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LOCAL CONVERGENCE OF MODIFIED HALLEY-LIKE METHODS WITH LESS COMPUTATION OF INVERSION

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Abstract. We present a local convergence analysis of a Modified Halley-Like Method of high convergence order in order to approximate a solution of a nonlinear equation in a Banach space. Our sufficient convergence conditions involve only hypotheses on the first Fréchet-derivative of the operator involved. Earlier studies use hypotheses up to the third Fréchet-derivative [26]. Numerical examples are also provided in this study.

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1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a subset D of a Banach space X with values in a Banach space Y.

Many problems in computational sciences and other disciplines can be brought in a form like (1.1) using mathematical modeling [3]. The solutions of equation (1.1) can rarely be found in closed form. That is why most solution methods for these equations are usually iterative. In particular, the practice of Numerical Functional Analysis for finding such solutions is essentially connected to Newton-like methods [1]-[27]. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analyses. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semi-local convergence analyses of Newton-like methods such as [1]-[27].

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We present a local convergence analysis for the modified Halley-Like Method [26] defined for each $n = 0, 1, 2, \cdots$ by

$$
\begin{aligned}
y_n &= x_n - F'(x_n)^{-1}F(x_n),\\
u_n &= x_n - \theta F'(x_n)^{-1}F(x_n),\\
&= y_n + (1-\theta)F'(x_n)^{-1}F(x_n),\\
z_n &= y_n - \gamma A_{\theta,n}F'(x_n)^{-1}F(x_n),\\
x_{n+1} &= z_n - \alpha B_{\theta,n}F'(x_n)^{-1}F(z_n),
\end{aligned}
\tag{1.2}
$$

where x_0 is an initial point, $\alpha, \gamma, \theta \in (-\infty, \infty) - \{0\}$ are given parameters, $H_{\theta,n} = \frac{1}{\theta}F'(x_n)^{-1}(F'(u_n) - F'(x_n))$, $A_{\theta,n} = I - \frac{1}{2}H_{\theta,n}(I - \frac{1}{2}H_{\theta,n})$ and $B_{\theta,n} = I - H_{1,n} + H_{\theta,n}^2$. The semi-local convergence of method (1.2) was studied in [26] in the special case when $\alpha = \gamma = 1$ and $\theta \in [0, 1]$. Moreover, if $\gamma = 1$, $\alpha = 0$ and $\theta \in (0, 1]$, the semi-local convergence of the resulting method (1.2) was given in [26].

The semi-local convergence results in [26] were given in a non-affine invariant form. However, the results obtained in our paper are given in affine invariant form. The sufficient semi-local convergence conditions (given in affine invariant form) used in [26] are $(\mathcal{C})$:

$(\mathcal{C}_1)$ There exists $F'(x_0)^{-1} \in L(Y, X)$ and $\|F'(x_0)^{-1}\| \le \beta$;

$(\mathcal{C}_2)$

$$\|F'(x_0)^{-1}F(x_0)\| \le \beta_1;$$

$(\mathcal{C}_3)$

$$\|F'(x_0)^{-1}F''(x)\| \le \beta_2 \quad \text{for each } x \in D;$$

$(\mathcal{C}_4)$

$$\|F'(x_0)^{-1}(F''(x) - F''(y))\| \le \beta_3\|x - y\|^q \quad \text{for each } x, y \in D, \quad \text{and some } q \in [0, 1].$$

Under the $(\mathcal{C})$ conditions for $\alpha = \gamma = 1$ and $\theta \in (0, 1]$ the convergence order was shown to be $3 + 2q$ in [26]. Moreover, for $\gamma = 1$, $\alpha = 0$ and $\theta \in (0, 1]$ the convergence order was shown to be $2 + q$ in [9].

Similar conditions have been used by several authors on other high convergence order methods [1]-[27]. The corresponding conditions for the local convergence analysis are given by simply replacing x_0 by x^* in the preceding $(\mathcal{C})$ conditions. These conditions, however, are very restrictive. As a motivational example, let us define the function f on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0\\ 0, & x = 0\end{cases}$$

Choose $x^* = 1$. We have that

$$
\begin{aligned}
f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \ F'(1) = 3,\\
f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x\\
f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22.
\end{aligned}
$$

Then, e.g, the function f cannot satisfy the condition $(\mathcal{C}_4)$, say for $q = 1$, since the function f''' is unbounded on D. In the present paper we only use hypotheses on the first Fréchet derivative (see conditions (2.12)-(2.15)). Notice that they used $\theta \in (0,1]$, whereas in this paper θ can belong in a wider interval than $(0,1]$ and $\gamma = \alpha = 1$ in [26]. This way we expand the applicability of method (1.2).

