Novi Sad J. Math.
Vol. 45, No. 2, 2015, 59-76

CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS ASSOCIATED WITH A UNIFICATION OF THE SRIVASTAVA-ATTIYA AND CHO-SAIGO-SRIVASTAVA OPERATORS[1]

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Abstract. We introduce a new subclass corresponding to the class of $k-$uniformly convex and starlike functions associated with Hurwitz-Lerch zeta functions and determine many properties like the coefficient estimates, extreme points, closure theorem, distortion bounds, radii of starlikeness and convexity. Furthermore, we obtain an integral transform results, neighborhood results, integral means inequalities and subordination results.

AMS Mathematics Subject Classification (2010): 30C45

Key words and phrases: Univalent; starlike; convex; uniformly convex; uniformly starlike; Hadamard product; integral means; subordination factor sequence; Hurwitz-Lerch zeta functions

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $\mathbb{U} = \{z \in \mathbb{C},\ |z| < 1\}$. Denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in \mathbb{U} \tag{1.2}$$

introduced and studied by Silverman [23]. A function $f(z) \in \mathcal{T}$ is starlike of order γ $(0 \le \gamma < 1)$ denoted by $\mathcal{T}^*(\gamma)$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma$$

and it is convex of order γ $(0 \le \gamma < 1)$ denoted by $\mathcal{C}(\gamma)$, if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma.$$

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The study of operators plays an important role in the geometric function theory and its related fields.Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better.

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \tag{1.3}$$

In the following we recall a general Hurwitz-Lerch zeta function $\Phi(z, s, a)$ defined in [27],

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \tag{1.4}$$

$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}, \text{when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1)$ where as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}$, $(\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, ...\}; \mathbb{N} := \{1, 2, 3, ...\})$. Several interesting properties and characteristics of the Hurwitz-Lerch zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [3, 28], Garg et al. [6], Lin and Srivastava [10] (see also [11, 20]).

In 2007, Srivastava and Attiya [26] introduced and investigated the linear operator:

$$\mathcal{J}_b^{\mu} : \mathcal{A} \to \mathcal{A}$$

defined, in terms of the Hadamard product (or convolution), by

$$\mathcal{J}_b^{\mu} f(z) = G_{b,\mu} * f(z) \tag{1.5}$$

$(z \in \mathbb{U}; b \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mu \in \mathbb{C}; f \in \mathcal{A})$, where, for convenience,

$$G_{\mu,b}(z) := (1+b)^{\mu}[\Phi(z, \mu, b) - b^{-\mu}] \quad (z \in \mathbb{U}). \tag{1.6}$$

It is easy to observe from (1.5) and (1.6) that, for $f \in \mathcal{A}$, we have

$$\mathcal{J}_b^{\mu} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^{\mu} a_n z^n. \tag{1.7}$$

Motivated essentially by the Srivastava-Attiya [26], Murugusundramoorthy [16, 17, 18, 19] introduced the generalized integral operator

$$\mathcal{J}_{\mu,b}^{m,\tau} f(z) = z + \sum_{n=2}^{\infty} C_n^m(b, \mu) a_n z^n, \tag{1.8}$$

where

$$C_n^m(b,\mu) = \left|\left(\frac{1+b}{n+b}\right)^{\mu} \frac{m!(n+\tau-2)!}{(\tau-2)!(n+m-1)!}\right| \tag{1.9}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mu \in \mathbb{C}, \tau \geq 2$ and $m > -1$. It is of interest to note that $J_{\mu,b}^{1,2}$ is the Srivastava-Attiya operator and $J_{0,b}^{m,\tau}$ is the well-known Choi-Saigo- Srivastava operator (see [3, 4]). Suitably specializing the parameters m, τ, μ and b in $\mathcal{J}_{\mu,b}^{m,\tau} f(z)$ we can get various integral operators as listed below.

1. For $\mu = 0$ and $b = 0$

$$\mathcal{J}_{0,0}^{1,2} f(z) := f(z). \tag{1.10}$$

2. For $\mu = 1$ and $b = 0$

$$\mathcal{J}_{1,0}^{1,2} f(z)(z) := \int_0^z \frac{f(t)}{t} dt = z + \sum_{n=2}^{\infty} \frac{1}{n} a_n z^n := \mathcal{L} f(z). \tag{1.11}$$

3. For $\mu = 1$ and $b = \nu (\nu > -1)$

$$\mathcal{J}_{1,\nu}^{1,2} f(z) := \frac{1+\nu}{z^{\nu}} \int_0^z t^{1-\nu} f(t) dt = z + \sum_{n=2}^{\infty} \left(\frac{1+\nu}{n+\nu}\right) a_n z^n := \mathcal{B}_{\nu} f(z). \tag{1.12}$$

4. For $\mu = \sigma$ $(\sigma > 0)$ and $b = 1$

$$\mathcal{J}_{\sigma,1}^{1,2} f(z) := z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^{\sigma} a_n z^n := \mathcal{I}^{\sigma} f(z). \tag{1.13}$$

Here $\mathcal{L} f(z)$ and $\mathcal{B}_{\nu} f(z)$ are the integral operators introduced by Alexandor [1] and Bernardi [2], respectively, and $\mathcal{I}^{\sigma} f$ is the Jung-Kim-Srivastava integral operator [12] closely related to some multiplier transformation studied by Fleet [5]. Analogously to the class studied by Rønning [21] and Kannas et al.,[8], making use of the operator $\mathcal{J}_{\mu,b}^{m,\tau}$, we introduce a new subclass of analytic functions with negative coefficients and discuss some interesting properties of this generalized function class.

