepsilon^\bigstar$ on $[a,b]$ such that

$$\left|\Xi_f\left(P[a,b]\backslash P[a,b]|_E\right) - \left(\Sigma_{F_{ex}}P[a,b] - \Sigma_{F_{ex}}P[a,b]|_E\right)\right| < \varepsilon\left|[a,b]\right|, \tag{3.8}$$

whenever $P[a,b]|_E \subset P[a,b]$ and $P[a,b] \in \mathcal{P}[a,b]$ is a $\delta_\varepsilon^\bigstar$-*fine* partition. In addition, f is totally Riemann integrable to the real number $\Im$ on $[a,b]$, so that for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon^* \equiv \underline{\delta}_\varepsilon^*$ on $[a,b]$ such that $|\Sigma_{F_{ex}}(P[a,b]) - \Im| < \varepsilon$, whenever $P[a,b]|_E \subset P[a,b]$ and $P[a,b] \in \mathcal{P}[a,b]$ is a δ_ε^*-*fine* partition. Choose a gauge $\delta_\varepsilon^\star \equiv \underline{\delta}_\varepsilon^\star$ as required in Definition 3.4 above. The constant function $\delta_\varepsilon(x) \equiv \underline{\delta}_\varepsilon = min\left(\underline{\delta}_\varepsilon^\bigstar, \underline{\delta}_\varepsilon^*, \underline{\delta}_\varepsilon^\star\right)$ is a gauge on $[a,b]$.

We now let $P[a,b] = \{([a_i,b_i], x_i) \mid i = 1,...,\nu\}$ be a δ_ε-fine partition of $[a,b]$ such that $P[a,b]|_E \subset P[a,b]$. It is readily seen that (remember $D_{ex}F(x) = 0$ if $x \in E$)

$$\begin{gathered}\left|\Xi_{D_{ex}F}(P[a,b]) - (\Im - \Re)\right| = \\ = \left|\Xi_f\left(P[a,b]\backslash P[a,b]|_E\right) - \left(\Sigma_{F_{ex}}P[a,b] - \Sigma_{F_{ex}}P[a,b]|_E\right) + \right. \\ \left. + \Xi_{D_{ex}F}\left(P[a,b]|_E\right) + \left[\Sigma_{F_{ex}}P[a,b] - \Im\right] - \left[\Sigma_{F_{ex}}\left(P[a,b]|_E\right) - \Re\right]\right| \le\end{gathered}$$

$$\le \left| \Xi_f \left(P\left[a,b\right] \backslash P\left[a,b\right]|_E \right) - \left(\Sigma_{F_{ex}} P\left[a,b\right] - \Sigma_{F_{ex}} P\left[a,b\right]|_E \right) \right| +$$
$$+ \left| \left[\Sigma_{F_{ex}} P\left[a,b\right] - \Im \right] \right| + \left| \Sigma_{F_{ex}} \left(P\left[a,b\right]|_E \right) - \Re \right| < \left(\left| \left[a,b\right] \right| + 2 \right) \varepsilon.$$

Therefore, f is Riemann integrable on $[a,b]$ and

$$vp \int_a^b f dx = \Im - \Re.$$

□

Remark 3.9. For an illustration of (3.7) we consider the Heaviside unit function defined by

$$F\left(x\right) = \begin{cases} 0, \text{ if } x \le 0 \\ 1, \text{ otherwise} \end{cases} .$$

Since $\Sigma_F \left(P\left[a,b\right]\right) \equiv 1$, whenever $P\left[a,b\right] \in \mathcal{P}\left[a,b\right]$, it follows from Definition 3.6 that $vt \int_a^b f dx = 1$, where

$$f\left(x\right) = \begin{cases} +\infty, \text{ if } x = 0 \\ 0, \text{ otherwise} \end{cases}$$

is the derivative of F and $[a,b]$ is a compact interval within which is the null point. In addition, $vp \int_a^b f dx = 0$, so that $\mathcal{R}\left(0\right) = 1$.

Let $[a,b]$ be as above. Consider the real-valued function $F\left(x\right) = 1/x$ that is differentiable to $f\left(x\right) = -\left(1/x^2\right)$ at all but the exceptional set $\{0\}$ of $[a,b]$. In spite of the fact that f is not integrable (in the sense of the generalized Riemann integrals) on $[a,b]$, it follows from Definition 3.6 that $vt \int_a^b f dx = \left(a-b\right)/\left(ab\right)$. The residual function $\mathcal{R}$ of F is not defined at the point $x = 0$, that is

$$\mathcal{R}\left(x\right) = \begin{cases} +\infty, \text{ if } x = 0 \\ 0, \text{ otherwise} \end{cases} .$$

Now, $vt \int_a^b f dx$ is reduced to the so-called indeterminate expression $\infty - \infty$ (here $vp \int_a^b f dx = -\infty$) that actually have, in this situation, the real numerical value of $\left(a-b\right)/\left(ab\right)$.