The paper is organized as follows. The local convergence of method (1.2) is given in Section 2, whereas the numerical examples are given in Section 3. Finally, some remarks are given in the concluding Section 4.

2. Local convergence analysis

We present the local convergence analysis of method (1.2) in this section. Denote by $U(v,\rho), \bar{U}(v,\rho)$ the open and closed balls, respectively, in X with center at $v \in X$ and of radius $\rho > 0$.

Let $L_0 > 0$, $L > 0$, $\theta \in (-\infty, \infty) - \{0\}$, α, $\gamma \in (-\infty,\infty)$ and $M > 0$ be given parameters. Define the following functions on the interval $[0, \frac{1}{L_0})$ by

$$\begin{aligned}
g_1(r) &= \frac{Lr}{2(1-L_0r)}, \\
g_2(r) &= g_1(r) + \frac{M|1-\theta|}{1-L_0r}, \\
g_3(r) &= \frac{L_0(1+g_2(r))}{2|\theta|(1-L_0r)}, \\
g_4(r) &= 1 + g_3(r)r + g_3^2(r)r^2, \\
g_5(r) &= g_1(r) + \frac{|\gamma| M g_4(r)}{1-L_0r}, \\
g_6(r) &= 1 + 2g_{1,3}(r)r + 4g_3^2(r)r^2, \\
g_{1,3}(r) &= \frac{L_0(1+g_1(r))}{2(1-L_0r)}
\end{aligned}$$

and

$$g_7(r) = [1 + \frac{|\alpha| M g_6(r)}{1-L_0r}] g_5(r).$$

Moreover, define the parameter

$$r_2 = \frac{2(1-M|1-\theta|)}{2L_0+L}.$$

Suppose

$$M|1-\theta| < 1.$$

Then, it follows from the definition of the the functions g_1 and g_2 that

$$0 < g_1(r) < 1, \text{ and } 0 < g_2(r) < 1, \text{ for each } r \in (0, r_2).$$

Evidently, $g_5(r) \in (0,1)$, if for each $r \in (0, r_5)$ and $r_5 < \frac{1}{L_0}$ to be determined, we have that

$$0 < g_1(r) + \frac{|\gamma| g_4(r) M}{1 - L_0 r} < 1 \text{ for each } \ r \in (0, r_5).$$

Define the the function p_5 on the interval $[0, \frac{1}{L_0}]$ by

$$p_5(r) = |\gamma| M g_4(r) - (1 - L_0 r)(1 - g_1(r)).$$

We have that

$$p_5((\frac{1}{L_0})^-) = |\gamma| M g_4((\frac{1}{L_0})^-) > 0.$$

Suppose that

$$|\gamma| M < 1.$$

Then, we have that

$$p_5(0) = M|\gamma| - 1 < 0.$$

It follows from the intermediate value theorem that the the function p_5 has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_5 the smallest such zero. Then, we have that

$$p_5(r) < 0 \Rightarrow 0 < g_5(r) < 1 \ \text{ for each } r \in (0, r_5).$$

Similarly, the the function $g_7 \in (0,1)$ for each $r \in (0, r_7)$ and $r_7 < \frac{1}{L_0}$ to be determined, if the the function $p_7(r) \in (0,1)$ for each $r \in [0, r_7]$, where

$$p_7(r) = (1 - L_0 r + |\alpha| M g_6(r)) g_5(r) - (1 - L_0 r).$$

We get that

$$p_7((\frac{1}{L_0})^-) = |\gamma| M g_6((\frac{1}{L_0})^-) g_5((\frac{1}{L_0})^-) > 0.$$

and

$$p_7(0) = (1 + |\alpha| M g_6(0)) |\gamma| g_5(0) - 1 = (1 + |\alpha| M) |\gamma| M - 1.$$