For $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$ and $k \geq 0$, we let $\mathcal{G}_{\mu,b}^{m,\tau}(\lambda, \gamma, k)$ be the subclass of $\mathcal{T}$ consisting of functions of the form (1.2) and satisfying the analytic criterion

$$\Re \left(\frac{F_{\lambda}(z)}{zF_{\lambda}'(z)} - \gamma\right) > k \left|\frac{F_{\lambda}(z)}{zF_{\lambda}'(z)} - 1\right|, \tag{1.14}$$

where

$$\frac{F_{\lambda}(z)}{zF_{\lambda}'(z)} = \frac{(1-\lambda)\mathcal{J}_{\mu,b}^{m,\tau} f(z) + \lambda z \left(\mathcal{J}_{\mu,b}^{m,\tau} f(z)\right)'}{z\left(\mathcal{J}_{\mu,b}^{m,\tau} f(z)\right)' + \lambda z^2 \left(\mathcal{J}_{\mu,b}^{m,\tau} f(z)\right)''}, (z \in \mathbb{U}) \tag{1.15}$$

and $\mathcal{J}_{\mu,b}^{m,\tau} f(z)$ is given by (1.8).

By suitably specializing the values of μ, α, β and λ, the class $\mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$ yields various subclasses of $\mathcal{T}$ associated with Hurwitz-Learch Zeta function. As illustrations, we present some examples

Example 1.1. For $\lambda = 0$ we let $\mathcal{GS}_{\mu,b}^{m,\tau}(\gamma,k)$ be the subclass of $\mathcal{T}$ consisting of functions of the form (1.2) and satisfying the analytic criterion

$$\Re\left(\frac{\mathcal{J}_{\mu,b}^{m,\tau} f(z)}{z(\mathcal{J}_{\mu,b}^{m,\tau} f(z))'} - \gamma\right) > k\left|\frac{\mathcal{J}_{\mu,b}^{m,\tau} f(z)}{z(\mathcal{J}_{\mu,b}^{m,\tau} f(z))'} - 1\right|, \quad z \in \mathbb{U}. \tag{1.16}$$

Further for $\lambda = 0$ and different choices of μ and b we can state various subclasses of $\mathcal{G}_{\mu,b}^{m,\tau}(\gamma,k)$.

Example 1.2. For $0 \leq \gamma < 1$ and if $\tau = 2$ and $m = 1, \lambda = 0$ with $\mu = 0, b = 0$ we let $\mathcal{G}_{0,0}^{1,2}(\gamma,k) \equiv \mathcal{GT}_p(\gamma,k)$ be the subclass of $\mathcal{T}$ consisting of functions of the form (1.2) and satisfying the analytic criterion

$$\Re\left(\frac{f(z)}{zf'(z)} - \gamma\right) > k\left|\frac{f(z)}{zf'(z)} - 1\right|, \quad z \in \mathbb{U}. \tag{1.17}$$

Example 1.3. For $0 \leq \gamma < 1$ and if $\tau = 2$ and $m = 1, \lambda = 0$ with $\mu = 1, b = 0$ we let $\mathcal{G}_{1,0}^{1,2}(\gamma,k) \equiv \mathcal{TL}(\gamma,k)$ be the subclass of $\mathcal{T}$ consisting of functions of the form (1.2) and satisfying the analytic criterion

$$\Re\left(\frac{\mathcal{L}f(z)}{z(\mathcal{L}f(z))'} - \gamma\right) > k\left|\frac{\mathcal{L}f(z)}{z(\mathcal{L}f(z))'} - 1\right|, \quad z \in \mathbb{U}$$

where $\mathcal{L}_b f(z)$ is defined by $\mathcal{L}f(z) := z - \sum\limits_{n=2}^{\infty}\left(\frac{1}{n}\right)a_n z^n$.

Example 1.4. For $0 \leq \gamma < 1$ and if $\tau = 2; m = 1; \lambda = 0$ with $b = \nu(\nu > -1)$, and $\mu = 1$, we let $\mathcal{G}_{1,\nu}^{1,2}(\gamma,k) \equiv \mathcal{BT}_\nu(\gamma,k)$ be the subclass of $\mathcal{T}$ consisting of functions of the form (1.2) and satisfying the analytic criterion

$$\Re\left(\frac{\mathcal{B}_\nu f(z)}{z(\mathcal{B}_\nu f(z))'} - \gamma\right) > k\left|\frac{\mathcal{B}_\nu f(z)}{z(\mathcal{B}_\nu f(z))'} - 1\right|, \quad z \in \mathbb{U}$$

where $\mathcal{B}_\nu f(z)$ is given by $\mathcal{B}_\nu f(z) := z - \sum\limits_{n=2}^{\infty}\left(\frac{1+\nu}{n+\nu}\right)a_n z^n$.

Note that the operator $\mathcal{B}_1$ was studied earlier by Libera [13] and Livingston [15].