Let $C : [0,1] \mapsto \mathbb{R}$ be the Cantor function, [2]. Its derivative c is not defined on the Cantor set $\mathcal{C}$. Since the Riemann integral of c (c_{ex} vanishes identically on $[0,1]$) is equal to zero on $[0,1]$ ($vp \int_0^1 c dx = 0$), it follows from Definition 3.6 and (3.7) that

$$\Re = vt \int_0^1 c dx = C\left(\left[a,b\right]\right) = 1,$$

where $\Re = \sum_{x \in \mathcal{C}} \mathcal{R}\left(x\right)$. So, the sum of the changes in the value of C over $\mathcal{C}$ is reduced to the so-called indeterminate expression $\infty \cdot 0$ (the residue function $\mathcal{R}$ of C vanishes identically on $[0,1]$ because C is continuous on $[0,1]$), that actually have, in this situation, the real numerical value of 1 (it means that C is not absolutely continuous and has no negligible variation on $\mathcal{C}$). Let's prove it once more. For the Cantor function with the total length of 2 on $[0,1]$ the total length of all line segments contained within $[0,1]\backslash\mathcal{C}$, on each of which C is constant, is as follows

$$\frac{1}{2}\sum_{n=1}^{+\infty}(\frac{2}{3})^n=\frac{1}{2}(3-1)=1.$$

Hence, the sum of the changes in the value of C over $\mathcal{C}$, is equal to $2-1$, meaning that $\sum_{x\in\mathcal{C}}\mathcal{R}(x)=1$.

We now turn to the proof of Theorem 3.7.

Proof. Fix some $\varepsilon>0$. By Definition 3.1 there exists a constant gauge $\delta^*_\varepsilon(x)\equiv\underline{\delta}^*_\varepsilon$ such that $|\Sigma_{\Delta\phi}(P[a,b]\backslash P[a,b]|_E)|<\varepsilon$, whenever $P[a,b]|_E\subset P[a,b]$ and $P[a,b]\in\mathcal{P}[a,b]$ is a δ_ε-*fine* partition. If $\delta_\varepsilon(x)\equiv\underline{\delta}_\varepsilon=min(\underline{\delta}^*_\varepsilon,\underline{\delta}^\star_\varepsilon)$, where $\underline{\delta}^\star_\varepsilon$ is a gauge as required in Definition 3.4, then $|\Sigma_{\Delta\phi}(P[a,b])-\Re|<2\varepsilon$, whenever $P[a,b]|_E\subset P[a,b]$ and $P[a,b]\in\mathcal{P}[a,b]$ is a δ_ε-*fine* partition. Therefore, it follows from Definition 3.6 that $d\phi$ being the limit of $\Delta\phi$ is totally Riemann integrable on $[a,b]$ and

$$vt\int_a^b d\phi=\Re.$$

Finally, based on the result of Lemma 3.8.

$$vp\int_a^b d\phi=0.$$

□

Remark 3.10. It is easy to see that the total Riemann integral has the linearity property. Hence, if $\Delta\phi$ has negligible variation on E, then

$$vt\int_a^b d\phi=0,$$

that is,

$$vt\int_a^b d(fg)=vt\int_a^b gdf+vt\int_a^b fdg. \tag{3.9}$$

Let f be the Peano kernel $P(t,x)$ defined by (1.2) and let F be a real-valued function with the primitive $\digamma$. The corresponding interval function $\Delta\phi$, defined by (3.1) for this pair of functions, has negligible variation on $E\cup\{t\}$, where $E\subset(a,b)\backslash\{t\}$, as the residual set of F, is a set of points of Lebesgue measure zero, at which F can take values $\pm\infty$ or not be defined at all. Since $vt\int_a^b d(PF)=P(t,b)F(b)-P(t,a)F(a)=0$ it follows that

$$vt\int_a^b FdP+vt\int_a^b PdF=0.$$

By Definitions 3.4 and 3.6

$$vt\int_a^b FdP = \frac{1}{b-a}vt\int_a^b Fdx - F(t),$$

taking into consideration the fact that the residue of the interval function $(FP)_{ex}(I)$ at the point t is $-F(t)$. Hence,

$$F(t) = \frac{1}{b-a}vt\int_a^b Fdx + vt\int_a^b PdF, \tag{3.10}$$

that represents totalization of the Montgomery identity (1.1).

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