Suppose that

$$(1 + |\alpha| M) |\gamma| M < 1.$$

Then, we have $p_7(0) < 0$. It follows that the the function p_7 has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_7 the smallest such zero. Then, we obtain that

$$p_7(0) < 0 \Rightarrow 0 < g_7(r) < 1, \ \text{ for each } r \in (0, r_7).$$

Set

$$r^* = \min\{r_2, r_5, r_7\}. \tag{2.1}$$

Then, we have that

$$0 < g_1(r) < 1, \tag{2.2}$$
$$0 < g_2(r) < 1 \tag{2.3}$$
$$0 < g_3(r) \tag{2.4}$$
$$0 < g_4(r) \tag{2.5}$$
$$0 < g_5(r) < 1 \tag{2.6}$$
$$0 < g_6(r) \tag{2.7}$$

and

$$0 < g_7(r) < 1, \text{ for each } r \in (0, r^*). \tag{2.8}$$

Next, we present the local convergence analysis of method (1.2).

Theorem 2.1. *Let $F : D \subseteq X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, parameters $L_0 > 0, L > 0$, $M > 0$, $\theta \in (-\infty, \infty) - \{0\}$ and α, $\gamma \in (-\infty, \infty)$ such that for each $x \in D$*

$$M|1 - \theta| < 1, \tag{2.9}$$

$$M|\gamma| < 1, \tag{2.10}$$

$$(1 + |\alpha|M)|\gamma|M < 1, \tag{2.11}$$

$$F(x^*) = 0, \ F'(x^*)^{-1} \in L(Y, X), \tag{2.12}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|, \tag{2.13}$$

$$\|F'(x^*)^{-1}(F(x) - F(x^*) - F'(x)(x - x^*)\| \leq \frac{L}{2}\|x - x^*\|^2, \tag{2.14}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq M \tag{2.15}$$

and

$$\bar{U}(x^*, r^*) \subseteq D, \tag{2.16}$$

where r^* is given in (2.1). Then, the sequence $\{x_n\}$ generated by method (1.2) for $x_0 \in U(x^*, r^*)$ is well defined, remains in $U(x^*, r^*)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \cdots$,

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r^*, \tag{2.17}$$

$$\|u_n - x^*\| \le g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.18}$$

$$\|H_{\theta,n}\| \le 2g_3(\|x_n - x^*\|)\|x_n - x^*\|, \tag{2.19}$$

$$\|A_{\theta,n}\| \le g_4(\|x_n - x^*\|) \tag{2.20}$$

$$\|z_n - x^*\| \le g_5(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.21}$$

$$\|B_{\theta,n}\| \le g_6(\|x_n - x^*\|) \tag{2.22}$$

and

$$\|x_{n+1} - x^*\| \le g_7(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|. \tag{2.23}$$

where the "g" the functions are defined above Theorem 2.1.

Proof. Using (2.13), the definition of r^* and the hypothesis $x_0 \in U(x^*, r^*)$ we get that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \le L_0\|x_0 - x^*\| < L_0 r^* < 1. \tag{2.24}$$

It follows from (2.24) and the Banach Lemma on invertible operators [3, 4] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \le \frac{1}{1 - L_0\|x_0 - x^*\|} < \frac{1}{1 - L_0 r^*}. \tag{2.25}$$

Hence, y_0 and u_0 are well defined. Using the first substep in method (1.2) for $n = 0$, (2.2), (2.14), (2.25) and the definition of the function g_1 we obtain in turn that

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= -F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)] \end{aligned}$$

so,

$$\begin{aligned} &\|y_0 - x^*\| \\ &\le \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)]\| \\ &\le \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*, \end{aligned}$$

which shows (2.17) for $n = 0$. We also have from the second substep of method (1.2) for $n = 0$, (2.9), (2.15), (2.17) and the definition of the functions g_1 and

g_2 that

$$\begin{aligned}\|u_0-x^*\| &\le \|y_0-x^*\|+|1-\theta|\|F'(x_0)^{-1}F'(x^*)\|\\ &\quad\times\|\int_0^1 F'(x^*+t(x_0-x^*)dt\|\|x_0-x^*\|\\ &\le [g_1(\|x_0-x^*\|)+\frac{M|1-\theta|}{1-L_0\|x_0-x^*\|}]\|x_0-x^*\|\\ &= g_2(\|x_0-x^*\|)\|x_0-x^*\|<\|x_0-x^*\|<r^*, \qquad (2.26)\end{aligned}$$

which shows (2.18) for $n=0$.