Example 1.5. For $0 \leq \gamma < 1$ and if $\tau = 2; \lambda = 0$ and $m = 1$ with $b = 1$, $\mu = \sigma\,(\sigma > 0)$, we let $\mathcal{G}_{\sigma,1}^{1,2}(\gamma,k) \equiv \mathcal{I}^\sigma(\gamma,k)$ be the subclass of $\mathcal{T}$ consisting of functions of the form (1.2) and satisfying the analytic criterion

$$\Re\left(\frac{\mathcal{I}^\sigma f(z)}{z(\mathcal{I}^\sigma f(z))'} - \alpha\right) > \beta\left|\frac{\mathcal{I}^\sigma f(z)}{z(\mathcal{I}^\sigma f(z))'} - 1\right|, \quad z \in \mathbb{U}$$

where $\mathcal{I}^\sigma f(z)$ is defined by $\mathcal{I}^\sigma f(z) := z - \sum\limits_{n=2}^{\infty}\left(\frac{2}{n+1}\right)^\sigma a_n z^n$.

Example 1.6. For $\lambda = 1$ we let $\mathcal{GC}^{m,\tau}_{\mu,b}(\gamma, k)$ be the subclass of $\mathcal{T}$ consisting of functions of the form (1.2) and satisfying the analytic criterion

$$\Re\left(\frac{(\mathcal{J}^{m,\tau}_{\mu,b} f(z))'}{[z(\mathcal{J}^{m,\tau}_{\mu,b} f(z))']'} - \gamma\right) > k \left|\frac{(\mathcal{J}^{m,\tau}_{\mu,b} f(z))'}{[z(\mathcal{J}^{m,\tau}_{\mu,b} f(z))']'} - 1\right|, \quad z \in \mathbb{U}.$$

Example 1.7. For $0 \le \gamma < 1$ and if $\tau = 2$ and $m = 1$ with $\mu = 1, b = 0$ we let $\mathcal{G}^{1,2}_{1,0}(\gamma, k) \equiv \mathcal{GUT}(\gamma, k)$ be the subclass of $\mathcal{T}$ consisting of functions of the form (1.2) and satisfying the analytic criterion

$$\Re\left(\frac{f'(z)}{[zf'(z)]'} - \gamma\right) > k \left|\frac{f'(z)}{[zf'(z)]'} - 1\right|, \quad z \in \mathbb{U}. \tag{1.18}$$

Indeed it follows from (1.17) and (1.18) that

$$f \in \mathcal{GUT}(\gamma, k) \Leftrightarrow zf' \in \mathcal{GT}_p(\gamma, k). \tag{1.19}$$

From Example 1.6, assuming the values of μ and b we can define the subclasses as stated in Examples 1.2 to 1.5.

The main object of this paper is to determine the coefficient estimates, distortion bounds, extreme points, closure theorems. Furthermore, we obtained integral transform results, neighborhood results, integral means inequalities and a subordination theorem for functions in the above mentioned class.

2. Characteristic properties of the class $\mathcal{G}^{m,\tau}_{\mu,b}(\lambda, \gamma, k)$

We recall the following lemmas, in order to prove our main results.

Lemma 2.1. *If γ is a real number and w is a complex number , then*

$$\Re(w) \ge \gamma \Leftrightarrow |w + (1 - \gamma)| - |w - (1 + \gamma)| \ge 0.$$

Lemma 2.2. *If w is a complex number and γ, k are real numbers, then*

$$\Re(w) \ge k|w - 1| + \gamma \Leftrightarrow \Re\{w(1 + ke^{i\theta}) - ke^{i\theta}\} \ge \gamma, \quad -\pi \le \theta \le \pi.$$

Theorem 2.3. *A function f of the form* (1.2) is in $\mathcal{G}^{m,\tau}_{\mu,b}(\lambda, \gamma, k)$ if and only if

$$\sum_{n=2}^{\infty}(1 + n\lambda - \lambda)|(1 + k) - n(\gamma + k)|\mathcal{C}^m_n(b, \mu)\ |a_n| \le 1 - \gamma, \tag{2.1}$$

where $0 \le \lambda \le 1$, $0 \le \gamma < 1$, $k \ge 0$ and $\mathcal{C}^m_n(b, \mu)$ is given by (1.9) .

Proof. Let a function f of the form (1.2) and such that $f \in \mathcal{T}$ satisfy the condition (2.1). We will show that (1.14) is satisfied and so $f \in \mathcal{G}^{m,\tau}_{\mu,b}(\lambda, \gamma, k)$. Using Lemma 2.2, it is enough to show that

$$\Re\left(\frac{F_\lambda(z)}{zF'_\lambda(z)}(1 + ke^{i\theta}) - ke^{i\theta}\right) > \gamma, \quad -\pi \le \theta \le \pi. \tag{2.2}$$

That is, suppose $f \in \mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$. Then by Lemma 2.2, we have (2.2). Choosing the values of z on the positive real axis the inequality (2.2) reduces to

$$\Re\left(\frac{(1-\gamma)-\sum\limits_{n=2}^{\infty}\left[(1+ke^{i\theta})-n(\gamma+ke^{i\theta})\right](1+\lambda n-\lambda)\mathcal{C}_n^m(b,\mu)|a_n|z^{n-1}}{1-\sum\limits_{n=2}^{\infty}n(1+n\lambda-\lambda)\mathcal{C}_n^m(b,\mu)a_nz^{n-1}}\right)\geq 0.$$

Since $\Re\,(-e^{i\theta}) \geq -e^{i0} = -1$, the above inequality reduces to

$$\Re\left(\frac{(1-\gamma)-\sum\limits_{n=2}^{\infty}(1+n\lambda-\lambda)[(k+1)-n(\gamma+k)]\mathcal{C}_n^m(b,\mu)a_nr^{n-1}}{1-\sum\limits_{n=2}^{\infty}n(1+n\lambda-\lambda)\mathcal{C}_n^m(b,\mu)a_nr^{n-1}}\right)\geq 0.$$

Letting $r \to 1^-$ and by the mean value theorem we get desired inequality (2.1).

Conversely, let (2.1) hold we will show that (1.14) is satisfied and so $f \in \mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$. In view of Lemma 2.1, $\Re(w) > \gamma \Leftrightarrow |w-(1+\gamma)| < |w+(1-\gamma)|$, it is enough to show that

$$\left|\frac{A(z)}{B(z)}-\left(1+k\left|\frac{A(z)}{B(z)}-1\right|+\gamma\right)\right|<\left|\frac{A(z)}{B(z)}+\left(1-k\left|\frac{A(z)}{B(z)}-1\right|-\gamma\right)\right|,$$

where

$$\begin{aligned} A(z) =& \ [(1-\lambda)\mathcal{J}_{\mu,b}^{m,\tau}f(z)+\lambda z(\mathcal{J}_{\mu,b}^{m,\tau}f(z))'] \\ =& \ z-\sum_{n=2}^{\infty}(1+\lambda n-\lambda)\mathcal{C}_n^m(b,\mu)|a_n|z^n \end{aligned}$$

and

$$\begin{aligned} B(z) &= [z(\mathcal{J}_{\mu,b}^{m,\tau}f(z))'+\lambda z^2(\mathcal{J}_{\mu,b}^{m,\tau}f(z))''] \\ &= z-\sum_{n=2}^{\infty}n(1+\lambda n-\lambda)\mathcal{C}_n^m(b,\mu)|a_n|z^n. \end{aligned}$$

Hence,we have

$$\begin{aligned} L &= \left|\frac{A(z)}{B(z)}-\left(1+k\left|\frac{A(z)}{B(z)}-1\right|+\gamma\right)\right| \\ &< \frac{|z|}{|B(z)|}\left|\gamma+\sum_{n=2}^{\infty}(1+n\lambda-\lambda)\left[n-1-\gamma+n(\gamma+k)\right]\mathcal{C}_n^m(b,\mu)a_nz^n\right| \\ &< \frac{|z|}{|B(z)|}\left|(2-\gamma)-\sum_{n=2}^{\infty}(1+n\lambda-\lambda)\left[(n+1+\gamma)-n(\gamma+k)\right]\mathcal{C}_n^m(b,\mu)a_nz^n\right| \\ &< R=\left|\frac{A(z)}{B(z)}+\left(1-k\left|\frac{A(z)}{B(z)}-1\right|-\gamma\right)\right|, \end{aligned}$$

and it is easy to show that $R-L>0$, by the given condition (2.1). This completes the proof. $\square$

In view of the Examples 1.1 and 1.6 in Section 1 and Theorem 2.3 we have the following theorems for the classes defined in these examples.

Theorem 2.4. *A function f of the form (1.2) is in $\mathcal{GS}_{\mu,b}^{m,\tau}(\gamma, k)$ $(0 \leq \gamma < 1, k \geq 0)$ if and only if*

$$\sum_{n=2}^{\infty} |(1+k) - n(\gamma+k)|\mathcal{C}_n^m(b,\mu)\ |a_n| \leq 1-\gamma, \tag{2.3}$$

where $\mathcal{C}_n^m(b,\mu)$is given by (1.9) .

Theorem 2.5. *A function $f(z)$ of the form (1.2) is in $\mathcal{GC}_{\mu,b}^{m,\tau}(\gamma, k)$ $(0 \leq \gamma < 1, k \geq 0)$ if and only if*

$$\sum_{n=2}^{\infty} n|(1+k) - n(\gamma+k)|\mathcal{C}_n^m(b,\mu)\ |a_n| \leq 1-\gamma, \tag{2.4}$$

where $\mathcal{C}_n^m(b,\mu)$is given by (1.9) .

Similarly, by choosing values of μ, τ, b and k one can easily state the necessary and sufficient conditions for functions in the classes defined in Examples 1.1 to 1.6 and Example 1.7.

Corollary 2.6. *If $f \in \mathcal{G}_{\mu,b}^{m,\tau}(\lambda, \gamma, k)$, then*

$$|a_n| \leq \frac{1-\gamma}{\Psi(\lambda,\gamma,k,n)}, \quad 0 \leq \lambda \leq 1,\ 0 \leq \gamma < 1, k \geq 0,$$

where

$$\Psi(\lambda,\gamma,k,n) = (1+n\lambda-\lambda)|(1+k) - n(\gamma+k)|\mathcal{C}_n^m(b,\mu), \tag{2.5}$$

and $\mathcal{C}_n^m(b,\mu)$ is given by (1.9). Equality holds for the function

$$f(z) = z - \frac{1-\gamma}{\Psi(\lambda,\gamma,k,n)} z^n.$$

For the sake of brevity we let

$$(i)\ \Psi(\lambda,\gamma,k,2) = (1+\lambda)|1-k-2\gamma|C_2^m(b,\mu), \tag{2.6}$$

and

$$(ii)\ \mathcal{C}_2^m(b,\mu) = |\left(\frac{1+b}{2+b}\right)^{\mu} \frac{\tau(\tau-1)}{(m+1)}|. \tag{2.7}$$

unless otherwise stated.