Next, we need an estimate on $\frac{1}{2}\|H_{\theta,0}\|$. We have from (2.4),(2.13), (2.25), (2.26) and the definition of the functions g_2 and g_3 that

$$\begin{aligned}\frac{1}{2}\|H_{\theta,0}\| &\le \frac{1}{2|\theta|}\|F'(x_0)^{-1}F'(x^*)\|(\|F'(x^*)^{-1}(F'(u_0)-F'(x^*))\|\\ &\quad+\|F'(x^*)^{-1}(F'(x_0)-F'(x^*))\|)\\ &\le \frac{L_0(\|u_0-x^*\|+\|x_0-x^*\|)}{2|\theta|(1-L_0\|x_0-x^*\|)}\\ &\le \frac{L_0(\|x_0-x^*\|+g_2(\|x_0-x^*\|)\|x_0-x^*\|)}{2|\theta|(1-L_0\|x_0-x^*\|)}\\ &\le \frac{L_0(1+g_2(\|x_0-x^*\|))\|x_0-x^*\|}{2|\theta|(1-L_0\|x_0-x^*\|)}\\ &= g_3(\|x_0-x^*\|)\|x_0-x^*\|, \qquad (2.27)\end{aligned}$$

which shows (2.19) for $n=0$. We also need an estimate on $\|A_{\theta,0}\|$. It follows from (2.27) and the definition of $A_{\theta,0}$, g_3, g_4 that

$$\begin{aligned}\|A_{\theta,0}\| &\le 1+\frac{1}{2}\|H_{\theta,0}\|+\frac{1}{4}\|H_{\theta,0}\|^2\\ &\le 1+g_3(\|x_0-x^*\|)\|x_0-x^*\|+g_3^2(\|x_0-x^*\|)\|x_0-x^*\|^2\\ &= g_4(\|x_0-x^*\|), \qquad (2.28)\end{aligned}$$

which shows (2.20) for $n=0$. Then, from the third substep of method (1.2) for $n=0$, (2.19), (2.20), (2.28) the definition of the functions g_1, g_5 and radius r^*, we have that

$$\begin{aligned}\|z_0-x^*\| &\le \|y_0-x^*\|+|\gamma|\|A_{\theta,0}\|\|F'(x_0)^{-1}F'(x^*)\|\\ &\quad\|\int_0^1 F'(x^*)^{-1}F'(x^*+t(x_0-x^*)dt\|\|x_0-x^*\|\\ &\le [g_1(\|x_0-x^*\|)+\frac{M|\gamma|g_4(\|x_0-x^*\|)}{1-L_0\|x_0-x^*\|}]\|x_0-x^*\|\\ &= g_5(\|x_0-x^*\|)\|x_0-x^*\|<\|x_0-x^*\|<r^*, \qquad (2.29)\end{aligned}$$

which shows (2.21) for $n=0$. Next, we need an estimate on $\|B_{\theta,0}\|$. We have by the definition of the operator $B_{\theta,0}$ and the functions $g_{1,3}$, g_3, g_6 that

$$\|B_{\theta,0}\|\le 1+2g_{1,3}(\|x_0-x^*\|)\|x_0-x^*\|+4g_3^2(\|x_0-x^*\|)\|x_0-x^*\|^2=g_6(\|x_0-x^*\|), \qquad (2.30)$$

which shows (2.22) for $n = 0$. Using the fourth substep in method (1.2) for $n = 0$, (2.3), (2.15), (2.21), (2.22), (2.29) the definition of the functions g_5, g_6, g_7 and radius r^*, we obtain that

$$\begin{aligned}
\|x_1 - x^*\| &\leq \|z_0 - x^*\| + |\alpha|\|B_{\theta,0}\|\|F'(x_0)^{-1}F'(x^*)\| \\
&\quad \|\int_0^1 F'(x^*)^{-1}F'(x^* + t(z_0 - x^*)dt\|\|z_0 - x^*\| \\
&\leq (1 + \frac{M|\alpha|g_6(\|x_0 - x^*\|)}{(1 - L_0\|x_0 - x^*\|)})\|z_0 - x^*\| \\
&= (1 + \frac{M|\alpha|g_6(\|x_0 - x^*\|)}{(1 - L_0\|x_0 - x^*\|)})g_5(\|x_0 - x^*\|)\|x_0 - x^*\|, \qquad (2.31)
\end{aligned}$$

which shows (2.23) for $n = 0$. By simply replacing y_0, u_0, z_0, x_1 by y_k, u_k, z_k, x_{k+1} in the preceding estimates we arrive at estimates (2.17)-(2.23). Finally, from the estimate $\|x_{k+1} - x^*\| < \|x_k - x^*\|$, we deduce that $\lim_{k\to\infty} x_k = x^*$. □