3. Distortion bounds, extreme points and closure theorem

By a routine procedure one can prove the distortion property and extreme points for function $f \in \mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$ so we state the results without proof.

Theorem 3.1. *Let the function f defined by (1.2) belong to $\mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$. Then we have*

$$r - \frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)} r^2 \leq |f(z)| \leq r + \frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)} r^2, \ |z| = r \tag{3.1}$$

and

$$1 - \frac{2(1-\gamma)}{\Psi(\lambda,\gamma,k,2)} r \leq |f'(z)| \leq 1 + \frac{2(1-\gamma)}{\Psi(\lambda,\gamma,k,2)} r, \ |z| = r. \tag{3.2}$$

Equalities are sharp for the function $f(z) = z - \frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)} z^2$, where $\Psi(\lambda,\gamma,k,2)$ is given by (2.6).

Theorem 3.2. *The extreme points of $\mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$ are*

$$f_1(z) = z \text{ and } f_n(z) = z - \frac{1-\gamma}{\Psi(\lambda,\gamma,k,n)} z^n, \quad \text{for } n = 2,3,4,\ldots. \tag{3.3}$$

where $\Psi(\lambda,\gamma,k,n)$ is defined in (2.5). Then $f \in \mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$ if and only if $f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z)$, for $\omega_n \geq 0$, and $\sum_{n=1}^{\infty} \omega_n = 1$.

Let the functions $f_j(z)$ $(j = 1,2)$ be defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j}\ z^n \text{ for } a_{n,\ j} \geq 0,\ z \in \mathbb{U}. \tag{3.4}$$

Theorem 3.3. *Let the functions $f_j(z)$ $(j = 1,2,\ldots m)$ defined by (3.4) be in the classes $\mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma_j,k)$ $(j = 1,2,\ldots m)$ respectively. Then the function $h(z)$ defined by*

$$h(z) = z - \frac{1}{m} \sum_{n=2}^{\infty} \left(\sum_{j=1}^{m} a_{n,j} \right) z^n$$

is in the class $\mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$, where $\gamma = \min_{1\leq j\leq m} \{\gamma_j\}$ with $-1 \leq \gamma_j < 1$.

Proof. Since $f_j(z) \in \mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma_j,k)$ $(j = 1,2,3,\ldots m)$, by applying Theorem 2.3

to 3.4, we observe that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\Psi(\lambda,\gamma,k,n)\left(\frac{1}{m}\sum_{j=1}^{m}a_{n,j}\right)\\
&\quad=\frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=2}^{\infty}\Psi(\lambda,\gamma,k,n)a_{n,j}\right)\\
&\quad\le\frac{1}{m}\sum_{j=1}^{m}(1-\gamma_j)\le 1-\gamma
\end{aligned}$$

whereand $\Psi(\lambda,\gamma,k,n)$ is defined in (2.5) which, in view of Theorem 2.3, again implies that $h(z)\in\mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$. So the proof is complete. $\square$

4. Integral Transform of the class $\mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$

In this section we prove that the class $\mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$ is closed under integral transform.

For $f\in\mathcal{A}$ we define the integral transform

$$\mathcal{V}_\nu(f)(z)=\int_0^1\nu(t)\frac{f(tz)}{t}dt,$$

where ν is a real valued, non-negative weight function normalized so that $\int_0^1\nu(t)dt=1$. Since special cases of $\nu(t)$ are particularly interesting, such as $\nu(t)=(1+c)t^c$, $c>-1$, for which $\mathcal{V}_\nu$ is known as the Bernardi operator, and

$$\nu(t)=\frac{(c+1)^\delta}{\nu(\delta)}t^c\left(\log\frac{1}{t}\right)^{\delta-1},\quad c>-1,\quad\delta\ge 0,$$

which gives the Komatu operator. For more details, see [9].

First we show that the class $\mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$ is closed under $\mathcal{V}_\nu(f)(z)$.

Theorem 4.1. *Let $f\in\mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$. Then $\mathcal{V}_\nu(f)(z)\in\mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$.*

Proof. By definition, we have

$$\begin{aligned}
\mathcal{V}_\nu(f)(z)&=\frac{(c+1)^\delta}{\nu(\delta)}\int_0^1(-1)^{\delta-1}t^c(\log t)^{\delta-1}\left(z-\sum_{n=2}^{\infty}|a_n|\;z^nt^{n-1}\right)dt\\
&=\frac{(-1)^{\delta-1}(c+1)^\delta}{\nu(\delta)}\lim_{r\to0^+}\left[\int_r^1t^c(\log t)^{\delta-1}\left(z-\sum_{n=2}^{\infty}|a_n|\;z^nt^{n-1}\right)dt\right].
\end{aligned}$$

A simple calculation gives

$$\mathcal{V}_\nu(f)(z)=z-\sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^\delta|a_n|\;z^n.$$

We need to prove that

$$\sum_{n=2}^{\infty} \frac{\Psi(\lambda,\gamma,k,n)}{1-\gamma}\left(\frac{c+1}{c+n}\right)^{\delta} |a_n| \ \le 1. \tag{4.1}$$

On the other hand by Theorem 2.3, $f \in \mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Psi(\lambda,\gamma,k,n)}{1-\gamma}|a_n| \ \le 1,$$

where $\Psi(\lambda,\gamma,k,n)$ is defined in (2.5). Hence $\frac{c+1}{c+n} < 1$. Therefore (4.1) holds and the proof is complete. $\square$

The above theorem yields the following two special cases.

Theorem 4.2. *If $f(z)$ is starlike of order γ then $\mathcal{V}_\nu(f)(z)$ is also starlike of order γ.*

Theorem 4.3. *If $f(z)$ is convex of order γ then $\mathcal{V}_\nu(f)(z)$ is also convex of order γ.*

Theorem 4.4. *Let $f \in \mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$. Then $\mathcal{V}_\nu(f)(z)$ is starlike of order $0 \le \xi < 1$ in $|z| < R_1$ where*

$$R_1 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\xi)\Psi(\lambda,\gamma,k,n)}{(n-\xi)(1-\gamma)}\right]^{\frac{1}{n-1}}, \ (n \ge 2)$$

where $\Psi(\lambda,\gamma,k,n)$ is defined in (2.5).

Proof. It is sufficient to prove

$$\left|\frac{z(\mathcal{V}_\nu(f)(z))'}{\mathcal{V}_\nu(f)(z)} - 1\right| < 1-\xi. \tag{4.2}$$

For the left hand side of (4.2), we have

$$\left|\frac{z(\mathcal{V}_\nu(f)(z))'}{\mathcal{V}_\nu(f)(z)} - 1\right| = \left|\frac{\sum\limits_{n=2}^{\infty}(1-n)\left(\frac{c+1}{c+n}\right)^{\delta} a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta} a_n z^{n-1}}\right|$$

$$\le \frac{\sum\limits_{n=2}^{\infty}(1-n)\left(\frac{c+1}{c+n}\right)^{\delta} |a_n| \ |z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^{\delta} |a_n| \ |z|^{n-1}}.$$

The last expression is less than $1-\xi$ since

$$|z|^{n-1} < \left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\xi)\Psi(\lambda,\gamma,k,n)}{(n-\xi)(1-\gamma)}.$$

Therefore the proof is complete. $\square$

Using the fact that $f(z)$ is convex if and only if $zf'(z)$ is starlike, we obtain the following.

Theorem 4.5. *Let $f \in \mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$. Then $\mathcal{V}_\nu(f)(z)$ is convex of order $0 \le \xi < 1$ in $|z| < R_2$ where*

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^\delta \frac{(1-\xi)\Psi(\lambda,\gamma,k,n)}{n(n-\xi)(1-\gamma)} \right]^{\frac{1}{n-1}}, \ (n \ge 2)$$

where $\Psi(\lambda,\gamma,k,n)$ is defined in (2.5).

5. Neighbourhood Results

In this section we discuss neighbourhood results of the class $\mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$. Following [7, 22], we define the $\delta-$ neighbourhood of function $f \in \mathcal{T}$ by

$$(5.1) \qquad N_\delta(f) := \left\{ h \in \mathcal{T} : h(z) = z - \sum_{n=2}^{\infty} |d_n| z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - d_n| \le \delta \right\}.$$

Particulary for the identity function $e(z) = z$, we have

$$(5.2) \qquad N_\delta(e) := \left\{ h \in \mathcal{T} : g(z) = z - \sum_{n=2}^{\infty} |d_n| z^n \text{ and } \sum_{n=2}^{\infty} n|d_n| \le \delta \right\}.$$

Theorem 5.1. *If*

$$(5.3) \qquad \delta := \frac{2(1-\gamma)}{\Psi(\lambda,\gamma,k,2)}$$

then $\mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k) \subset N_\delta(e)$, where $\Psi(\lambda,\gamma,k,2)$ is defined in (2.6).

Proof. For $f \in \mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$, Lemma 2.3 immediately yields

$$\Psi(\lambda,\gamma,k,2) \sum_{n=2}^{\infty} |a_n| \le 1-\gamma,$$

so that

$$(5.4) \qquad \sum_{n=2}^{\infty} |a_n| \le \frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)}.$$

On the other hand, from (2.1) and (5.4) that

$$
\begin{aligned}
&-(k+\gamma)(1+\lambda)\mathcal{C}_2^m(b,\mu)\sum_{n=2}^{\infty} n|a_n| \\
&\le (1-\gamma)-(1+\lambda)(1+k)\mathcal{C}_2^m(b,\mu)\sum_{n=2}^{\infty}|a_n| \\
&\le (1-\gamma)-(1+\lambda)(1+k)\mathcal{C}_2^m(b,\mu) \\
&\qquad\times\frac{(1-\gamma)}{(1+\lambda)|1-k-2\gamma|\mathcal{C}_2^m(b,\mu)} \\
&\le \frac{-2(1-\gamma)(k+\gamma)}{|1-k-2\gamma|},
\end{aligned}
$$

that is

$$
\sum_{n=2}^{\infty} n|a_n| \le \frac{2(1-\gamma)}{(1+\lambda)|1-k-2\gamma|\mathcal{C}_2^m(b,\mu)} := \frac{2(1-\gamma)}{\Psi(\lambda,\gamma,k,2)} := \delta \tag{5.5}
$$

which, in view of the definition (5.2), proves Theorem 5.1. □

Now we determine the neighborhood for the class $\mathcal{G}_{\mu,b}^{m,\tau}(\rho,\lambda,\gamma,k)$ which we define as follows. A function $f\in\mathcal{T}$ is said to be in the class $\mathcal{G}_{\mu,b}^{m,\tau}(\rho,\lambda,\gamma,k)$ if there exists a function $h\in\mathcal{G}_{\mu,b}^{m,\tau}(\rho,\lambda,\gamma,k)$ such that

$$
\left|\frac{f(z)}{h(z)}-1\right| < 1-\rho, \quad (z\in\mathbb{U},\quad 0\le\rho<1). \tag{5.6}
$$