Remark 2.2. *1. In view of (2.13) and the estimate*

$$\begin{aligned}
\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\
&\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \\
&\leq 1 + L_0\|x - x^*\|
\end{aligned}$$

condition (2.15) can be dropped and M can be replaced by

$$M(r) = 1 + L_0 r.$$

Moreover, condition (2.14) can be replaced by the popular but stronger conditions

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\| \text{ for each } x, y \in D \qquad (2.32)$$

or

$$\|F'(x^*)^{-1}(F'(x^* + t(x - x^*)) - F'(x))\| \leq L(1 - t)\|x - x^*\| \text{ for each } x, y \in D \text{ and } t \in [0, 1].$$

2. The results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

3. The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method(GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [3, 4].

4. *The radius r_A given by*

$$r \leq r_A = \frac{1}{L_0 + \frac{L}{2}}. \tag{2.33}$$

was shown by us to be the convergence radius of Newton's method [3, 4]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{2.34}$$

under the conditions (2.13) and (2.32). It follows from (2.1) and (2.33) that the convergence radius r^* of the method (1.2) cannot be larger than the convergence radius r_A of the second order Newton's method (2.33). As already noted in [3, 4] r_A is at least as large as the convergence ball given by Rheinboldt [3, 4]

$$r_R = \frac{2}{3L}. \tag{2.35}$$

In particular, for $L_0 < L$ we have that

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \ \textit{as} \ \frac{L_0}{L} \to 0.$$

That is, our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [3, 4].

5. *It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger ($\mathcal{C}$) conditions used in [26]. Moreover, we can compute the computational order of convergence (COC) defined by*

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds given in [26] involving estimates up to the second Fréchet derivative of operator F.

3. Numerical Examples

We present numerical examples in this section.

Example 3.1. Let $X = Y = \mathbb{R}^2, D = \bar{U}(0,1), x^* = 0$ and define the function F on D by

$$F(x) = (\sin x, \frac{1}{3}(e^x + 2x - 1)). \tag{3.1}$$

Then, using (2.9)-(2.15), we get $L_0 = L = 1$, $M = \frac{1}{3}(e+2)$, $\theta = \frac{3}{4}$, $\gamma = \frac{3}{5}$, $\alpha = \frac{3}{100}$. Then, by (2.1) we obtain

$$r^* = 0.3161 < r_R = r_A = 0.6667$$

Example 3.2. Let $X = Y = \mathbb{R}^3$, $D = \overline{U}(0,1)$. Define F on D for $v = x, y, z)$ by

$$F(v) = (e^x - 1, \frac{e-1}{2}y^2 + y, z). \tag{3.2}$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0,0,0)$, $F'(x^*) = F'(x^*)^{-1} = diag\{1,1,1\}$, $L_0 = e - 1 < L = e$, $M = e$, $\theta = \frac{3}{4}$, $\gamma = \frac{3}{10}$, $\alpha = \frac{3}{100}$. Then, by (2.1) we obtain

$$r^* = 0.2136 < r_R = 0.2453 < r_A = 0.3249.$$

Example 3.3. Returning back to the motivational example at the introduction of this study, we see that conditions (2.12)–(2.15) are satisfied for $x^* = 1$, $f'(x^*) = 3$, $f(1) = 0$, $L_0 = L = 146.6629073$ and $M = 101.5578008$. Hence, the results of Theorem 2.1 can apply but not the ones in [26]. In particular, for $\theta = 0.9902$, $\alpha = 0.008$ and $\gamma = 0.005$ hypotheses (2.9)-(2.15) are satisfied. Moreover, we obtain

$$r^* = 0.0032 < r_R = 0.0045 \leq r_A = 0.0045.$$

4. Conclusion

We present a local convergence analysis of Modified Halley-Like Methods with less computation of inversion in order to approximate a solution of an equation in a Banach space setting. Earlier convergence analysis is based on Lipschitz and Holder-type hypotheses up to the second Fréchet-derivative [1]–[27]. In this paper the local convergence analysis is based only on Lipschitz hypotheses of the first Fréchet-derivative. Hence, the applicability of these methods is expanded under less computational cost of the constants involved in the convergence analysis.

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