Theorem 5.2. *If $h\in\mathcal{G}_{\mu,b}^{m,\tau}(\rho,\lambda,\gamma,k)$ and*

$$
\rho = 1-\frac{\delta\Psi(\lambda,\gamma,k,2)}{2[(\Psi(\lambda,\gamma,k,2)-(1-\gamma)]} \tag{5.7}
$$

then

$$
N_\delta(h)\subset\mathcal{G}_{\mu,b}^{m,\tau}(\rho,\lambda,\gamma,k) \tag{5.8}
$$

where $\Psi(\lambda,\gamma,k,2)$ is defined in (2.6).

Proof. Suppose that $f\in N_\delta(h)$. Then we find from (5.1) that

$$
\sum_{n=2}^{\infty} n|a_n-d_n|\le\delta,
$$

which implies the coefficient inequality

$$
\sum_{n=2}^{\infty}|a_n-d_n|\le\frac{\delta}{2}.
$$

Next, since $h \in \mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$, we have

$$\sum_{n=2}^{\infty} d_n = \frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)}$$

so that

$$\begin{aligned}
\left|\frac{f(z)}{h(z)}-1\right| &< \frac{\sum\limits_{n=2}^{\infty}|a_n-d_n|}{1-\sum\limits_{n=2}^{\infty}d_n} \\
&\leq \frac{\delta}{2}\cdot\frac{\Psi(\lambda,\gamma,k,2)}{\Psi(\lambda,\gamma,k,2)-(1-\gamma)} \\
&\leq \frac{\delta\Psi(\lambda,\gamma,k,2)}{2[(\Psi(\lambda,\gamma,k,2)-(1-\gamma)]} \\
&= 1-\rho
\end{aligned}$$

provided that ρ is given precisely by (5.7). Thus, by definition, we have $f \in \mathcal{G}_{\mu,b}^{m,\tau}(\rho,\lambda,\gamma,k)$ for ρ given by (5.7). This completes the proof. □

6. Integral Means

In [23], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$. He applied this function to resolve his integral means inequality, conjectured in [24] and settled in [25], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\eta} d\theta,$$

for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < r < 1$. In [25], he also proved his conjecture for the subclasses $\mathcal{T}^*(\gamma)$, the class of starlike functions, and $\mathcal{C}(\gamma)$, the class of convex functions with negative coefficients.

We recall the following definition and lemma to prove our result on integral means inequality.

Definition 6.1. (Subordination Principle)[14]. For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h, denoted by $g \prec h$, if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$ and $g(z) = h(w(z))$, for all $z \in \mathbb{U}$.

Lemma 6.2. *[14]. If the functions f and g are analytic in $\mathbb{U}$ with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\eta} d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\eta} d\theta. \tag{6.1}$$

Applying Lemma 6.2, Lemma 2.3 and Theorem 3.2, we prove Silverman's conjecture for the functions in the family $\mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$.

Theorem 6.3. *Suppose $f \in \mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$, $\eta > 0$, $0 \le \lambda \le 1$, $0 \le \gamma < 1$, $k \ge 0$ and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)} z^2,$$

where $\Psi(\lambda,\gamma,k,2)$ is defined in (2.6). Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \le \int_0^{2\pi} |f_2(z)|^\eta \, d\theta. \tag{6.2}$$

Proof. For $f \in \mathcal{T}$, (6.2) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n z^{n-1} \right|^\eta d\theta \le \int_0^{2\pi} \left| 1 - \frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)} z \right|^\eta d\theta.$$

By Lemma 6.2, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)} w(z), \tag{6.3}$$

and using (2.1), we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{n=2}^{\infty} \frac{\Psi(\lambda,\gamma,k,n)}{1-\gamma} a_n z^{n-1} \right| \\ &\le |z| \sum_{n=2}^{\infty} \frac{\Psi(\lambda,\gamma,k,n)}{1-\gamma} |a_n| \\ &\le |z|. \end{aligned}$$

This completes the proof . □

7. Subordination Results

Now we recall the following results due to Wilf [29], which are much required for our study.

Definition 7.1. (Subordinating Factor Sequence). A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 = 1$ is regular, univalent and convex in $\mathbb{U}$, we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z), z \in \mathbb{U}. \tag{7.1}$$

Lemma 7.2. *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re \left(1 + 2\sum_{n=1}^{\infty} b_n z^n\right) > 0, \quad z \in \mathbb{U}. \tag{7.2}$$

Theorem 7.3. *Let $f \in \mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$ and g be any function in the usual class of convex functions $\mathcal{C}$. Then we have*

$$\frac{\Psi(\lambda,\gamma,k,2)}{2[1-\gamma+\Psi(\lambda,\gamma,k,2)]}(f*g)(z) \prec g(z) \tag{7.3}$$

where $0 \leq \gamma < 1$; $k \geq 0$ and $0 \leq \lambda \leq 1$, and

$$\Re\ (f(z)) > -\frac{[1-\gamma+\Psi(\lambda,\gamma,k,2)}{\Psi(\lambda,\gamma,k,2)}, \quad z \in \mathbb{U}. \tag{7.4}$$

The constant factor $\frac{\Psi(\lambda,\gamma,k,2)}{2[1-\gamma+\Psi(\lambda,\gamma,k,2)]}$ in (7.3) cannot be replaced by a larger number.

Proof. Let $f \in \mathcal{G}_{\mu,b}^{m,\tau}(\lambda,\gamma,k)$ and suppose that $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{C}$. Then

$$\begin{aligned}&\frac{\Psi(\lambda,\gamma,k,2)}{2[1-\gamma+\Psi(\lambda,\gamma,k,2)]}(f*g)(z)\\ &= \frac{\Psi(\lambda,\gamma,k,2)}{2[1-\gamma+\Psi(\lambda,\gamma,k,2)]}\left(z + \sum_{n=2}^{\infty} b_n a_n z^n\right).\end{aligned} \tag{7.5}$$

Thus, by Definition 7.1, the subordination result holds true if

$$\left\{\frac{\Psi(\lambda,\gamma,k,2)}{2[1-\gamma+\Psi(\lambda,\gamma,k,2)]}\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 7.2, this is equivalent to the following inequality

$$\Re\left(1 + \sum_{n=1}^{\infty} \frac{\Psi(\lambda,\gamma,k,2)}{[1-\gamma+\Psi(\lambda,\gamma,k,2)]} a_n z^n\right) > 0, \quad z \in \mathbb{U}. \tag{7.6}$$

By noting the fact that $\frac{\Psi(\lambda,\gamma,k,n)}{(1-\gamma)}$ is increasing function for $n \geq 2$ and in particular

$$\frac{\Psi(\lambda,\gamma,k,2)}{1-\gamma} \leq \frac{\Psi(\lambda,\gamma,k,n)}{1-\gamma}, \quad n \geq 2,$$

therefore, for $|z| = r < 1$, we have

$$\begin{aligned}
&\Re\left(1 + \frac{\Psi(\lambda,\gamma,k,2)}{[1-\gamma+\Psi(\lambda,\gamma,k,2)]}\sum_{n=1}^{\infty} a_n z^n\right) \\
&= \Re\left(1 + \frac{\Psi(\lambda,\gamma,k,2)}{[1-\gamma+\Psi(\lambda,\gamma,k,2)]}z + \frac{\sum\limits_{n=2}^{\infty}\Psi(\lambda,\gamma,k,2)a_n z^n}{[1-\gamma+\Psi(\lambda,\gamma,k,2)]}\right) \\
&\geq 1 - \frac{\Psi(\lambda,\gamma,k,2)}{[1-\gamma+\Psi(\lambda,\gamma,k,2)]}r - \frac{\sum\limits_{n=2}^{\infty}|\Psi(\lambda,\gamma,k,n)a_n|\, r^n}{[1-\gamma+\Psi(\lambda,\gamma,k,2)]} \\
&\geq 1 - \frac{\Psi(\lambda,\gamma,k,2)}{[1-\gamma+\Psi(\lambda,\gamma,k,2)]}r - \frac{1-\gamma}{[1-\gamma+\Psi(\lambda,\gamma,k,2)]}r \\
&> 0, \quad |z| = r < 1,
\end{aligned}$$

where we have also made use of the assertion (2.1) of Theorem 2.3. This evidently proves the inequality (7.6) and hence also the subordination result (7.3) asserted by Theorem 2.3.

The inequality (7.4) follows from (7.3) by taking

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in C.$$

Next we consider the function

$$F(z) := z - \frac{1-\gamma}{\Psi(\lambda,\gamma,k,2)}z^2$$

where $0 \leq \gamma < 1,\ k \geq 0, 0 \leq \lambda < 1$ and $\Psi(\lambda,\gamma,k,2)$ is given by (2.6). Clearly $F \in \mathcal{G}_m^{*l}(\lambda,\gamma,k)$. For this function (7.3) becomes

$$\frac{\Psi(\lambda,\gamma,k,2)}{2[1-\gamma+\Psi(\lambda,\gamma,k,2)]}F(z) \prec \frac{z}{1-z}.$$

It is easily verified that

$$\min\left\{\Re\left(\frac{\Psi(\lambda,\gamma,k,2)}{2[1-\gamma+\Psi(\lambda,\gamma,k,2)]}F(z)\right)\right\} = -\frac{1}{2}, \quad z \in \mathbb{U}.$$

This shows that the constant $\frac{\Psi(\lambda,\gamma,k,2)}{2[1-\gamma+\Psi(\lambda,\gamma,k,2)]}$ cannot be replaced by any larger one. □

Concluding Remarks. In fact, suitably specializing the values of λ, γ and k the results presented in this paper would find further applications for the class of univalent starlike functions with negative coefficients stated in Examples 1.1 to 1.7 in Section 1.

Acknowledgement

I record my sincere thanks to the referees for their valuable suggestions.

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Received by the editors April 26, 2014